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Time Delay Effect on Two Mathematical Fishery Models

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Abstract

We investigate the stability and bifurcation dynamics of two distinct bio-economic fishery models incorporating temporal delays that capture realistic biological and economic processes. The first model introduces delay in fishing effort costs, representing the depreciation dynamics of economic investments in fishery operations. The second model incorporates delay in the population harvesting term, reflecting the biological reality of age-selective fishing practices and temporal lags in population response to harvesting pressure. Our findings provide quantitative guidelines for fishery management, establishing mathematical criteria that balance economic viability with ecological sustainability. The contrasting behaviors of these two delay mechanisms highlight the importance of identifying the dominant temporal processes in specific fishery systems for effective management strategies.

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1. Introduction

Around the world, fisheries are essential to the survival of both human communities and ecosystems. Fish populations are, however, dropping as a result of overfishing and poor management, endangering the delicate balance of marine ecosystems. In order to tackle these issues, mathematical modeling has become a potent instrument for comprehending fish population dynamics and fishing effort effects. We explore the field of mathematical fisheries modeling in this article, concentrating on the formulas that control fish populations and fishing activity [1, 2, 3].

Numerous factors, such as natural growth rates, death rates, reproduction rates, and interactions with other species, all have an impact on fish populations. Differential equations are frequently used in mathematical models of fish populations to represent these dynamics. To more accurately represent the circumstances

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found in the real world, this models can be expanded to incorporate elements like competition, predation, and environmental fluctuation [1, 2, 3, 4, 5, 6].

Fish population dynamics are significantly influenced by death in addition to growth. Predation, illness, and old age are examples of natural mortality, but the effects of fishing operations are represented by fishing mortality. Since fishing mortality has a direct impact on the sustainability of fish stocks, it is a fundamental concept in fisheries management. Researchers can evaluate the effects of fishing by utilizing models that combine both natural and fishing-related mortality rates [1, 2, 3].

A common mathematical model used to describe the growth of a fish population in a fishery involves a system of ordinary differential equations (ODEs) known as Lotka-Volterra's predator-prey models [5, 6]. In this model, there are growth and death functions that govern the dynamics of the fish population.

The catch per unit effort (CPUE) model [7], which links the quantity of fish caught to the degree of fishing effort expended, is one often used method for quantifying fishing effort. Researchers can assess the link between fishing effort and fish population by examining historical CPUE data and environmental parameters. This can provide valuable information about the efficacy of various fishing tactics and the possibility of over-exploitation[4, 7, 8].

Utilizing bioeconomic models, which integrate biological dynamics with market demand, resource rents, and fishing costs, is one method of integration. With the long-term sustainability of fish supplies guaranteed, these models can assist in identifying the best harvesting practices. Through the consideration of feedback loops including fishing effort, fish populations, and economic incentives, bioeconomic models offer significant insights into the trade-offs associated with decisions on fisheries management [9, 10].

The foundation of our model lies in two fundamental equations capturing the dynamics of fish population and fishing effort. The first equation governs the rate of change of the fish population, reflecting the intricate interactions between reproduction, mortality, and environmental factors. The second equation characterizes the rate of change of fishing effort, representing the adaptive behavior of fishers in response to economic incentives, regulatory measures, and technological advancements. In general this type of models take the following form :

$$\begin{cases} \dfrac{dn}{dt} = G(n) - D(n,e), \\ \\ \dfrac{de}{dt} = F(n,e) - C(e), \end{cases}$$

where, n, e represent fish density and fishing effort rate respectively. The function $G(n)$ corresponds to the growth of the fish population while $D(n,e)$ corresponds to its mortality natural it is or not. The revenue generated from the fishing effort is formulated by the $F(n,e)$ function and for the costs of the fishing effort, they are denoted by the $C(e)$ function.

Accounting for the economic depreciation of fishing effort is one important area where time delays are critical. Models can better represent the underlying economic dynamics of fisheries by include delays that account for the continuous costs related to fishing activities [5, 6]. This makes it possible for scholars and decision-makers to evaluate the long-term sustainability of fishing operations in addition to their short-term profitability. This will be our first scenario to treat and to highlight the significance of adding a time delay into such model.

Moreover, time delays play a key role in modeling the harvesting of fish populations selectively according to age classes [8]. Certain ages or sizes of fish are preferred to be harvested in various fisheries; this preference is frequently caused by market demand or laws designed to protect breeding populations [1, 2, 3]. Nevertheless, the consequences of this kind of selective harvesting could not show up right away and can take some time to show themselves. Models can replicate the delayed effects of selective harvesting on population dynamics and ecosystem health by including time delays that represent the growth and reproductive cycles of fish populations. This is going to be treated as a second case in this article.

2. Economic Fishery Model with Cost Delay Dynamics

Consider a fishery ecosystem where $n(t)$ represents the fish population density and $e(t)$ denotes the fishing effort intensity at time t. We incorporate a temporal delay $\tau \geq 0$ in the cost structure to model the realistic scenario where fishing expenditures exhibit delayed depreciation effects. This formulation yields the following dynamical system:

$$\begin{cases} \dfrac{dn}{dt} = n\left(g(n) - qe\right), \\ \dfrac{de}{dt} = pqne - ce(t-\tau), \end{cases} \tag{2.1}$$

where $g(n)$ represents the intrinsic growth function of the fish population, and qne denotes the harvesting rate based on the catch-per-unit-effort (CPUE) principle [7]. The economic parameters p and c correspond respectively to the unit price of harvested fish and the cost coefficient associated with fishing effort.

Remark 2.1. The delayed differential system (2.1) requires initial history functions defined as $n_t(\theta) = n(t+\theta)$ and $e_t(\theta) = e(t+\theta)$ for $t \geq 0$ and $\theta \in [-\tau, 0]$. We specify these through $n_0(\theta) = \phi_1(\theta) \geq 0$ and $e_0(\theta) = \phi_2(\theta) \geq 0$, where $(\phi_1, \phi_2) = \Phi \in C([-\tau, 0], \mathbb{R}_+^2)$ ensures solution existence and uniqueness.

2.1. *Equilibrium Analysis and Linearization*

The interior equilibrium (n^*, e^*) of system (2.1) satisfies the algebraic conditions:

$$\begin{cases} 0 = n^*\left(g(n^*) - qe^*\right), \\ 0 = e^*(pqn^* - c). \end{cases} \tag{2.2}$$

From the second equation, assuming $e^* \neq 0$, we obtain $n^* = \dfrac{c}{pq}$. Substituting into the first equation yields $e^* = \dfrac{g(n^*)}{q}$, provided $g(n^*) > 0$.

Remark 2.2. The system also admits boundary equilibria: the trivial equilibrium $(0, 0)$ representing ecosystem collapse, and potential equilibria of the form $(\tilde{n}, 0)$ where $g(\tilde{n}) = 0$, corresponding to carrying capacity states with no fishing activity. These equilibria represent degenerate cases outside our primary interest.

Introducing perturbation variables $N = n - n^*$ and $E = e - e^*$, the linearized system around (n^*, e^*) becomes:

$$\begin{cases} \dfrac{dN}{dt} = g'(n^*)n^*N - qn^*E, \\ \dfrac{dE}{dt} = pqe^*N + pqn^*E - cE(t-\tau). \end{cases} \tag{2.3}$$

2.2. *Stability Analysis via Characteristic Equation*

The characteristic equation of system (2.3) takes the form:

$$\Delta(\lambda, \tau) = \lambda^2 + \alpha_1\lambda + \alpha_2 + \left[c\lambda + \alpha_3\right]e^{-\lambda\tau} = 0, \tag{2.4}$$

where the coefficients are defined as:

$$\alpha_1 = -n^* \left[g'(n^*) + pq\right], \tag{2.5}$$

$$\alpha_2 = pqn^* \left[qe^* + g'(n^*)n^*\right], \tag{2.6}$$

$$\alpha_3 = -cg'(n^*)n^*. \tag{2.7}$$

We begin by establishing stability in the absence of delay.

Theorem 2.3 (Stability without Delay)**.** *The equilibrium* (n^*, e^*) *is locally asymptotically stable when* $\tau = 0$ *if and only if:*

$$\alpha_1 + c > 0 \quad \textit{and} \quad \alpha_2 + \alpha_3 > 0. \tag{2.8}$$

Proof. Setting $\tau = 0$ in (2.4) yields:

$$\lambda^2 + (\alpha_1 + c)\lambda + (\alpha_2 + \alpha_3) = 0. \tag{2.9}$$

By the Routh-Hurwitz criterion, both roots have negative real parts if and only if the trace $-(\alpha_1 + c) < 0$ and the determinant $\alpha_2 + \alpha_3 > 0$, which directly gives conditions (2.8). ☐

2.3. Critical Delay Analysis

For $\tau > 0$, we seek purely imaginary roots $\lambda = i\omega$ with $\omega > 0$. Substituting into (2.4) and separating real and imaginary parts:

$$\begin{cases} \alpha_3 \cos(\omega\tau) + c\omega \sin(\omega\tau) = \omega^2 - \alpha_2, \\ c\omega \cos(\omega\tau) - \alpha_3 \sin(\omega\tau) = -\alpha_1\omega. \end{cases} \tag{2.10}$$

Eliminating the trigonometric terms by squaring and adding, we obtain the frequency equation:

$$\omega^4 + (\alpha_1^2 - c^2 - 2\alpha_2)\omega^2 + (\alpha_2^2 - \alpha_3^2) = 0. \tag{2.11}$$

Let $z = \omega^2$. Then equation (2.11) becomes:

$$z^2 + (\alpha_1^2 - c^2 - 2\alpha_2)z + (\alpha_2^2 - \alpha_3^2) = 0. \tag{2.12}$$

Based on the above, we arrive at the following theorem.

Theorem 2.4 (Delay-Independent Stability)**.** *If conditions* (2.8) *hold along with:*

$$\alpha_1^2 - c^2 - 2\alpha_2 > 0 \quad \textit{and} \quad \alpha_2^2 - \alpha_3^2 > 0, \tag{2.13}$$

then (n^*, e^*) *remains locally asymptotically stable for all* $\tau \geq 0$.

Proof. Under conditions (2.13), equation (2.12) has no positive real solutions, implying that (2.4) has no purely imaginary roots for any $\tau > 0$. Combined with stability at $\tau = 0$, continuity arguments ensure stability persists for all delays. ☐

2.4. Hopf Bifurcation Analysis

When $\alpha_2^2 - \alpha_3^2 < 0$, equation (2.12) admits a unique positive solution:

$$\omega_0^2 = \frac{-(\alpha_1^2 - c^2 - 2\alpha_2) + \sqrt{(\alpha_1^2 - c^2 - 2\alpha_2)^2 - 4(\alpha_2^2 - \alpha_3^2)}}{2}. \tag{2.14}$$

The corresponding critical delay values are:

$$\tau_{0k} = \frac{1}{\omega_0} \arctan\left(\frac{\omega_0(\alpha_1\alpha_3 - c\alpha_2 + c\omega_0^2)}{\alpha_1 c\omega_0^2 + (\alpha_2 - \omega_0^2)\alpha_3}\right) + \frac{2k\pi}{\omega_0}, \quad k = 0, 1, 2, \ldots \tag{2.15}$$

To examine the transversality condition, we differentiate the characteristic equation (2.4) with respect to the delay τ. This leads to

$$\left[2\lambda + \alpha_1 + ce^{-\lambda\tau} - \tau(c\lambda + \alpha_3)e^{-\lambda\tau}\right]\frac{d\lambda}{d\tau} = \lambda e^{-\lambda\tau}(c\lambda + \alpha_3). \tag{2.16}$$

Making use of the identity

$$e^{-\lambda\tau} = -\frac{\lambda^2 + \alpha_1\lambda + \alpha_2}{c\lambda + \alpha_3}, \tag{2.17}$$

we arrive at the following expression for the inverse derivative:

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + \alpha_1}{-\lambda(\lambda^2 + \alpha_1\lambda + \alpha_2)} + \frac{c}{\lambda(c\lambda + \alpha_3)} - \frac{\tau}{\lambda}. \tag{2.18}$$

The foregoing analysis allows us to establish the following result.

Theorem 2.5 (Hopf Bifurcation)**.** *Assume conditions* (2.8) *and* $\alpha_2^2 - \alpha_3^2 < 0$ *hold. Then* (n^*, e^*) *is asymptotically stable for* $\tau < \tau_0$ *and undergoes a Hopf bifurcation at* $\tau = \tau_0$*, where* $\tau_0 = \tau_{00}$ *represents the smallest positive critical delay.*

Proof. The transversality condition requires:

$$\operatorname{sign}\left\{\frac{d(\operatorname{Re}\lambda)}{d\tau}\right\}_{\lambda=i\omega_0} = \operatorname{sign}\left\{\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\lambda=i\omega_0} \tag{2.19}$$

$$= \operatorname{sign}\left[\frac{2(\omega_0^2 - \alpha_2) + \alpha_1^2 - c^2}{\alpha_3^2 + c^2\omega_0^2}\right] > 0. \tag{2.20}$$

This condition is satisfied under our assumptions, confirming the Hopf bifurcation and the emergence of periodic solutions. □

2.5. Economic Interpretation of Stability Conditions

The stability analysis provides crucial insights for fishery management. The fundamental stability conditions (2.8) can be rewritten in terms of the cost parameter c:

$$c > \max\left\{n^*\left[g'(n^*) + pq\right], \frac{pq\left[qe^* + g'(n^*)n^*\right]}{g'(n^*)}\right\}. \tag{2.21}$$

This threshold condition (2.21) establishes that sustainable fishery equilibrium requires sufficiently high operational costs relative to economic incentives.

For delay-independent stability, the additional constraint (2.13) requires:

$$c^2 < \min\left\{\left(n^*[g'(n^*) + pq]\right)^2 + 2pqn^*[qe^* + g'(n^*)n^*], \left(\frac{pq[qe^* + g'(n^*)n^*]}{g'(n^*)}\right)^2\right\}. \tag{2.22}$$

These conditions define a feasible cost interval ensuring stability regardless of cost delay duration, providing valuable guidance for fishery economic policy.

2.6. Numerical Illustration

As an example, we consider the growth function to be :

$$g(x) = rx(1 - x),$$

and we choose history functions to be $n(t) = 0.5$ and $e(t) = 0.1$ for all $t \leq 0$. The parameters values are taken as follows :

Parameter	p	q	r	c
Value	4	0.9	1	0.6

For the Case $\tau = 0$, numerical simulations show the following.

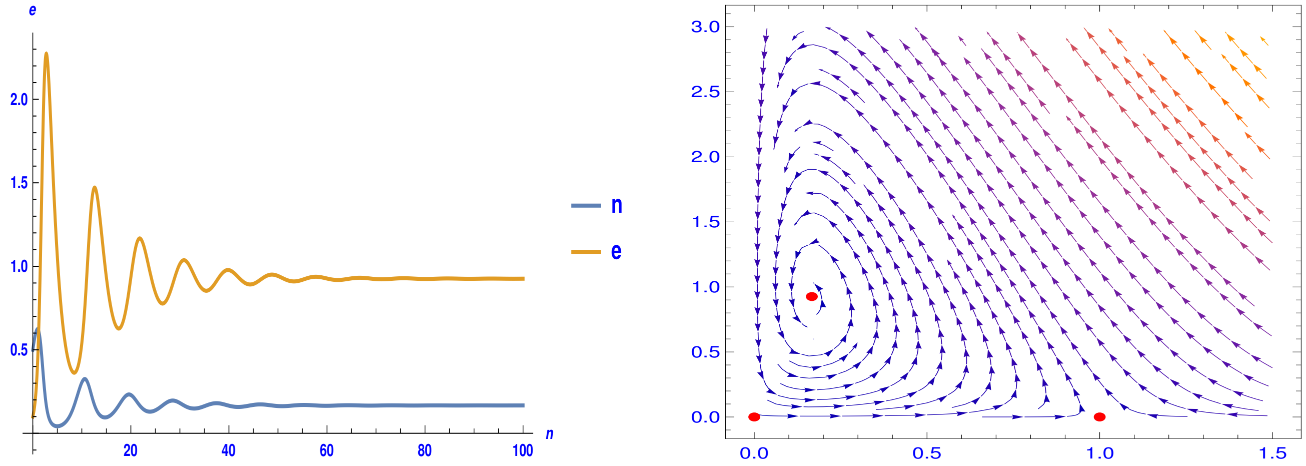

Figure 1: Convergence of the fishery and fishing effort to the equilibrium point (left) and stream plot for the $n - e$ plan of the considered system (right).

The red dots in Figure (1) right represent equilibrium point. It is observed from both plots of this figure that in the absence of time delay, trajectories converge to the stable interior equilibrium.

In the other hand for the case $\tau \neq 0$, numerical simulations show the following.

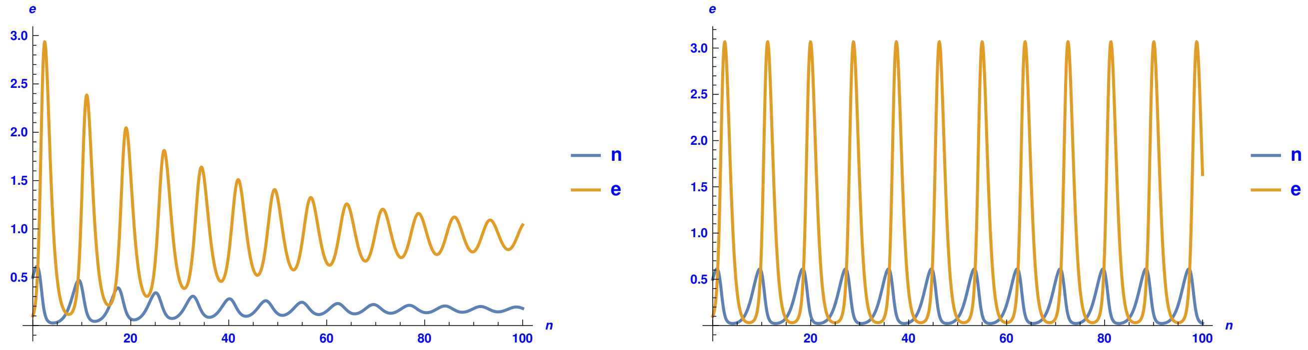

Figure 2: One observes convergence of the fishery and fishing effort to the equilibrium point for the choice of $\tau = 0.61 < \tau_0$ (left) and emergence of periodic solution on the contrary for $\tau = 0.67 > \tau_0$ (right).

As soon as the delay is introduced, slight changes in the trajectories convergence start to appear. For values of the delay below the critical value τ_0, the interior equilibrium is still stable and attract solutions. For values greater than τ_0, periodic trajectories appear and instability occurs.

Remark 2.6. The global stability of the interior equilibrium in the absence of delay can be established using Lyapunov function techniques, though this analysis extends beyond the current scope. The delay-induced destabilization represents a fundamental limitation of economic systems with temporal cost structures.

3. Selective Harvesting Model with Population Delay

We now examine a fundamentally different delay mechanism that captures the biological reality of selective harvesting practices. In this formulation, the delay $\tau \geq 0$ appears in the population mortality term, reflecting the temporal lag between fishing effort deployment and the actual harvesting of fish from specific age cohorts. This biological delay represents the time required for fishing gear to effectively target and capture fish that were present in the system at an earlier time.

The mathematical model takes the form:

$$\begin{cases} \dfrac{dn}{dt} = ng(n) - qen(t-\tau), \\ \dfrac{de}{dt} = e(pqn - c), \end{cases} \tag{3.1}$$

where the harvesting term $qen(t-\tau)$ explicitly incorporates the population density from time $t-\tau$, representing the fact that current fishing effort affects the fish population that existed τ time units ago. This captures scenarios such as:

- Age-selective fishing where mature fish caught today represent recruitment that occurred τ periods earlier;
- Seasonal fishing patterns where current effort impacts populations that established their density in previous seasons;
- Migration-based fisheries where effort affects populations from earlier spatial distributions.

A crucial observation is that the introduction of delay in the harvesting term does not alter the equilibrium structure of the system. The interior equilibrium (n^*, e^*) continues to satisfy the same algebraic conditions:

$$\begin{cases} 0 = n^*(g(n^*) - qe^*), \\ 0 = e^*(pqn^* - c), \end{cases} \tag{3.2}$$

yielding $n^* = \dfrac{c}{pq}$ and $e^* = \dfrac{g(n^*)}{q}$ as before.

3.1. Linearization and Characteristic Analysis

To understand the dynamic behavior near equilibrium, we introduce perturbation variables $N = n - n^*$ and $E = e - e^*$. The linearization of system (3.1) around (n^*, e^*) yields:

$$\begin{cases} \dfrac{dN}{dt} = N[g(n^*) + g'(n^*)n^*] - qn^*E - qe^*N(t-\tau), \\ \dfrac{dE}{dt} = pqe^*N. \end{cases} \tag{3.3}$$

This linearized system reveals the key difference from the cost-delay model: the delay now appears directly in the population dynamics through the term $qe^*N(t-\tau)$, creating a more complex coupling between current population changes and historical population states.

The characteristic equation associated with system (3.3) is:

$$\Delta(\lambda, \tau) = \lambda^2 + \beta_1\lambda + \beta_2 + \beta_3\lambda e^{-\lambda\tau} = 0, \tag{3.4}$$

where the coefficients are defined as:

$$\beta_1 = -[g(n^*) + g'(n^*)n^*], \tag{3.5}$$

$$\beta_2 = pqn^*\beta_3, \tag{3.6}$$

$$\beta_3 = qe^*. \tag{3.7}$$

Understanding these coefficients provides biological insight: β_1 represents the net population growth effect, β_2 captures the economic-biological coupling strength, and β_3 quantifies the direct harvesting impact.

3.2. Stability Analysis without Delay

When $\tau = 0$, the characteristic equation (3.4) simplifies to:

$$\lambda^2 + (\beta_1 + \beta_3)\lambda + \beta_2 = 0. \tag{3.8}$$

Using the equilibrium relations from system (3.2), we can establish that:

$$\beta_1 + \beta_3 = -[g(n^*) + g'(n^*)n^*] + qe^* = -g'(n^*)n^*, \tag{3.9}$$

$$\beta_2 = pq^2n^*e^* > 0. \tag{3.10}$$

The simplification in equation (3.9) follows from the equilibrium condition $g(n^*) = qe^*$, which eliminates the intrinsic growth term and leaves only the density-dependent component.
We first address the case of stability without delay.

Theorem 3.1 (Stability Condition for Selective Harvesting)**.** *When $\tau = 0$, the equilibrium (n^*, e^*) is locally asymptotically stable if and only if:*

$$g'(n^*) < 0. \tag{3.11}$$

Proof. From the Routh-Hurwitz criterion applied to equation (3.8), stability requires both the trace and determinant conditions. Since $\beta_2 > 0$ from equation (3.10), we need $\beta_1 + \beta_3 > 0$. From equation (3.9), this translates to $-g'(n^*)n^* > 0$, which, given that $n^* > 0$, is equivalent to $g'(n^*) < 0$. □

This result reveals a fundamental biological principle: stability in selective harvesting systems requires that the fish population exhibits density-dependent regulation (decreasing per capita growth rate) at the equilibrium density. This condition ensures that the population has inherent self-regulating mechanisms that can counteract harvesting perturbations.

3.3. Critical Delay Analysis and Hopf Bifurcations

For $\tau > 0$, we seek conditions under which purely imaginary eigenvalues $\lambda = i\omega$ (with $\omega > 0$) exist. Substituting into equation (3.4) and separating real and imaginary parts:

$$\begin{aligned} \frac{\beta_2 - \omega^2}{-\omega\beta_3} &= \sin(\omega\tau), \\ -\frac{\omega\beta_1}{\omega\beta_3} &= \cos(\omega\tau). \end{aligned} \tag{3.12}$$

The existence and multiplicity of solutions directly relate to the number of critical delays at which Hopf bifurcations can occur.

Eliminating the trigonometric functions by squaring and adding equations (3.12), we obtain the frequency equation:

$$\omega^4 + (\beta_1^2 - \beta_3^2 - 2\beta_2)\omega^2 + \beta_2^2 = 0. \tag{3.13}$$

Let $z = \omega^2$. Then equation (3.13) becomes a quadratic in z:

$$z^2 + (\beta_1^2 - \beta_3^2 - 2\beta_2)z + \beta_2^2 = 0. \tag{3.14}$$

The discriminant of this quadratic is:

$$\Delta_z = (\beta_1^2 - \beta_3^2 - 2\beta_2)^2 - 4\beta_2^2. \tag{3.15}$$

Theorem 3.2 (Delay-Independent Stability for Selective Harvesting)**.** *If the stability condition* (3.11) *holds along with:*

$$\beta_1^2 - \beta_3^2 - 2\beta_2 > 0, \tag{3.16}$$

then (n^*, e^*) *remains locally asymptotically stable for all* $\tau \geq 0$.

Proof. Under condition (3.16), equation (3.14) has no positive real solutions, implying no purely imaginary eigenvalues exist for any $\tau > 0$. Combined with stability at $\tau = 0$, continuity ensures persistent stability. ☐

The selective harvesting model exhibits richer bifurcation behavior. When condition (3.16) is violated, multiple scenarios emerge:

- **Case 1:** Single Critical Frequency.
  If $\beta_1^2 - \beta_3^2 - 2\beta_2 < 0$ and $\Delta_z = 4\beta_2^2$, then equation (3.14) admits exactly one positive solution $\omega_0^2 = -\dfrac{\beta_1^2 - \beta_3^2 - 2\beta_2}{2}$.
- **Case 2:** Multiple Critical Frequencies.
  If $\beta_1^2 - \beta_3^2 - 2\beta_2 < 0$ and $\Delta_z > 4\beta_2^2$, then two positive solutions exist:

$$\omega_\pm^2 = \frac{-(\beta_1^2 - \beta_3^2 - 2\beta_2) \pm \sqrt{(\beta_1^2 - \beta_3^2 - 2\beta_2)^2 - 4\beta_2^2}}{2}. \tag{3.17}$$

For each critical frequency ω_k, the corresponding critical delays are:

$$\tau_k^{(j)} = \frac{1}{\omega_k}\arctan\left(-\frac{\omega_k^2 - \beta_2}{\beta_1\omega_k}\right) + \frac{2j\pi}{\omega_k}, \quad j = 0, 1, 2, \ldots \tag{3.18}$$

3.4. Transversality and Stability Switching

To determine the nature of bifurcations, we compute the transversality condition. Differentiating equation (3.4) with respect to τ:

$$\left[2\lambda + \beta_1 + \beta_3 e^{-\lambda\tau} - \tau\beta_3\lambda e^{-\lambda\tau}\right]\frac{d\lambda}{d\tau} = \beta_3\lambda^2 e^{-\lambda\tau}. \tag{3.19}$$

Using the relation $e^{-\lambda\tau} = -\dfrac{\lambda^2 + \beta_1\lambda + \beta_2}{\beta_3\lambda}$, we derive:

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + \beta_1}{-\lambda(\lambda^2 + \beta_1\lambda + \beta_2)} + \frac{1}{\lambda} - \frac{\tau}{\lambda}. \tag{3.20}$$

The transversality condition becomes:

$$\operatorname{sign}\left\{\frac{d(\operatorname{Re}\lambda)}{d\tau}\right\}_{\lambda = i\omega_k} = \operatorname{sign}\left[\frac{2(\omega_k^2 - \beta_2) + \beta_1^2 - \beta_3^2}{\beta_3^2\omega_k^2}\right]. \tag{3.21}$$

From the previous conclusions, we can state the following theorem.

Theorem 3.3 (Multiple Hopf Bifurcations in Selective Harvesting)**.** *Assume condition* (3.11) *holds and that two positive frequencies* ω_+ *and* ω_- *exist satisfying equation* (3.17)*. Then there exists a finite number of stability switches as* τ *increases, with Hopf bifurcations occurring at critical delays* $\tau_k^{(\pm)}$ *defined by equation* (3.18)*.*

Proof. The transversality conditions can be verified by direct computation. For ω_+, we have:

$$\left.\frac{d(\operatorname{Re}\lambda)}{d\tau}\right|_{\tau=\tau_k^{(+)}} = \frac{\sqrt{(\beta_3^2-\beta_1^2+2\beta_2)^2-4\beta_2^2}}{\beta_3^2\omega_+^2} > 0,$$

indicating destabilization. For ω_-, the sign is negative, indicating restabilization. This creates alternating stability regions as τ increases. ☐

3.5. Example

Illustrating this part make use of the previous example with the following changes in the parameters values :

Parameter	p	q	r	c
Value	2	0.8	2	0.5

Numerical simulations show the following cases on which the conclusions are the same as previously and hence being omitted.

- Case $\tau = 0$. As in the theory, it is predicted that convergence to the interior equilibrium occur.

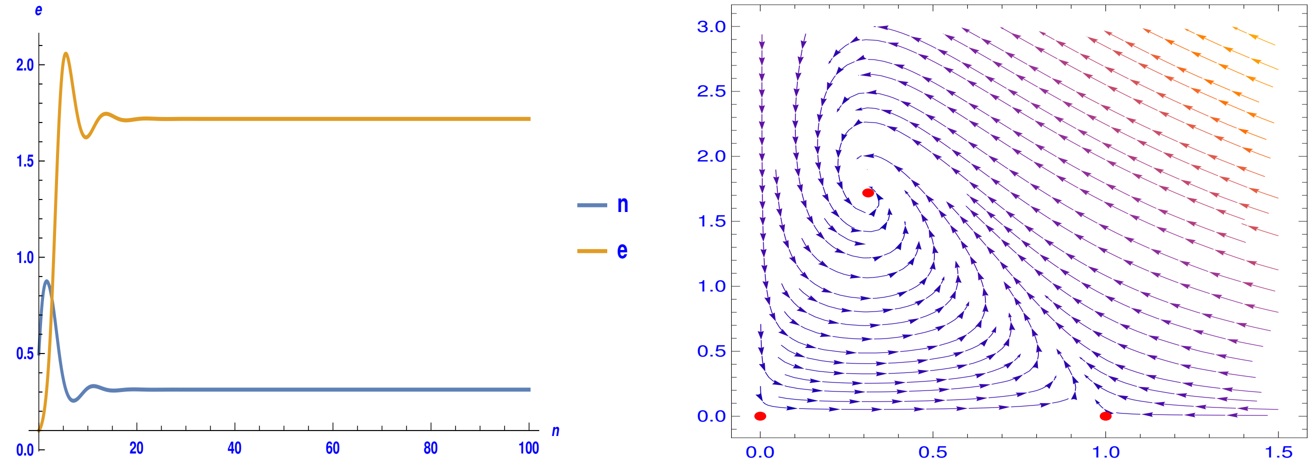

Figure 3: Convergence of the fishery and fishing effort to the equilibrium point (left) and stream plot for the $n - e$ plan of the considered system (right).

- Case $\tau \neq 0$. Periodic trajectories emerge as soon as the delay value is taken greater than the critical value τ_0.

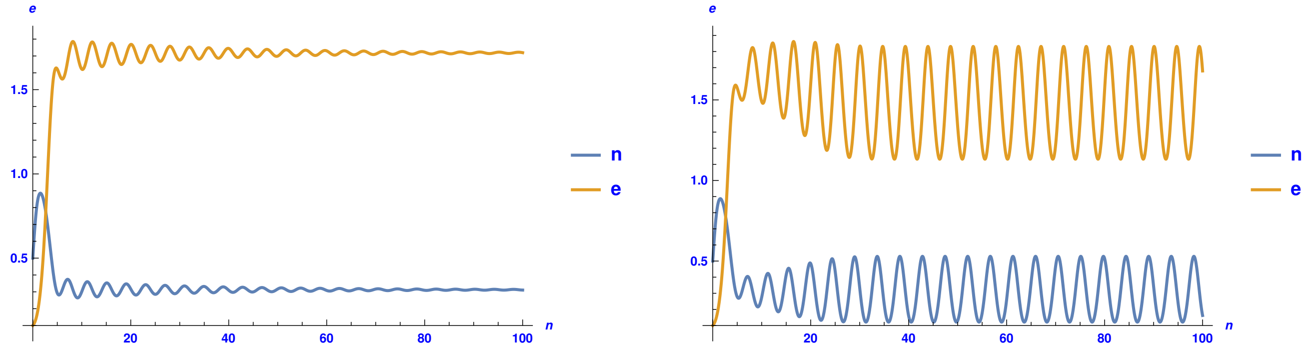

Figure 4: Convergence of the fishery and fishing effort to the equilibrium point for the choice of $\tau = 0.6 < \tau_0$ (left) and emergence of periodic solution on the contrary for $\tau = 0.7 > \tau_0$ (right).

4. Discussion

Our analysis of two delay-induced fishery models reveals that the location of delay economic cost depreciation versus biological harvesting fundamentally alters system dynamics and stability. Cost delays lead to a critical threshold triggering Hopf bifurcations and predictable oscillations, providing clear management targets. The biological stability condition $g'(n^*) < 0$ signifies harvesting within the density-dependent growth phase, essential for natural regulation and preventing large oscillations. Economically, rapid implementation of cost controls is crucial to avoid destabilizing delays, while adaptive management strategies can exploit stability switches in harvesting delays.

Mathematically, frequency domain analysis and bifurcation theory clarify how delay placement shapes stability, offering a framework applicable to other bio-economic systems. Limitations include deterministic assumptions and constant delays; future work should address stochasticity, variable delays, and global dynamics to enhance realism.

5. Conclusion

This study has investigated the stability and bifurcation dynamics of two bioeconomic fishery models, each incorporating a distinct type of temporal delay. By differentiating between a delay in fishing effort costs reflecting economic depreciation and a delay in the harvesting term capturing age-selective practices we have shown that the origin of the time lag fundamentally alters the system's behavior.

Our analysis establishes rigorous, quantitative stability criteria directly linked to key management parameters. These findings translate into practical guidelines: effective management strategies must be tailored to the dominant temporal processes in a given fishery, as the two delay mechanisms yield contrasting dynamics. A one-size-fits-all approach is therefore insufficient.

Beyond its practical applications, this work demonstrates how delay differential equations can address complex challenges in renewable resource management. Future research should incorporate additional realism such as environmental variability and spatial structure to further enhance our capacity for ensuring the long-term health and resilience of aquatic ecosystems.

References

[1] J. D. Murray , 2002, *Mathematical Biology: I. An Introduction*, Interdisciplinary Applied Mathematics, 2nd edn., Springer, New York. 1

[2] C.W. Clark, 1990. *Mathematical Bioeconomis. The Optimal Management of Renewable Resources. 2nd ed.* John Wiley & Sons, Inc., New York. 1

[3] D. Gascuel, 2009. *Exploitation des ressources marines : quand la crise écologique compromet l'alimentation des pays du Sud*, POUR, 202, (2009), 49-56. DOI 10.3917/pour.202.0049 1

[4] C. Clark, 1976. *Mathematical bioeconomics: the optimal management of renewable resources*, New York, Wiley-Interscience. 1

[5] J.D. Murray, 1989. *Mathematical Biology*, Springer-Verlag, Berlin Heidelberg. 1

[6] P. Auger, C. Lett, J.-C. Poggiale, 2015. *Mathematical modelling in ecology, courses and exercises corrected.* Dunod, Paris. 1

[7] C.W. Clark, 1979. Mathemalical Bioeconomics, The Optimal Management of Renewable Resources. John Wiley & Sons, New York. 1, 2

[8] T. K. Kar, 2003. *Selective Harvesting in a Prey-Predator with Time Delay*, Mathematical and Computer Modelling 38, 449-458. 1

[9] A. Moussaoui, A. Ducrot, A. Moulai-Khatir, P. Auger, 2023. *A model of a fishery with fish storage and variable price involving delay equations*, Mathematical Biosciences, 362. https://doi.org/10.1016/j.mbs.2023.109022 1

[10] A. Moulai-Khatir, A. Moussaoui, P. Auger, 2023. *Maximum sustainable yield for a fishery with variable price*, Journal of Biological Systems, Vol. 31, No. 4, 1433–1453. 1

[11] J.K. Hale, 1969. *Ordinary Differential Equations*, Wiley, New York.

[12] Y. Kuang, 1993. *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, San Diego.

[13] J.M. Cushing, 1977. *Integro-Differential Equations and Delay Models in Population Dynamics*, Springer-Verlag. Heidelberg.

[14] J.M. Cushing and M. Saleem, 1982. *A predator-prey model with age structure*, J. Math. Biol. 14, 231-250.

[15] S. Chakraborty, S. Pal, N. Bairagi, 2012. *Predator–prey interaction with harvesting: Mathematical study with biological ramifications*, Appl Math Model 36:4044–4059.

[16] P. Allen, JM. McGlade, 1987. Modelling complex human systems: A fisheries example, Eur J Oper Res 30:147–167.

[17] M.H. Holden, M. McDonald-Madden, 2017. *High prices for rare species can drive large population extinct: the anthropogenic effect revisited*, J. Theoret. Biol. 429, 170-180.

[18] S. Ruan, 1995. *The effect of delays on stability and persistence in plankton models*, Nonlinear Anal. 24, 575-585.

[19] A. Moussaoui, P. Auger, 2021. *A bioeconomic model of a fishery with saturated catch and variable price: Stabilizing effect of marine reserves on fishery dynamics*, Ecol. Complex., 45.

[20] A. Ghouali, A. Moussaoui, P. Auger, T.N. Huu, (2022). *Optimal placement of marine protected areas to avoid the extinction of the fish stock*, J. Biol. Systems 30 (2), 1-15.