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A Numerical Scheme for Time Fractional Stochastic Advection-Diffusion Equation with Vieta-Lucas Polynomials

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Abstract

In this paper, a numerical method is implemented to solve time fractional stochastic Advection-Diffusion equation. The aforementioned equation plays a pivotal role across a spectrum of applied sciences. Vieta-Lucas polynomials are employed as basic functions to achieve numerical solutions. The central idea of our method involves the approximation of Brownian motion via the application of Gauss-Legendre quadrature, thereby simplifying computational efforts. Within the framework of the proposed technique, operational matrices derived from the aforementioned polynomials are utilized. It is pertinent to note that the stochastic model under consideration is transformed into a simpler system via the application of the derived operational matrices. Subsequently, Newton's method is employed to ascertain the numerical solution for the specified model. Theoretical validation of the error bound and convergence analysis for the referenced scheme has been established. In addition, proofs pertaining to the existence and uniqueness of the solutions for the equation under investigation have been furnished. To demonstrate the precision and efficacy of the suggested approach, several test problems have been employed.

Keywords: Fractional stochastic Advection-Diffusion equation, Vieta-Lucas polynomials, Brownian motion, Operational matrix

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1. Introduction

Fractional differential equations have seen widespread application across diverse disciplines within applied sciences, encompassing fields such as physics, chemistry, engineering, biology, and electromagnetism, among others [7, 12, 16, 29, 30]. A multitude of mathematical physics problems are characterized by Partial Differential Equations (PDEs) that incorporate derivatives of fractional order. The Advection-Diffusion equation stands as a salient partial differential equation that characterizes a myriad of phenomena, including

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but not limited to, oil reservoir simulations, moisture migration in soil matrices, mass and energy transference, the conveyance of pollutants in fluvial environments, and the diffusion of solutes within groundwater systems [8, 9, 14, 27]. Energy within a physical system undergoes transformation through processes of advection and diffusion. Advection pertains to the transport of particles from areas from greater to lesser concentration, while diffusion is associated with the dispersion of particles throughout a fluid as a result of stochastic motion. An array of numerical methodologies has been advanced to acquire the solutions of equations characterized by both integer and fractional order. Mohebbi and Dehghan [24] derived a high-order efficient compact solution for the heat equation as well as the Advection-Diffusion equations. Dehghan [5] examined weighted finite difference methodologies for resolving this problem and subsequently investigated the computational solution of three-dimensional variant of this equation [6]. In [17], Korkmaz and Dag implemented B-spline techniques to numerically resolve this equation. Esmaeelzade et al. [1] devised a numerical methodology to solve the fractional-order Advection-Diffusion equation. Furthermore, Safdar et al. [32] advanced a collocation technique predicated upon shifted Chebyshev polynomials of the fourth type for solving this equation. Salama and Ali [33] delineated an efficacious methodology for the solutions of the two-dimensional kind of this equation of time-fractional order. Balasim et al. [2] scrutinized an iterative scheme to derive numerical solutions for the aforementioned equation. It merits attention that stochastic partial differential equations (SPDEs) of both integer and fractional order have assumed prominent roles in a multitude of applied science domains in recent years. The aforementioned issues arise from the incorporation of an uncertainty term into partial differential equations. Furthermore, these problems frequently exhibit dependence on an origin of noise, such as Gaussian white noise, which is affected by specific probabilistic laws. These equations are utilized to model a plethora of applications across various disciplines including mathematical sciences, economics, biology, turbulence phenomena, medical science, and engineering [19, 26, 28]. In numerous stochastic models, the exact solution is not invariably accessible, thereby precipitating an escalating interest in the development of numerical methodologies. The quest for new, efficient numerical techniques naturally motivates researchers in the field. Literature has engaged in discussions on numerical solutions for diverse forms of such equations, encompassing stochastic differential equations (SDEs), stochastic integral equations (SIEs), and stochastic integro-differential equations (SIDEs) of both integer and fractional orders. For example, Khodabin et al. [15] documented the numerical solutions for stochastic Volterra-Fredholm integral equations, while Mohammadi [23] delineated a computational approach for resolving stochastic Itô-Volterra integral equations. In 2017, Mirzaee and Hamzeh [20] employed a stochastic operational matrix method to solve stochastic differential equations driven by fractional Brownian motion. Subsequently, in 2019, Heydari et al. [10] discerned numerical solutions for nonlinear stochastic differential equations characterized by variable-order fractional Brownian motion. In 2020, Sayevand et al. [34] presented a study examining stochastic fractional integro-differential equations. The following year, Shirzadia and Dehghan [35] conducted an investigation of stochastic partial differential equations. Progressing into 2022, Banihashemi et al. [3] implemented Legendre polynomials for the resolution of fractional stochastic systems incorporating mixed delays. Most recently, in 2023, Singh and Ray [36] utilized Lucas polynomials to address the solution of stochastic Itô-Volterra integral equations. The stochastic Advection-Diffusion equation of fractional order represents a significant category of stochastic partial differential equations (SPDEs) that manifest in numerous mathematical physics problems. Considerable specialized endeavors have been directed towards solving the aforementioned equation.
Thus, the focus of this study is the time-fractional stochastic Advection-Diffusion equation, which is articulated as follows:

$$D_\tau^\theta z(\xi,\tau)+\sigma_1\frac{\partial z(\xi,\tau)}{\partial\xi}=(\sigma_2+\sigma_3\dot{B}(\tau))\frac{\partial^2 z(\xi,\tau)}{\partial\xi^2}+h(\xi,\tau),\ \ \xi,\tau\in[0,1], \tag{1.1}$$

accompanied by initial and boundary conditions

$$z(\xi,0)=\kappa_0(\xi), \tag{1.2}$$

$$z(0,\tau)=\kappa_1(\tau),\qquad z(1,\tau)=\kappa_2(\tau), \tag{1.3}$$

wherein σ_1 denotes the advection coefficient, while σ_2 and σ_3 represent the diffusion term coefficients. The functions $\kappa_0(\xi), \kappa_1(\tau)$ and $\kappa_2(\tau)$ are continuous. The source function $h(\xi,\tau)$ is characterized by sufficient smoothness. In this context, the term $\dot{B}(\tau) = \frac{dB(\tau)}{d\tau}$ represents white noise, where $B(\tau)$ signifies a Brownian motion. Also, D_τ^θ denotes the Caputo fractional derivative of $z(\xi,\tau)$ of order θ. In numerous environmental problems, $z(\xi,\tau)$ denotes the concentration of a pollutant or contaminant material at the spatial location ξ and time τ. Occasionally, the solutions pertain to the transport of mass, heat, water, or energy across different media, which may include draining films or soil [14, 27]. Researchers have extensively engaged in the development of numerical methods for solving such equations. Mirzaee and Samdyar [21] investigated a combined approach of meshless and finite difference methods for solving fractional stochastic Advection-Diffusion equations. Moreover, Mirzaee et al. [22] reported the application of a finite difference method grounded in spline approximation for this class of equations. Yazdani and Azimi [4] conducted a study employing the cubic trigonometric B-spline collocation approach to derive solutions of Eq. (1.1). Furthermore, Heydari [11] carried out a numerical investigation of this equation utilizing second kind Chebyshev wavelets. The present paper introduces an efficacious approach for acquiring numerical solutions to this problem through the application of Vieta-Lucas polynomials.
The organization of this paper is as follows. Section 2 introduces the requisite background on fundamental definitions of fractional and stochastic calculus necessary for the remainder of the paper. It also discusses a type of orthogonal polynomial, specifically the Vieta-Lucas polynomials, and the approximation of functions. Section 3 elucidates the operational matrix of fractional integral, the operational matrix of derivative and the operational matrix of the product. Implementation of the suggested approach to the stochastic fractional Advection-Diffusion equation is detailed in section 4. Theoretical proof of the error estimation and convergence analysis of the suggested scheme is elaborated in section 5. Section 6 is devoted to establishing the existence and uniqueness of solutions for the equations under scrutiny. Section 7 provides instances to demonstrate the efficacy of the suggested plan. Finally, the paper ends with section 8, summarizing the findings.

2. Preliminaries

This section outlines essential definitions and properties of fractional operators, stochastic calculus, and Vieta-Lucas polynomials, which will be utilized in the subsequent sections.

2.1. Fractional calculus

Definition 2.1. The Riemann-Liouville integral of order $\theta > 0$ for the function $z(\tau)$ is exhibited as [30]

$$I_\tau^\theta z(\tau) = \frac{1}{\Gamma(\theta)} \int_0^\tau \frac{z(\eta)}{(\tau-\eta)^{1-\theta}} d\eta. \tag{2.1}$$

Definition 2.2. The Caputo derivative of order θ for the function $z(\tau)$ is defined as [30]

$$D_\tau^\theta z(\tau) = \frac{1}{\Gamma(n-\theta)} \int_0^\tau \frac{z(\eta)}{(\tau-\eta)^{1+\theta-n}} d\eta, \ n-1 < \theta < n, \ n \in \mathbb{N}. \tag{2.2}$$

Additionally

$i)$ $D_\tau^\theta C = 0$, (C is a constant),

$$ii)\ D_\tau^\theta \tau^\gamma = \begin{cases} 0, & \gamma \in \mathbb{N}, \quad \gamma < \lceil \theta \rceil, \\ \frac{\Gamma(\gamma+1)}{\Gamma(1+\gamma-\theta)} \tau^{\gamma-\theta}, \quad \gamma \in \mathbb{N}, \ \gamma \geqslant \lceil \theta \rceil, & \text{or} \quad \gamma \notin \mathbb{N}, \ \gamma > \lfloor \theta \rfloor, \end{cases}$$

$$iii)\ I_\tau^\theta D_\tau^\theta h(\tau) = h(\tau) - \sum_{k=0}^{n-1} h^{(k)}(0^+) \frac{\tau^k}{k!}, \qquad n-1 < \theta \leqslant n. \tag{2.3}$$

2.2. Stochastic calculus

Definition 2.3. More precisely, a real-valued stochastic process $B(\tau)$, $\tau \in [0,T]$ is known as the Brownian motion with the properties in the following [19]:
i) For $\tau \geqslant 0$, $B(\tau)$ is a continuous function of τ.
ii) For $0 \leqslant \tau_0 < \tau \leqslant T$, $B(\tau) - B(\tau_0)$, is independent of the past. As a results, $B(0) = 0$ (with the probability 1)
iii) For $0 \leqslant \tau_0 < \tau \leqslant T$, $B(\tau) - B(\tau_0)$ has normal distribution with mean zero and variance $\tau - \tau_0$. Such that, $B(\tau) - B(\tau_0) \sim \sqrt{\tau - \tau_0}\, \mathcal{N}(0,1)$, where $\mathcal{N}(0,1)$ denotes a normally distributed random variable with zero mean and unit variance.

Definition 2.4. The Itô integral of $h \in \upsilon(\tau_0, T)$ is given by [26]

$$\int_{\tau_0}^{T} h(\tau, w)\; dB_\tau(w) = \lim_{n\to\infty} \int_{\tau_0}^{T} \varrho_n(\tau, w)\; dB_\tau(w),$$

where ϱ_n is the sequence of elementary functions of

$$E\left(\int_{\tau_0}^{T} (h - \varrho_n)^2 d\tau\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Two important properties of the Itô integrals are satisfied the following equalities:

$$E\left[\left(\int_{\tau_0}^{T} h(\tau, w) dB_\tau(w)\right)^2\right] = E\left[\int_{\tau_0}^{T} h^2(\tau, w) d\tau\right], \tag{2.4}$$

$$E\left[\int_{\tau_0}^{T} h^2(\tau, w) d\tau\right] < \infty. \tag{2.5}$$

2.3. Vieta-Lucas polynomials

The Vieta-Lucas polynomials (VLPs) $\mathrm{VL}_n(\tau)$ are defined by the recursive formula as [13]

$$\mathrm{VL}_n(\tau) = \tau \mathrm{VL}_{n-1}(\tau) - \mathrm{VL}_{n-2}(\tau), \quad n \geqslant 2,$$

in which $\mathrm{VL}_0(\tau) = 2$, $\mathrm{VL}_1(\tau) = \tau$ and n denote the degree of Vieta-Lucas polynomials. The general terms of $\mathrm{VL}_n(\tau)$ with weight function $w(\tau) = \frac{1}{\sqrt{4-\tau^2}}$ on $[-1,1]$ are given by

$$\mathrm{VL}_n(\tau) = \sum_{k=0}^{\lceil \frac{n}{2} \rceil} (-1)^k \frac{n\Gamma(n-k)}{\Gamma(k+1)\Gamma(n+1-2k)} \tau^{n-2k}, \quad n \geqslant 1,$$

where $\lceil \frac{n}{2} \rceil$ is a celling function.
The Vieta-Lucas polynomials on the interval $[0,1]$ are introduced using the change of variable τ to $4\tau - 2$ as following

$$\mathrm{VL}^*_n(\tau) = 2n \sum_{k=0}^{n} (-1)^k \frac{4^{n-k}\Gamma(2n-k)}{\Gamma(k+1)\Gamma(2n-2k+1)} \tau^{n-k}, \quad n \geqslant 1, \tag{2.6}$$

with

$$\mathrm{VL}^*_0(\tau) = 2, \qquad \mathrm{VL}^*_1(\tau) = 4\tau - 2.$$

Moreover, another representation of relation (2.6) is expressed as

$$\mathrm{VL}^*_n(\tau) = \sum_{k=0}^{n} (-1)^{n-k} \frac{4^k (2n)\Gamma(n+k)}{\Gamma(2k+1)\Gamma(n-k+1)} \tau^k, \quad n \geqslant 1. \tag{2.7}$$

These polynomials are orthogonal with the weight function $w(\tau) = \frac{1}{\sqrt{\tau-\tau^2}}$ on $[0,1]$ in the following form:

$$\begin{aligned}\langle \mathrm{VL}_n^*(\tau), \mathrm{VL}_m^*(\tau)\rangle &= \int_0^1 w(\tau)\mathrm{VL}_n^*(\tau)\mathrm{VL}_m^*(\tau)d\tau \\ &= \begin{cases} 4\pi, & m = n = 0, \\ 2\pi, & m = n \neq 0, \\ 0, & m \neq n. \end{cases}\end{aligned} \tag{2.8}$$

2.4. The function approximation

Any function $z(\tau) \in L^2[0,1]$ is expressed in terms of the VLPs

$$z(\tau) = \sum_{i=0}^{\infty} z_i \mathrm{VL}_i^*(\tau). \tag{2.9}$$

In practice, by truncating the infinite series in (2.9), $z(\tau)$ is estimated as follows:

$$z(\tau) \simeq \sum_{i=0}^{n} z_i \mathrm{VL}_i^*(\tau) = Z^T \psi(\tau), \tag{2.10}$$

where

$$Z = [z_0, \ldots, z_n]^T, \tag{2.11}$$
$$\psi(\tau) = [\mathrm{VL}_0^*(\tau), \mathrm{VL}_1^*(\tau), \ldots, \mathrm{VL}_n^*(\tau)]^T, \tag{2.12}$$

then

$$Z = Q^{-1}\langle z(\tau), \psi(\tau)\rangle. \tag{2.13}$$

where Q is an $(n+1)\times(n+1)$ matrix and is defined as

$$Q = \langle \psi(\tau), \psi^T(\tau)\rangle = \int_0^1 \frac{\psi(\tau)\psi^T(\tau)}{\sqrt{\tau-\tau^2}} d\tau = \gamma_i \pi, \tag{2.14}$$

that

$$\gamma_i = \begin{cases} 4, & i = 0, \\ 2, & i \geqslant 1. \end{cases}$$

An arbitrary function $z(\xi,\tau)$ defined over $L^2([0,1]\times[0,1])$, can be expanded as:

$$z(\xi,\tau) \simeq \sum_{i=0}^{n}\sum_{j=0}^{n} z_{i,j}\mathrm{VL}_i^*(\xi)\mathrm{VL}_j^*(\tau) = \psi^T(\xi) Z \psi(\tau), \tag{2.15}$$

where $Z = [z_{i,j}]_{(n+1)\times(n+1)}$ and $z_{i,j} = \dfrac{\langle \mathrm{VL}_i^*(\xi), \langle z(\xi,\tau), \mathrm{VL}_j^*(\tau)\rangle\rangle}{\langle \mathrm{VL}_i^*(\xi), \mathrm{VL}_i^*(\xi)\rangle\langle \mathrm{VL}_j^*(\tau), \mathrm{VL}_j^*(\tau)\rangle}$.

Remark 2.5. The Brownian motion $B(\tau)$ is expanded as

$$B(\tau) \simeq \beta^T \psi(\tau), \tag{2.16}$$

where $\beta^T = [\beta_0, \beta_1, \cdots, \beta_n]$ and the unkwon coefficients $\beta_i, i = 0, 1, \cdots, n$ can be computed by using Eq. (2.13) and Gauss-Legendre quadrature reule [37]

$$\beta_i = \frac{1}{4\pi}\sum_{j=1}^{m} \omega_j w(\frac{1}{2}\eta_j + \frac{1}{2}) B(\frac{1}{2}\eta_j + \frac{1}{2}) \mathrm{VL}_i^*(\frac{1}{2}\eta_j + \frac{1}{2}), \quad i = 0, 1, \ldots, n. \tag{2.17}$$

3. Operational matrix of Vieta-Lucas polynomials

This section aims to determine the derivative operational matrix, the operational matrix for the fractional integral, and the product matrix specifically for Vieta-Lucas polynomials.

Theorem 3.1. *The fractional integral of the vector $\psi(\tau)$ defined in (2.12) is expressed by the VLPs as follows [38]*

$$I^{\theta}\psi(\tau) = \mathbb{I}^{\theta}\ \psi(\tau), \tag{3.1}$$

where $\mathbb{I}^{\theta}$ represents the $(n+1)\times(n+1)$ operational matrix of fractional integral of order $\theta > 0$ which is given by:

$$\mathbb{I}^{\theta} = \left[\Pi_{i,0}, \Pi_{i,1}, \cdots, \Pi_{i,n}\right], \tag{3.2}$$

which $\Pi_{i,j}$ is given for $i = 0$ as

$$\Pi_{0,j} = \begin{cases} \dfrac{\Gamma(\theta+\frac{1}{2})}{\sqrt{\pi}\Gamma^2(\theta+1)}, & j = 0, \\ \sum_{r=0}^{j} \frac{(-1)^{j-r}4^r(2j)\Gamma(j+r)\Gamma(\theta+r+\frac{1}{2})}{\sqrt{\pi}\Gamma(\theta+1)\Gamma(2r+1)\Gamma(j-r+1)\Gamma(\theta+r+1)}, & j \geqslant 1, \end{cases}$$

and for $i \geq 1$

$$\Pi_{i,j} = \begin{cases} \sum_{k=0}^{i} \frac{(-1)^{i-k}4^k ik!\Gamma(i+k)\Gamma(k+\theta+\frac{1}{2})}{\sqrt{\pi}\Gamma(2k+1)\Gamma(i-k+1)\Gamma^2(k+\theta+1)}, & j = 0, \\ \sum_{k=0}^{i}\sum_{r=0}^{j} \frac{(-1)^{i-k+j-r}4^{k+r}2ijk!\Gamma(i+k)\Gamma(j+r)\Gamma(k+\theta+r+\frac{1}{2})}{\sqrt{\pi}\Gamma(2k+1)\Gamma(2r+1)\Gamma(i-k+1)\Gamma(j-r+1)\Gamma(k+\theta+1)\Gamma(k+\theta+r+1)}, & j \geqslant 1. \end{cases}$$

Proof: Applying the fractional integral on Vieta-Lucas polynomials for $i = 0, 1, \cdots, n$ defined in (2.7), we achieve

$$\begin{aligned} I^{\theta}\mathrm{VL}_0^*(\tau) &= \frac{2}{\Gamma(\theta+1)}\tau^{\theta}, \\ I^{\theta}\mathrm{VL}_i^*(\tau) &= \sum_{k=0}^{i}(-1)^{i-k}\frac{4^k(2i)\Gamma(i+k)}{\Gamma(2k+1)\Gamma(i-k+1)}I^{\theta}\tau^k \\ &= \sum_{k=0}^{i}(-1)^{i-k}\frac{4^k(2i)k!\Gamma(i+k)}{\Gamma(2k+1)\Gamma(i-k+1)\Gamma(k+\theta+1)}\tau^{k+\theta}, \quad i \geqslant 1. \end{aligned} \tag{3.3}$$

Now, we approximate the function $\tau^{k+\theta},\ k = 0, 1, \cdots, n$ using the VLPs as follows:

$$\tau^{k+\theta} \simeq \sum_{j=0}^{n}\xi_{k,j}\mathrm{VL}_j^*(\tau), \quad k = 0, 1, \cdots, n. \tag{3.4}$$

where

$$\xi_{k,j} = \frac{1}{\gamma_j\pi}\int_0^1 \frac{\eta^{k+\theta}\mathrm{VL}_j^*(\eta)}{\sqrt{\eta-\eta^2}}d\eta, \tag{3.5}$$

By substituting the values of $\mathrm{VL}_j^*(\tau)$ into Eq. (3.5), we have

$$\xi_{k,j} = \begin{cases} \dfrac{1}{2\sqrt{\pi}}\dfrac{\Gamma(k+\theta+\frac{1}{2})}{\Gamma(k+\theta+1)}, & j = 0, \\ \dfrac{1}{\sqrt{\pi}}\sum_{r=0}^{j}\frac{(-1)^{j-r}4^r j\Gamma(j+r)\Gamma(k+\theta+r+\frac{1}{2})}{\Gamma(2r+1)\Gamma(j-r+1)\Gamma(k+\theta+r+1)}, & j \geqslant 1. \end{cases}$$

By substituting Eq. (3.4) into (3.3), we have

$$I^{\theta}\mathrm{VL}_i^*(\tau) = \sum_{j=0}^{n} \Pi_{i,j} \mathrm{VL}_j^*(\tau), \tag{3.6}$$

where for $i = 0$

$$\Pi_{0,j} = \begin{cases} \dfrac{\Gamma(\theta + \frac{1}{2})}{\sqrt{\pi}\Gamma^2(\theta + 1)}, & j = 0, \\ \sum_{r=0}^{j} \frac{(-1)^{j-r} 4^r (2j)\Gamma(j+r)\Gamma\left(r+\theta+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(\theta+1)\Gamma(2r+1)\Gamma(j-r+1)\Gamma(\theta+r+1)}, & j \geqslant 1. \end{cases}$$

and for $i \geq 1$

$$\Pi_{i,j} = \begin{cases} \sum_{k=0}^{i} \frac{(-1)^{i-k} 4^k ik! \Gamma(i+k)\Gamma(k+\theta+\frac{1}{2})}{\sqrt{\pi}\Gamma(2k+1)\Gamma(i-k+1)\Gamma^2(k+\theta+1)}, & j = 0, \\ \sum_{k=0}^{i} \sum_{r=0}^{j} \frac{(-1)^{i-k+j-r} 4^{k+r} 2ijk! \Gamma(i+k)\Gamma(j+r)\Gamma(k+\theta+r+\frac{1}{2})}{\sqrt{\pi}\Gamma(2k+1)\Gamma(2r+1)\Gamma(i-k+1)\Gamma(j-r+1)\Gamma(k+\theta+1)\Gamma(k+\theta+r+1)}, & j \geqslant 1. \end{cases}$$

which completes the proof.

Theorem 3.2. *The first-order derivative of the vector $\psi(\tau)$ is expressed by the VLPs as following [31, 38]*

$$\frac{d\psi(\tau)}{d\tau} = \mathbb{D}^1 \psi(\tau), \tag{3.7}$$

where $\mathbb{D}$ is called the derivative operational matrix which is given by:

$$\mathbb{D}^1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \Theta_{1,0} & \Theta_{1,1} & \cdots & \Theta_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \Theta_{i,0} & \Theta_{i,1} & \cdots & \Theta_{i,n} \\ \vdots & \vdots & \ddots & \vdots \\ \Theta_{n,0} & \Theta_{n,1} & \cdots & \Theta_{n,n} \end{pmatrix},$$

with

$$\Theta_{i,j} = \begin{cases} \sum_{k=0}^{i} \frac{(-1)^{i-k} 4^k ik\Gamma(i+k)\Gamma(k-\frac{1}{2})}{\sqrt{\pi}\Gamma(2k+1)\Gamma(i-k+1)\Gamma(k)}, & j = 0, \\ \sum_{k=0}^{i} \sum_{r=0}^{j} \frac{(-1)^{i-k+j-r} 4^{k+r} 2ijk\Gamma(i+k)\Gamma(j+r)\Gamma(k+r-\frac{1}{2})}{\sqrt{\pi}\Gamma(2k+1)\Gamma(2r+1)\Gamma(i-k+1)\Gamma(j-r+1)\Gamma(k+r)}, & j \geqslant 1. \end{cases}$$

Proof: By using Eq. (2.7), we achieve

$$\frac{d\mathrm{VL}_i^*(\tau)}{d\tau} = \sum_{k=1}^{i} (-1)^{i-k} \frac{4^k (2i) k \Gamma(i+k)}{\Gamma(2k+1)\Gamma(i-k+1)} \tau^{k-1}, \qquad i \geqslant 1. \tag{3.8}$$

Now, the function $\tau^{k-1},\ k = 1, \cdots, n$ can be as following:

$$\tau^{k-1} \simeq \sum_{j=0}^{n} \chi_{k,j} \mathrm{VL}_j^*(\tau), \tag{3.9}$$

where

$$\chi_{k,j} = \frac{1}{\gamma_j \pi} \int_0^1 \frac{\eta^{k-1} \mathrm{VL}_j^*(\eta)}{\sqrt{\eta - \eta^2}} d\eta,$$

the values of $\chi_{k,j}$ can be computed

$$\chi_{k,j}=\begin{cases}\dfrac{1}{2\sqrt{\pi}}\dfrac{\Gamma(k-\frac{1}{2})}{\Gamma(k)}, & j=0,\\ \dfrac{1}{\sqrt{\pi}}\sum_{r=0}^{j}\frac{(-1)^{j-r}4^{r}j\Gamma(j+r)\Gamma(k+r-\frac{1}{2})}{\Gamma(2r+1)\Gamma(j-r+1)\Gamma(k+r)}, & j\geqslant 1.\end{cases}$$

substituting Eq. (3.9) into (3.8), we have

$$\frac{d\mathrm{VL}_i^*(\tau)}{d\tau}=\sum_{j=0}^{n}\Theta_{i,j}\mathrm{VL}_j^*(\tau),\quad i\geqslant 1, \tag{3.10}$$

which

$$\Theta_{i,j}=\begin{cases}\sum_{k=0}^{i}\frac{(-1)^{i-k}4^{k}ik\Gamma(i+k)\Gamma(k-\frac{1}{2})}{\sqrt{\pi}\Gamma(2k+1)\Gamma(i-k+1)\Gamma(k)}, & j=0,\\ \sum_{k=0}^{i}\sum_{r=0}^{j}\frac{(-1)^{i-k+j-r}4^{k+r}2ijk\Gamma(i+k)\Gamma(j+r)\Gamma(k+r-\frac{1}{2})}{\sqrt{\pi}\Gamma(2k+1)\Gamma(2r+1)\Gamma(i-k+1)\Gamma(j-r+1)\Gamma(k+r)}, & j\geqslant 1.\end{cases}$$

Hence, the proof is completed.
Generally, one has

$$\frac{d^r\psi(\tau)}{d\tau^r}=\mathbb{D}^r\psi(\tau), \tag{3.11}$$

where $\mathbb{D}^r$ is rth power of $\mathbb{D}^1$.

Lemma 3.3. *The vector $\psi(\tau)$ defined in (2.12) is approximated based on the vector of the monomial basis function*

$$\psi(\tau)=\mathbb{G}\mathbb{T}_n(\tau). \tag{3.12}$$

Here, $\mathbb{T}_n(\tau)$ is defined as

$$\mathbb{T}_n(\tau)=\begin{bmatrix}1 & \tau & \cdots & \tau^n\end{bmatrix}^T, \tag{3.13}$$

and $\mathbb{G}=[g_{i,j}]$ in (3.12) is an $(n+1)$ order lower triangular matrix with

$$g_{i,j}=\begin{cases}2, & i=j=1,\\ \frac{(-1)^{i-j}4^{j-1}(2i-2)(i+j-3)!}{(i-j)!(2j-2)!}, & 2\leqslant i\leqslant n,\quad 1\leqslant j\leqslant i,\\ 0, & \textit{otherwise}.\end{cases}$$

Proof: The components of the matrix $\mathbb{G}$ is readily calculated with the use of Eq. (2.7).

Theorem 3.4. *Let Z is an arbitrary $(n+1)\times 1$ matrix, then the operational matrix of product for the Vieta-Lucas polynomials is given as*

$$Z^T\psi(\tau)\psi^T(\tau)\simeq\psi^T(\tau)\hat{Z}, \tag{3.14}$$

that $\hat{Z}$ is the $(n+1)\times(n+1)$ operational matrix of product

Proof:

$$\begin{aligned}Z^T\psi(\tau)\psi^T(\tau) &= Z^T\psi(\tau)(\mathbb{T}_n^T(\tau)\mathbb{G}^T)\\ &= \left[Z^T\psi(\tau),Z^T\psi(\tau)\tau,\cdots,Z^T\psi(\tau)\tau^n\right]\mathbb{G}^T\\ &= \left[\sum_{i=0}^{n}z_i\mathrm{VL}_i^*(\tau),\sum_{i=0}^{n}z_i\tau\mathrm{VL}_i^*(\tau),\ldots,\sum_{i=0}^{n}z_i\tau^n\mathrm{VL}_i^*(\tau)\right]\mathbb{G}^T.\end{aligned} \tag{3.15}$$

Now, we approximate $\tau^p \mathrm{VL}_i^*(\tau)$ in terms of Vieta-Lucas polynomials as

$$\tau^p \mathrm{VL}_i^*(\tau) \simeq \sum_{j=0}^{n} \lambda_{p,i,j} \mathrm{VL}_j^*(\tau), \quad p = i = 0, 1, \cdots, n, \tag{3.16}$$

where

$$\lambda_{p,i,j} = \frac{1}{\gamma_j \pi} \int_0^1 \frac{\eta^p \mathrm{VL}_i^*(\eta) \mathrm{VL}_j^*(\eta)}{\sqrt{\eta - \eta^2}} d\eta,$$

for $i = 0$

$$\begin{aligned}\lambda_{p,0,j} &= \frac{1}{\gamma_j \pi} \int_0^1 \frac{2\eta^p \mathrm{VL}_j^*(\eta)}{\sqrt{\eta - \eta^2}} d\eta \\ &= \begin{cases} \dfrac{1}{\sqrt{\pi}} \frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)}, & j = 0, \\ \dfrac{1}{\sqrt{\pi}} \sum_{r=0}^{j} \frac{(-1)^{j-r} 4^r 2j \Gamma(j+r) \Gamma(p+r+\frac{1}{2})}{\Gamma(2r+1)\Gamma(j-r+1)\Gamma(p+r+1)}, & j \geq 1. \end{cases}\end{aligned}$$

for $i \geq 1$

$$\begin{aligned}\lambda_{p,i,j} &= \frac{1}{\gamma_j \pi} \int_0^1 \frac{\eta^p \mathrm{VL}_i^*(\eta) \mathrm{VL}_j^*(\eta)}{\sqrt{\eta - \eta^2}} d\tau \\ &= \begin{cases} \sum_{k=0}^{i} \frac{(-1)^{i-k} 4^k i \Gamma(i+k) \Gamma(p+k+\frac{1}{2})}{\sqrt{\pi}\Gamma(2k+1)\Gamma(i-k+1)\Gamma(p+k+1)}, & j = 0, \\ \sum_{k=0}^{i} \sum_{r=0}^{j} \frac{(-1)^{i-k+j-r} 4^{k+r} 2ij \Gamma(i+k)\Gamma(j+r)\Gamma(p+k+r+\frac{1}{2})}{\sqrt{\pi}\Gamma(2k+1)\Gamma(2r+1)\Gamma(i-k+1)\Gamma(j-r+1)\Gamma(p+k+r+1)}, & j \geq 1. \end{cases}\end{aligned}$$

Therefore, we have

$$\begin{aligned}\sum_{i=0}^{n} z_i \tau^p \mathrm{VL}_i^*(\tau) &\simeq \sum_{i=0}^{n} z_i \left(\sum_{j=0}^{n} \lambda_{p,i,j} \mathrm{VL}_j^*(\tau) \right) = \sum_{j=0}^{n} \mathrm{VL}_j^*(\tau) \left(\sum_{i=0}^{n} z_i \lambda_{p,i,j} \right) \\ &= \psi^T(\tau) \left[\sum_{i=0}^{n} z_i \lambda_{p,i,0}, \sum_{i=0}^{n} z_i \lambda_{p,i,1}, \cdots, \sum_{i=0}^{n} z_i \lambda_{p,i,n} \right]^T \\ &= \psi^T(\tau) \left[\Lambda_{p,0}, \Lambda_{p,1}, \cdots, \Lambda_{p,n} \right] Z = \psi^T(\tau) \bar{W}_{p+1},\end{aligned} \tag{3.17}$$

where

$$\begin{aligned}&\Lambda_{p,i} = \left[\lambda_{p,i,0}, \lambda_{p,i,1}, \cdots, \lambda_{p,i,n} \right]^T, \\ &\bar{W}_{p+1} = \left[\Lambda_{p,0}, \Lambda_{p,1}, \cdots, \Lambda_{p,n} \right] Z, \qquad p = 0, 1, \ldots, n.\end{aligned}$$

Now, assuming $\bar{W} = \left[\bar{W}_1, \bar{W}_2, \cdots, \bar{W}_{n+1} \right]$ and using Eqs. (3.15) and (3.17), we achieve

$$\begin{aligned}Z^T \psi(\tau) \psi^T(\tau) &= \left[\sum_{i=0}^{n} z_i \mathrm{VL}_i^*(\tau), \sum_{i=0}^{n} z_i \tau \mathrm{VL}_i^*(\tau), \ldots, \sum_{i=0}^{n} z_i \tau^n \mathrm{VL}_i^*(\tau) \right] \mathbb{G}^T \\ &\simeq \psi^T(\tau) \left[\bar{W}_1, \bar{W}_2, \cdots, \bar{W}_{n+1} \right] \mathbb{G}^T = \psi^T(\tau) \bar{W} \mathbb{G}^T,\end{aligned}$$

therefore, we have the operational matrix of product as

$$\hat{Z} = \bar{W} \mathbb{G}^T.$$

4. Description of the proposed method

In this section, we describe a numerical scheme to solve the Eq. (1.1) under the conditions (1.2) and (1.3). To this end, we approximate the $\frac{\partial^2 z(\xi,\tau)}{\partial \xi^2}$ in terms of Vieta-Lucas polynomial as follows

$$\frac{\partial^2 z(\xi,\tau)}{\partial \xi^2} = \sum_{i=0}^{n}\sum_{j=0}^{n} z_{i,j}\mathrm{VL}_i^*(\xi)\mathrm{VL}_j^*(\tau) = \psi^T(\xi)Z\psi(\tau), \tag{4.1}$$

employing the integral operator with respect ξ on both sides of (4.1) and in view of (3.1), we conclude

$$\frac{\partial z(\xi,\tau)}{\partial \xi} = \frac{\partial z(0,\tau)}{\partial \xi} + \psi^T(\xi)\mathbb{I}^T Z\psi(\tau), \tag{4.2}$$

which $\mathbb{I}$ is the operational matrix of integral with $\theta = 1$. In order to obtain $z(\xi,\tau)$, by integrating of (4.2) and using the boundary condition given by (1.3), we get

$$z(\xi,\tau) = \kappa_1(\tau) + \xi\frac{\partial z(0,\tau)}{\partial \xi} + \psi^T(\xi)\left(\mathbb{I}^T\right)^2 Z\psi(\tau), \tag{4.3}$$

replacing $\xi = 1$ in Eq. (4.3) and based on Eq. (1.3), we obtain

$$\frac{\partial z(0,\tau)}{\partial \xi} = \kappa_2(\tau) - \kappa_1(\tau) - \psi^T(1)\left(\mathbb{I}^T\right)^2 Z\psi(\tau), \tag{4.4}$$

approximating $\kappa_1(\tau), \kappa_2(\tau)$ by means of the Vieta-Lucas polynomials and substituting Eq. (4.4) in Eq. (4.2), we have

$$\begin{aligned}\frac{\partial z(\xi,\tau)}{\partial \xi} &= \psi^T(\xi)K_2\psi(\tau) - \psi^T(\xi)K_1\psi(\tau) - \psi^T(1)\left(\mathbb{I}^T\right)^2 Z\psi(\tau)\\ &+ \psi^T(\xi)\mathbb{I}^T Z\psi(\tau)\\ &= \psi^T(\xi)\left(K_2 - K_1 - \psi^T(1)\left(\mathbb{I}^T\right)^2 Z + \mathbb{I}^T Z\right)\psi(\tau)\\ &= \psi^T(\xi)M_1\psi(\tau),\end{aligned} \tag{4.5}$$

where $M_1 = K_2 - K_1 - \psi^T(1)\left(\mathbb{I}^T\right)^2 Z + \mathbb{I}^T Z$. Substituting Eq. (4.4) in Eq. (4.3), lead to

$$\begin{aligned}z(\xi,\tau) &= \kappa_1(\tau) + \xi\left(\kappa_2(\tau) - \kappa_1(\tau) - \psi^T(1)\left(\mathbb{I}^T\right)^2 Z\psi(\tau)\right)\\ &+ \psi^T(\xi)\left(\mathbb{I}^T\right)^2 Z\psi(\tau) = \psi^T(\xi)K_1\psi(\tau) + \psi^T(\xi)H_2\psi(\tau)\\ &- \psi^T(\xi)H_1\psi(\tau) - \psi^T(\xi)E\psi^T(1)\left(\mathbb{I}^T\right)^2 Z\psi(\tau) + \psi^T(\xi)\left(\mathbb{I}^T\right)^2 Z\psi(\tau)\\ &= \psi^T(\xi)\left(K_1 + H_2 - H_1 - E\psi^T(1)\left(\mathbb{I}^T\right)^2 Z + \left(\mathbb{I}^T\right)^2 Z\right)\psi(\tau)\\ &= \psi^T(\xi)M_2\psi(\tau),\end{aligned} \tag{4.6}$$

where $M_2 = K_1 + H_2 - H_1 - E\psi^T(1)\left(\mathbb{I}^T\right)^2 Z + \left(\mathbb{I}^T\right)^2 Z$ and

$$\begin{aligned}\xi\kappa_1(\tau) &= \psi^T(\xi)H_1\psi(\tau),\\ \xi\kappa_2(\tau) &= \psi^T(\xi)H_2\psi(\tau),\\ \xi &= \psi^T(\xi)E.\end{aligned}$$

Now applying the operator I_τ^θ on Eq. (1.1), we get

$$\begin{aligned}z(\xi,\tau) &= z(\xi,0) + I_\tau^\theta h(\xi,\tau) - \sigma_1 I_\tau^\theta\frac{\partial z(\xi,\tau)}{\partial \xi} + \sigma_2 I_\tau^\theta\frac{\partial^2 z(\xi,\tau)}{\partial \xi^2}\\ &+ \frac{\sigma_3}{\Gamma(\theta)}\int_0^\tau (\tau-\eta)^{\theta-1}\frac{\partial^2 z(\xi,\eta)}{\partial \xi^2}dB(\eta).\end{aligned} \tag{4.7}$$

Substituting Eqs. (4.1), (4.5) and Eq. (4.6) in the stochastic integral equation (4.7), we get

$$\begin{aligned}\psi^T(\xi)M_2\psi(\tau) &= \psi^T(\xi)K_0\psi(\tau)+\psi^T(\xi)H\mathbb{I}^\theta\psi(\tau)-\sigma_1\psi^T(\xi)M_1\mathbb{I}^\theta\psi(\tau)\\ &+ \sigma_2\psi^T(\xi)Z\mathbb{I}^\theta\psi(\tau)\\ &+ \frac{\sigma_3}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\psi^T(\xi)Z\psi(\eta)dB(\eta)\end{aligned} \tag{4.8}$$

where $\kappa_0=\psi^T(\xi)K_0\psi(\tau)$ and $h(\xi,\tau)=\psi^T(\xi)H\psi(\tau)$. Hence using Eq. (2.16) and applying the operational matrix of derivative (3.7) in the integral term of Eq. (4.8), we get

$$\begin{aligned}\psi^T(\xi)M_2\psi(\tau) &= \psi^T(\xi)K_0\psi(\tau)+\psi^T(\xi)F\mathbb{I}^\theta\psi(\tau)-\sigma_1\psi^T(\xi)M_1\mathbb{I}^\theta\psi(\tau)\\ &+ \sigma_2\psi^T(\xi)Z\mathbb{I}^\theta\psi(\tau)+\sigma_3\psi^T(\xi)Z\hat{\Delta}^T\mathbb{I}^\theta\psi(\tau),\end{aligned}$$

or

$$\psi^T(\xi)\left(M_2-K_0-F\mathbb{I}^\theta+\sigma_1M_1\mathbb{I}^\theta-\sigma_2Z\mathbb{I}^\theta-\sigma_3Z\hat{\Delta}^T\mathbb{I}^\theta\right)\psi(\tau)=0,$$

in which $\Delta=\mathbb{D}^{1^T}\beta$ and $\hat{\Delta}$ is the operational matrix of product for the Vieta-Lucas polynomials. Therefore, by solving the following system

$$M_2-K_0-F\mathbb{I}^\theta+\sigma_1M_1\mathbb{I}^\theta-\sigma_2Z\mathbb{I}^\theta-\sigma_3U\hat{\Delta}^T\mathbb{I}^\theta=0,$$

the unknown parameters of Z is determined. Then, the approximate values of $z(\xi,\tau)$ can be obtained from Eq. (4.6).

5. Error bound and convergence analysis

Theorem 5.1. *Let $z(\tau)\in C^{n+1}[0,1]$ and $\Upsilon=Span\{VL_0^*(\tau),VL_1^*(\tau),\dots,VL_n^*(\tau)\}$. If $z_n(\tau)$ is the best approximation of $z(\tau)$ out of Υ, then, the error bound is as*

$$\|z(\tau)-z_n(\tau)\|\leqslant\frac{M}{(n+1)!}\sqrt{\pi}, \tag{5.1}$$

where $M=\max_{\tau\in[0,1]}z^{(n+1)}(\tau)$.

Proof: Using Taylor's series expansion, the function $z(\tau)$ is expanded as

$$z(\tau)=z(0)+\tau z'(0)+\frac{\tau^2}{2!}z''(0)+\ldots+\frac{\tau^n}{n!}z^{(n)}(0)+\frac{\tau^{n+1}}{(n+1)!}z^{(n+1)}(\zeta),$$

where $\zeta\in(0,1)$. Let

$$\bar{z}_n(\tau)=z(0)+\tau z'(0)+\frac{\tau^2}{2!}z''(0)+\ldots+\frac{\tau^n}{n!}z^{(n)}(0).$$

Since, $z_n(\tau)$ is the best approximation of $z(\tau)$, so, we have

$$\begin{aligned}\|z(\tau)-z_n(\tau)\|^2\leq&\|z(\tau)-\bar{z}_n(\tau)\|^2=\int_0^1|z(\tau)-\bar{z}_n(\tau)|^2w(\tau)dt\\ =&\int_0^1\left|\frac{\tau^{n+1}}{(n+1)!}z^{(n+1)}(\zeta)\right|^2\frac{1}{\sqrt{\tau-\tau^2}}d\tau\\ \leqslant&\frac{M^2}{[(n+1)!]^2}\int_0^1\frac{1}{\sqrt{\tau-\tau^2}}d\tau=\frac{M^2}{[(n+1)!]^2}\pi.\end{aligned}$$

Therefore, we get

$$\|z(\tau)-z_n(\tau)\|\leqslant\frac{M}{(n+1)!}\sqrt{\pi}.$$

So, we have achieved the require result.

Theorem 5.2. *Suppose $z(\xi,\tau)$ be a sufficiently function in $\Omega = [0,1] \times [0,1]$, $\{VL_i(\xi)\}_{i=0}^n \subset L^2[0,1]$, $\{VL_i(\tau)\}_{i=0}^n \subset L^2[0,1]$, $\chi = Span[VL_0(\xi), VL_1(\xi), \dots$
$, VL_n(\xi)]$, $\Upsilon = Span[VL_0(\tau), VL_1(\tau), \dots, VL_n(\tau)]$ and $z_n(\xi,\tau)$ be the best approximation of $z(\xi,\tau)$ out of $\chi \times \Upsilon$. Now assume*

$$\begin{aligned} \max_{(\xi,\tau)\in\Omega} \left| \frac{\partial^{n+1} z(\xi,\tau)}{\partial \xi^{n+1}} \right| &\leqslant \mathcal{C}_1, \\ \max_{(\xi,\tau)\in\Omega} \left| \frac{\partial^{n+1} z(\xi,\tau)}{\partial \tau^{n+1}} \right| &\leqslant \mathcal{C}_2, \\ \max_{(\xi,\tau)\in\Omega} \left| \frac{\partial^{2n+2} z(\xi,\tau)}{\partial \xi^{n+1}\partial \tau^{n+1}} \right| &\leqslant \mathcal{C}_3, \end{aligned} \tag{5.2}$$

then, the error bound is as

$$\|z(\xi,\tau) - z_n(\xi,\tau)\|_2 \leqslant \mathcal{C} \left(\frac{1}{(n+1)!2^{2n}} + \frac{1}{((n+1)!)^2\, 2^{4n+2}} \right) \pi,$$

where $\mathcal{C} = \max\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$.

Proof: Let ξ_i and τ_j, $i,j = 0,1,\dots,n$ are the roots of $(n+1)$ degree shifted Chebyshev polynomials on $[0,1]$. Also, $P_{n,n}z$ is the interpolation polynomial of $z(\xi,\tau)$ at (ξ_i,τ_j), then [25]

$$\begin{aligned} z(\xi,\tau) - P_{n,n}z = &\frac{1}{(n+1)!} \frac{\partial^{n+1} z(\zeta,\tau)}{\partial \xi^{n+1}} \prod_{i=0}^{n} (\xi - \xi_i) \\ &+ \frac{1}{(n+1)!} \frac{\partial^{n+1} z(\xi,\eta)}{\partial \tau^{n+1}} \prod_{j=0}^{n} (\tau - \tau_j) \\ &- \frac{1}{((n+1)!)^2} \frac{\partial^{2n+2} z(\acute{\zeta},\acute{\eta})}{\partial \xi^{n+1}\partial \tau^{n+1}} \prod_{i=0}^{n} (\xi - \xi_i) \prod_{j=0}^{n} (\tau - \tau_j), \end{aligned}$$

where $\zeta, \acute{\zeta}, \eta, \acute{\eta} \in [0,1]$. So, we obtain

$$\begin{aligned} |z(\xi,\tau) - P_{n,n}z| \leqslant &\max_{(\xi,\tau)\in\Omega} \left| \frac{\partial^{n+1} z(\xi,\tau)}{\partial \xi^{n+1}} \right| \frac{\prod_{i=0}^{n} |\xi - \xi_i|}{(n+1)!} \\ &+ \max_{(\xi,\tau)\in\Omega} \left| \frac{\partial^{n+1} z(\xi,\tau)}{\partial \tau^{n+1}} \right| \frac{\prod_{j=0}^{n} |\tau - \tau_i|}{(n+1)!} \\ &+ \max_{(\xi,\tau)\in\Omega} \left| \frac{\partial^{2n+2} z(\xi,\tau)}{\partial \xi^{n+1}\partial \tau^{n+1}} \right| \frac{\prod_{i=0}^{n} |\xi - \xi_i| \prod_{j=0}^{n} |\tau - \tau_i|}{((n+1)!)^2}. \end{aligned}$$

By using Eq. (5.2) and utilizing the estimations for Chebyshev interpolation nodes on the Ω, [18].

$$|z(\xi,\tau) - P_{n,n}z| \leqslant \frac{\mathcal{C}_1}{(n+1)!2^{2n+1}} + \frac{\mathcal{C}_2}{(n+1)!2^{2n+1}} + \frac{\mathcal{C}_3}{((n+1)!)^2\, 2^{4n+2}}.$$

Since $z_n(\xi,\tau)$ is the best approximation of $z(\xi,\tau)$, so that the following holds

$$\begin{aligned}
&\|z(\xi,\tau)-z_n(\xi,\tau)\|^2 \leqslant \|z(\xi,\tau)-P_{n,n}z\|^2\\
&=\int_0^1\int_0^1 |z(\xi,\tau)-P_{n,n}z|^2\,w(\xi)w(\tau)d\xi d\tau\\
&\leqslant \int_0^1\int_0^1\Big(\frac{\mathcal{C}_1}{(n+1)!2^{2n+1}}+\frac{\mathcal{C}_2}{(n+1)!2^{2n+1}}+\frac{\mathcal{C}_3}{((n+1)!)^2\,2^{4n+2}}\Big)^2 w(\xi)w(\tau)d\xi d\tau\\
&\leqslant \left(\frac{\mathcal{C}_1}{(n+1)!2^{2n+1}}+\frac{\mathcal{C}_2}{(n+1)!2^{2n+1}}+\frac{\mathcal{C}_3}{((n+1)!)^2\,2^{4n+2}}\right)^2\int_0^1\int_0^1 w(\xi)w(\tau)d\xi d\tau\\
&\leqslant \left(\frac{\mathcal{C}_1}{(n+1)!2^{2n+1}}+\frac{\mathcal{C}_2}{(n+1)!2^{2n+1}}+\frac{\mathcal{C}_3}{((n+1)!)^2\,2^{4n+2}}\right)^2\pi^2
\end{aligned}$$

therefore, there exists $\mathcal{C}$ such that

$$\|z(\xi,\tau)-z_n(\xi,\tau)\| \leqslant \mathcal{C}\left(\frac{1}{(n+1)!2^{2n}}+\frac{1}{((n+1)!)^2\,2^{4n+2}}\right)\pi.$$

Theorem 5.3. *(Convergence Analysis): Suppose $z(\xi,\tau)$ be the exact solution of Eq. (1.1) and $z_n(\xi,\tau)$ be its approximate solution. Furthermore, assume the following hypotheses hold:*
H1) $|\frac{\partial z}{\partial \xi}-\frac{\partial w}{\partial \xi}| < K_1|z-w|$,
H2) $|\frac{\partial^2 z}{\partial \xi^2}-\frac{\partial^2 w}{\partial \xi^2}| < K_2|z-w|$,
H3) $5\left(\sigma_1^2\varpi_1K_1^2+\sigma_2^2\varpi_1K_2^2+\sigma_3^2\varpi_2K_2^2\right)<1$,
which K_1,K_2 are positive constants and by the theorems 5.1 and 5.2

$$\|\kappa_0(\xi)-\kappa_{0_n}(\xi)\| \leqslant \rho(n), \tag{5.3}$$

$$\|h(\xi,\tau)-h_n(\xi,\tau)\| \leqslant \varrho(n), \tag{5.4}$$

where $\kappa_{0_n}(\xi)$ and $h_n(\xi,\tau)$ are approximation of $\kappa_0(\xi)$ and $h(\xi,\tau)$, respectively and

$$\rho(n)=\frac{M}{(n+1)!}\sqrt{\pi},\qquad \varrho(n)=\mathcal{C}\Big(\frac{1}{(n+1)!2^{2n}}+\frac{1}{((n+1)!)^2 2^{4n+2}}\Big). \tag{5.5}$$

Then, $z_n(\xi,\tau)$ convergence to $z(\xi,\tau)$ as $n\to\infty$ in L^2 and $\|z\|=(E|z|^2)^{\frac{1}{2}}$.

Proof: Applying the operator I_τ^θ on Eq. (1.1), we get

$$\begin{aligned}
z(\xi,\tau)=\kappa_0(\xi) \quad &+\quad \frac{1}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}h(\xi,\eta)d\eta\\
&-\quad \frac{\sigma_1}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\frac{\partial z(\xi,\eta)}{\partial\xi}d\eta\\
&+\quad \frac{\sigma_2}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}d\eta\\
&+\quad \frac{\sigma_3}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}dB(\eta),
\end{aligned}\tag{5.6}$$

and the approximate solution of Eq. (5.6) is

$$\begin{aligned}
z_n(\xi,\tau) \quad &= \quad \kappa_{0_n}(\xi)+\frac{1}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}h_n(\xi,\eta)d\eta\\
&- \quad \frac{\sigma_1}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\frac{\partial z_n(\xi,\eta)}{\partial\xi}d\eta\\
&+ \quad \frac{\sigma_2}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\frac{\partial^2 z_n(\xi,\eta)}{\partial\xi^2}d\eta\\
&+ \quad \frac{\sigma_3}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\frac{\partial^2 z_n(\xi,\eta)}{\partial\xi^2}dB(\eta).
\end{aligned}\tag{5.7}$$

Substracting Eq. (5.7) from Eq. (5.6) and let $e_n(\xi,\tau)=z(\xi,\tau)-z_n(\xi,\tau)$.
Using the relation $\left(\sum_{i=0}^{n}\varsigma_i\right)^2 \leqslant n\left(\sum_{i=0}^{n}\varsigma_i^2\right)$, the following inequality is obtained:

$$\begin{aligned} E|e_n(\xi,\tau)|^2 \;&\leqslant 5\Bigg(E|\kappa_0(\xi)-\kappa_{0_n}(\xi)|^2 \\ &+\; E\Big|\frac{1}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\left(h(\xi,\eta)-h_n(\xi,\eta)\right)d\eta\Big|^2 \\ &+\; E\Big|\frac{\sigma_1}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\left(\frac{\partial z(\xi,\eta)}{\partial x}-\frac{\partial z_n(\xi,\eta)}{\partial \xi}\right)d\eta\Big|^2 \\ &+\; E\Big|\frac{\sigma_2}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\left(\frac{\partial^2 z(\xi,\eta)}{\partial \xi^2}-\frac{\partial^2 z_n(\xi,\eta)}{\partial \xi^2}\right)d\eta\Big|^2 \\ &+\; E\Big|\frac{\sigma_3}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\left(\frac{\partial^2 z(\xi,\eta)}{\partial \xi^2}-\frac{\partial^2 z_n(\xi,\eta)}{\partial \xi^2}\right)dB(\eta)\Big|^2\Bigg). \end{aligned} \tag{5.8}$$

From Eq. (5.3), it follows that

$$E|\kappa_0(\xi)-\kappa_{0_n}(\xi)|^2=\|\kappa_0(\xi)-\kappa_{0_n}(\xi)\|^2\leqslant\rho^2(n). \tag{5.9}$$

Using Cauchy-Schwartz inequality and Eq. (5.4), we obtain

$$\begin{aligned} E\Big|\frac{1}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\left(h(\xi,\eta)-h_n(\xi,\eta)\right)d\eta\Big|^2 &\leqslant \frac{1}{\Gamma^2(\theta)}\big(\int_0^\tau(\tau-\eta)^{2(\theta-1)}d\eta\big) \\ &\times\big(\int_0^\tau E\big|h(\xi,\eta)-h_n(\xi,\eta)\big|^2 d\eta\big) \\ &\leqslant \varpi_1\big\|h(\xi,\eta)-h_n(\xi,\eta)\big\|^2\leqslant\varpi_1\varrho^2(n), \end{aligned} \tag{5.10}$$

where ϖ_1 is a positive constant depended on θ. In the continuation of the proof, using Cauchy-Schwartz inequality and the hypothesis H1, we get

$$\begin{aligned} E\Big|\frac{\sigma_1}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\left(\frac{\partial z(\xi,\eta)}{\partial \xi}-\frac{\partial z_n(\xi,\eta)}{\partial \xi}\right)d\eta\Big|^2 &\leqslant \frac{\sigma_1^2}{\Gamma^2(\theta)} \\ &\times\big(\int_0^\tau(\tau-\eta)^{2(\theta-1)}d\eta\big)E\Big(\int_0^\tau\Big|\frac{\partial z(\xi,\eta)}{\partial \xi}-\frac{\partial z_n(\xi,\eta)}{\partial \xi}\Big|^2 d\eta\Big) \\ &\leqslant \sigma_1^2\varpi_1\left(\int_0^\tau E\Big|\frac{\partial z(\xi,\eta)}{\partial \xi}-\frac{\partial z_n(\xi,\eta)}{\partial \xi}\Big|^2 d\eta\right) \\ &\leqslant \sigma_1^2\varpi_1K_1^2\|z-z_n\|^2. \end{aligned} \tag{5.11}$$

Again, using the same methodology as previously described and the hypothesis H2, we have

$$E\Big|\frac{\sigma_2}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\left(\frac{\partial^2 z(\xi,\eta)}{\partial \xi^2}-\frac{\partial^2 z_n(\xi,\eta)}{\partial \xi^2}\right)d\eta\Big|^2\leqslant\sigma_2^2\varpi_1K_2^2\|z-z_n\|^2. \tag{5.12}$$

Now, by using the Itô isometry property, following is obtained

$$\begin{aligned} &E\Big|\frac{\sigma_3}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\left(\frac{\partial^2 z(\xi,\eta)}{\partial \xi^2}-\frac{\partial^2 z_n(\xi,\eta)}{\partial \xi^2}\right)dB(\eta)\Big|^2 \\ &\qquad\leqslant \frac{\sigma_3^2}{\Gamma^2(\theta)}\varpi_2E\Big|\int_0^\tau\left(\frac{\partial^2 z(\xi,\eta)}{\partial \xi^2}-\frac{\partial^2 z_n(\xi,\eta)}{\partial \xi^2}\right)dB(\eta)\Big|^2 \\ &\qquad\leqslant \frac{\sigma_3^2}{\Gamma^2(\theta)}\varpi_2\int_0^\tau E\Big|\frac{\partial^2 z(\xi,\eta)}{\partial \xi^2}-\frac{\partial^2 z_n(\xi,\eta)}{\partial \xi^2}\Big|^2 d\eta \\ &\qquad\leqslant \sigma_3^2\varpi_2K_2^2\|z-z_n\|^2, \end{aligned} \tag{5.13}$$

where ϖ_2 is a positive constant depended on θ. Now, substituting Eqs. (5.9)-(5.13) in Eq. (5.8), we get

$$E\left|e_n(\xi,\tau)\right|^2=\left\|e_n(\xi,\tau)\right\|^2\leqslant 5\Big(\rho^2(n)+\varpi_1\varrho^2(n)+\big(\sigma_1^2\varpi_1K_1^2+\sigma_2^2\varpi_1K_2^2+\sigma_3^2\varpi_2K_2^2\big)\left\|e_n(\xi,\tau)\right\|^2\Big),$$

therefore

$$\|e_n(\xi,\tau)\|\leqslant\sqrt{\frac{5\left(\rho^2(n)+\varpi_1\varrho^2(n)\right)}{1-5\left(\sigma_1^2\varpi_1K_1^2+\sigma_2^2\varpi_1K_2^2+\sigma_3^2\varpi_2.K_2^2\right)}}.$$

It implies that, if $n\to\infty$ then $\|e_n(\xi,\tau)\|=\|z(\xi,\tau)-z_n(\xi,\tau)\|\to 0$, Therefore $z_n(\xi,\tau)$ convergence to $z(\xi,\tau)$ in L^2.

6. Existence and uniqueness of solutions

In this section, we provide the existence and uniqueness result of the solutions for the problem (1.1) under the conditions (1.2) and (1.3). We introduced the Banach space of all continuous function with the norm

$$\|z\|_{\hat{X}}:=\sup_{(\xi,\tau)\in\Omega}\|z(\xi,\tau)\|_{L^2}=\sup_{(\xi,\tau)\in\Omega}\left(E|z(\xi,\tau)|^2\right)^{\frac{1}{2}}<\infty.$$

Lemma 6.1. *Let $z(\xi,\tau)$ is a solution of the problem (1.1) under the conditions (1.2) and (1.3) if and only if it is a solution of the following stochastic integral equation*

$$\begin{aligned} z(\xi,\tau) &= \kappa_0(\xi)+I_\tau^\theta h(\xi,\tau)-\sigma_1 I_\tau^\theta\frac{\partial z(\xi,\tau)}{\partial\xi}+\sigma_2 I_\tau^\theta\frac{\partial^2 z(\xi,\tau)}{\partial\xi^2}\\ &+ \frac{\sigma_3}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}dB(\eta). \end{aligned}\tag{6.1}$$

Proof: Assume $z(\xi,\tau)$ satisfies in Eq. (1.1), then by applying I_τ^θ on Eq. (1.1) and using Eq. (2.3), we have

$$\begin{aligned} z(\xi,\tau) &= z(\xi,0)+I_\tau^\theta h(\xi,\tau)-\sigma_1 I_\tau^\theta\frac{\partial z(\xi,\tau)}{\partial\xi}+\sigma_2 I_\tau^\theta\frac{\partial^2 z(\xi,\tau)}{\partial\xi^2}\\ &+ \frac{\sigma_3}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}dB(\eta), \end{aligned}\tag{6.2}$$

by substituting the condition Eq. (1.2), Eq. (6.1) can be obtained. It is clear that if $z(\xi,\tau)$ satisfies Eq. (6.1) then Eq. (1.1) hold.
We propound the following assumption:
(A1) suppose that there exists the constants m_1, m_2, $m_3>0$ such that

$$\left|\frac{\partial z(\xi,\tau)}{\partial\xi}\right|<m_1,\quad\left|\frac{\partial^2 z(\xi,\tau)}{\partial\xi^2}\right|<m_2,\quad|h(\xi,\tau)|<m_3.$$

Theorem 6.2. *Suppose that the assumption A1 hold, then the problem (1.1) under the conditions (1.2) and (1.3) has at least one solution.*

Proof: We define the operator $\mathcal{A}:\hat{X}\to\hat{X}$ as:

$$\begin{aligned}\mathcal{A}z(\xi,\tau)&=\kappa_0(\xi)+\frac{1}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}h(\xi,\eta)d\eta\\ &-\frac{\sigma_1}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\frac{\partial z(\xi,\eta)}{\partial\xi}d\eta+\frac{\sigma_2}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}d\eta\\ &+\frac{\sigma_3}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}dB(\eta).\end{aligned}\tag{6.3}$$

Also, we consider set $U := \Big\{z \in \hat{X} : \|z-\kappa_0\|_{\hat{X}} \leqslant \rho^*,\ \rho^*>0\Big\}$. It is evident that U is a closed and convex subset of the Banach space of all continuous function with the norm $\|z\|_{\hat{X}}$. Choose $\varpi_2\sigma_3^2 m_2^2 \leqslant \frac{\rho^*}{4}$, $\varpi_1(m_3^2+\sigma_1^2 m_1^2+\sigma_2^2 m_2^2) \leqslant \frac{\rho^*}{12}$. where ϖ_1, ϖ_2 are positive constants depended on θ.
The subsequent objective is to demonstrate the continuity of the operator $\mathcal{A}$. For this purpose, a sequence z_n within U is considered, which converges to $z \in U$, to illustrate the continuity of $\mathcal{A}$. So, using the relation $\left(\sum_{i=0}^n \varsigma_i\right)^2 \leqslant n\left(\sum_{i=0}^n \varsigma_i^2\right)$.

$$\begin{aligned}\big|\mathcal{A}z_n(\xi,\tau)-\mathcal{A}z(\xi,\tau)\big|^2 \leqslant 3\Bigg(&\Big|\frac{\sigma_1}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\Big(\frac{\partial z_n(\xi,\eta)}{\partial\xi}-\frac{\partial z(\xi,\eta)}{\partial\xi}\Big)d\eta\Big|^2\\ &+\Big|\frac{\sigma_2}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\Big(\frac{\partial^2 z_n(\xi,\eta)}{\partial\xi^2}-\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}\Big)d\eta\Big|^2\\ &+\Big|\frac{\sigma_3}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\Big(\frac{\partial^2 z_n(\xi,\eta)}{\partial\xi^2}-\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}\Big)dB(\eta)\Big|^2\Bigg),\end{aligned} \tag{6.4}$$

by taking expectation at both sides of Eq. (6.4) and using from the assumptions H1 and H2 of Theorem 5.3 and Cauchy-Schwartz inequality and Itô isometry property, we obtain

$$\begin{aligned}E\big|\mathcal{A}z_n(\xi,\tau)-\mathcal{A}z(\xi,\tau)\big|^2 \leqslant 3\Bigg(&\sigma_1^2\varpi_1\int_0^\tau E\Big|\frac{\partial z_n(\xi,\eta)}{\partial\xi}-\frac{\partial z(\xi,\eta)}{\partial\xi}\Big|^2 d\eta\\ &+\sigma_2^2\varpi_1\int_0^\tau E\Big|\frac{\partial^2 z_n(\xi,\eta)}{\partial\xi^2}-\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}\Big|^2 d\eta\\ &+\frac{\sigma_3^2}{\Gamma^2(\theta)}\varpi_2\int_0^\tau E\Big|\frac{\partial^2 z_n(\xi,\eta)}{\partial\xi^2}-\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}\Big|^2 d\eta\Bigg)\\ &\leqslant 3\Big(\sigma_1^2\varpi_1 K_1^2\big\|z_n-z\big\|^2+\sigma_2^2\varpi_1 K_2^2\big\|z_n-z\big\|^2+\sigma_3^2\varpi_2 K_2^2\big\|z_n-z\big\|^2\Big),\end{aligned} \tag{6.5}$$

Hence

$$\big\|\mathcal{A}z_n(\xi,\tau)-\mathcal{A}z(\xi,\tau)\big\|_{\hat{X}} \leqslant 3\Big(\sigma_1^2\varpi_1 K_1^2+\sigma_2^2\varpi_1 K_2^2+\sigma_3^2\varpi_2 K_2^2\Big)\big\|z_n-z\big\|_{\hat{X}}. \tag{6.6}$$

Because z_n is convergent to z, it means that $\|z_n-z\|\to 0$. The right side of the inequlity is convergent to zero. Therefore, the operator $\mathcal{A}$ is continuous.
Next, we prove that $\mathcal{A}(U)\subset U$. For any $z\in U$, we have

$$\begin{aligned}\big|\mathcal{A}z(\xi,\tau)-\kappa_0(\xi)\big| \leqslant \Big|\frac{1}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\Big(&h(\xi,\eta)-\sigma_1\frac{\partial z(\xi,\eta)}{\partial\xi}\\ &+\sigma_2\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}\Big)d\eta\Big|+\Big|\frac{\sigma_3}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}dB(\eta)\Big|\end{aligned} \tag{6.7}$$

Thus, using the relation $\left(\sum_{i=0}^n \varsigma_i\right)^2 \leqslant n\left(\sum_{i=0}^n \varsigma_i^2\right)$ and by taking expectation, we have

$$\begin{aligned}E\big|\mathcal{A}z(\xi,\eta)-\kappa_0(\xi)\big|^2 \leqslant 2\Bigg(&E\Big|\frac{1}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\big(h(\xi,\eta)-\sigma_1\frac{\partial z(\xi,\eta)}{\partial\xi}\\ &+\sigma_2\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}\big)d\eta\Big|^2+E\Big|\frac{\sigma_3}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}dB(\eta)\Big|^2\Bigg).\end{aligned} \tag{6.8}$$

We use the assumption A1 and Cauchy-Schwartz inequality and the Itô isometry property. we obtain

$$\begin{aligned}
E\big|\mathcal{A}z(\xi,\tau)-\kappa_0(\xi)\big|^2 &\leqslant 2\Bigg(\frac{1}{\Gamma^2(\theta)}\Big(\int_0^\tau(\tau-\eta)^{2(\theta-1)}d\eta\Big)\\
&\times E\Big(\int_0^\tau\Big|h(\xi,\eta)-\sigma_1\frac{\partial z(\xi,\eta)}{\partial\xi}+\sigma_2\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}\Big|^2 d\eta\Big)\\
&+\frac{\sigma_3^2}{\Gamma^2(\theta)}\varpi_2 E\Big|\int_0^\tau\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}dB(\eta)\Big|^2\Bigg)\\
&\leqslant 2\Bigg(3\varpi_1\Big(\int_0^\tau\Big(E\big|h(\xi,\eta)\big|^2+\sigma_1^2E\Big|\frac{\partial z(\xi,\eta)}{\partial\xi}\Big|^2\\
&+\sigma_2^2E\Big|\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}\Big|^2\Big)d\eta\Big)\\
&+\sigma_3^2\varpi_2\Big(\int_0^\tau E\Big|\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}\Big|^2d\eta\Big)\Bigg)\\
&\leqslant 2\Big(3\varpi_1(m_3^2+\sigma_1^2\ m_1^2+\sigma_2^2m_2^2)+\sigma_3^2\varpi_2m_2^2\Big)\\
&\leqslant 2\Big(\frac{3\rho^*}{12}+\frac{\rho^*}{4}\Big)\leqslant\rho^*.
\end{aligned}\tag{6.9}$$

Therefore

$$\|\mathcal{A}z(\xi,\tau)-\kappa_0(\xi)\|_{\hat{X}}=\sup_{(\xi,\tau)\in\Omega}E\big|\mathcal{A}z(\xi,\tau)-\kappa_0(\xi)\big|^2\leqslant\rho^*,\tag{6.10}$$

which implies that $\mathcal{A}(U)\subset U$, i.e. $\mathcal{A}$ maps the set U to itself. Now, we prove that the operator $\mathcal{A}:\hat{X}\to\hat{X}$ is bounded on U.

$$\begin{aligned}
|\mathcal{A}z(\xi,\tau)| &\leqslant|\kappa_0(\xi)|\\
&+\left|\frac{1}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\left(h(\xi,\eta)-\sigma_1\frac{\partial z(\xi,\eta)}{\partial\xi}+\sigma_2\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}\right)d\eta\right|\\
&+\left|\frac{\sigma_3}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}dB(\eta)\right|.
\end{aligned}\tag{6.11}$$

So, we obtain

$$\begin{aligned}
E|\mathcal{A}z(\xi,\tau)|^2 &\leqslant 3\Bigg(E\,|\kappa_0(\xi)|^2+\frac{1}{\Gamma^2(\theta)}\Big(\int_0^\tau(\tau-\eta)^{2(\theta-1)}d\eta\Big)\\
&\times E\Big(\int_0^\tau\Big|h(\xi,\eta)-\sigma_1\frac{\partial z(\xi,\eta)}{\partial\xi}+\sigma_2\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}\Big|^2 d\eta\Big)\\
&+\frac{\sigma_3^2}{\Gamma^2(\theta)}\varpi_2 E\Big|\int_0^\tau\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}dB(\eta)\Big|^2\Bigg)\\
&\leqslant 3\Bigg(E\,|\kappa_0(\xi)|^2+3\varpi_1\Big(\int_0^\tau\Big(E\big|h(\xi,\eta)\big|^2+\sigma_1^2E\Big|\frac{\partial z(\xi,\eta)}{\partial\xi}\Big|^2\\
&+\sigma_2^2E\Big|\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}\Big|^2\Big)d\eta\Big)+\sigma_3^2\varpi_2\Big(\int_0^\tau E\Big|\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}\Big|^2d\eta\Big)\Bigg)\\
&\leqslant 3\Big(\|\kappa_0(\xi)\|_{\hat{X}}+3\varpi_1(m_3^2+\sigma_1^2\ m_1^2+\sigma_2^2m_2^2)+\sigma_3^2\varpi_2m_2^2\Big)\\
&\leqslant 3\Big(\|\kappa_0(\xi)\|_{\hat{X}}+\rho^*\Big).
\end{aligned}\tag{6.12}$$

Thus

$$\|\mathcal{A}z(\xi,\tau)\|_{\hat{X}} \leqslant L$$

where $L = 3\Big(\|\kappa_0(\xi)\|_{\hat{X}} + \rho^*\Big)$. Lastly, we prove that the operator $\mathcal{A}: \hat{X} \to \hat{X}$ is equicontinuous on U. For any $(\xi,\tau_1),(\xi,\tau_2) \in \Omega$ such that $0 \leqslant \tau_1 < \tau_2 \leqslant 1$, we have

$$\begin{aligned}
\big|\mathcal{A}u(\xi,\tau_2) - \mathcal{A}u(\xi,\tau_1)\big| &\leqslant \Big|\frac{1}{\Gamma(\theta)}\int_0^{\tau_2}(\tau_2-\eta)^{\theta-1}\Big(h(\xi,\eta) - \sigma_1\frac{\partial z(\xi,\eta)}{\partial \xi} \\
&+ \sigma_2\frac{\partial^2 z(\xi,\eta)}{\partial \xi^2}\Big)d\eta \\
&- \frac{1}{\Gamma(\theta)}\int_0^{\tau_1}(\tau_1-\eta)^{\theta-1}\Big(h(\xi,\eta) - \sigma_1\frac{\partial z(\xi,\eta)}{\partial \xi} + \sigma_2\frac{\partial^2 z}{\partial \xi^2}\Big)d\eta\Big| \\
&+ \Big|\frac{\sigma_3}{\Gamma(\theta)}\int_0^{\tau_2}(\tau_2-\eta)^{\theta-1}\frac{\partial^2 z(\xi,\eta)}{\partial \xi^2}dB(\eta) \\
&- \frac{\sigma_3}{\Gamma(\theta)}\int_0^{\tau_1}(\tau_1-\eta)^{\theta-1}\frac{\partial^2 z(\xi,\eta)}{\partial \xi^2}dB(\eta)\Big| \\
&\leqslant \Big|\frac{1}{\Gamma(\theta)}\int_0^{\tau_1}\Big((\tau_2-\eta)^{\theta-1} - (\tau_1-\eta)^{\theta-1}\Big) \\
&\times \Big(h(\xi,\eta) - \sigma_1\frac{\partial z(\xi,\eta)}{\partial \xi} + \sigma_2\frac{\partial^2 z(\xi,\eta)}{\partial \xi^2}\Big)d\eta \\
&+ \frac{1}{\Gamma(\theta)}\int_{\tau_1}^{\tau_2}(\tau_2-\eta)^{\theta-1}\Big(h(\xi,\eta) - \sigma_1\frac{\partial z(\xi,\eta)}{\partial \xi} + \sigma_2\frac{\partial^2 z(\xi,\eta)}{\partial \xi^2}\Big)d\eta\Big| \\
&+ \Big|\frac{\sigma_3}{\Gamma(\theta)}\int_0^{\tau_1}\Big((\tau_2-\eta)^{\theta-1} - (\tau_1-\eta)^{\theta-1}\Big)\frac{\partial^2 z(\xi,\eta)}{\partial \xi^2}dB(\eta) \\
&+ \frac{\sigma_3}{\Gamma(\theta)}\int_{\tau_1}^{\tau_2}(\tau_2-\eta)^{\theta-1}\frac{\partial^2 z(\xi,\eta)}{\partial \xi^2}dB(\eta)\Big|. \qquad (6.13)
\end{aligned}$$

By taking expectation of the Eq. (6.13), we obtain

$$\begin{aligned}
E\big|\mathcal{A}z(\xi,\tau_2) - \mathcal{A}z(\xi,\tau_1)\big|^2 &\leqslant 4\Bigg(E\Big|\frac{1}{\Gamma(\theta)}\int_0^{\tau_1}\Big((\tau_2-\eta)^{\theta-1} - (\tau_1-\eta)^{\theta-1}\Big) \\
&\times \Big(h(\xi,\eta) - \sigma_1\frac{\partial z(\xi,\eta)}{\partial \xi} + \sigma_2\frac{\partial^2 z(\xi,\eta)}{\partial \xi^2}\Big)d\eta\Big|^2 \\
&+ E\Big|\frac{1}{\Gamma(\theta)}\int_{\tau_1}^{\tau_2}(\tau_2-\eta)^{\theta-1}\Big(h(\xi,\eta) - \sigma_1\frac{\partial z(\xi,\eta)}{\partial \xi} + \sigma_2\frac{\partial^2 z(\xi,\eta)}{\partial \xi^2}\Big)d\eta\Big|^2 \\
&+ E\Big|\frac{\sigma_3}{\Gamma(\theta)}\int_0^{\tau_1}\Big((\tau_2-\eta)^{\theta-1} - (\tau_1-\eta)^{\theta-1}\Big)\frac{\partial^2 z(\xi,\eta)}{\partial \xi^2}dB(\eta)\Big|^2 \\
&+ E\Big|\frac{\sigma_3}{\Gamma(\theta)}\int_{\tau_1}^{\tau_2}(\tau_2-\eta)^{\theta-1}\frac{\partial^2 z(\xi,\eta)}{\partial \xi^2}dB(\eta)\Big|^2\Bigg), \qquad (6.14)
\end{aligned}$$

using Cauchy-Schwartz inequality and Itô isometry property, we conclude

$$
\begin{aligned}
&E\big|\mathcal{A}z(\xi,\tau_2)-\mathcal{A}z(\xi,\tau_1)\big|^2 \leqslant 4\Big(\int_0^{\tau_1}\big((\tau_2-\eta)^{\theta-1}-(\tau_1-\eta)^{\theta-1}\big)^2 d\eta\Big)\\
&\times E\Big(\int_0^{\tau_1}\Big|h(\xi,\eta)-\sigma_1\frac{\partial z(\xi,\eta)}{\partial\xi}+\sigma_2\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}\Big|^2 d\eta\Big)\\
&+4\Big(\int_{\tau_1}^{\tau_2}((\tau_2-\eta)^{\theta-1})^2 d\eta\Big)E\Big(\int_{\tau_1}^{\tau_2}\Big|h(\xi,\eta)-\sigma_1\frac{\partial z(\xi,\eta)}{\partial\xi}+\sigma_2\\
&\times\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}\Big|^2 d\eta\Big)+4\sigma_3^2\Big(\int_0^{\tau_1}\big((\tau_2-\eta)^{\theta-1}-(\tau_1-\eta)^{\theta-1}\big)^2\\
&\times\big|\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}\big|^2 d\eta\Big)+4\sigma_3^2\Big(\int_{\tau_1}^{\tau_2}((\tau_2-\eta)^{\theta-1})^2\big|\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}\big|^2 d\eta\Big)\\
&\leqslant 12\Big(\int_0^{\tau_1}\big((\tau_2-\eta)^{\theta-1}-(\tau_1-\eta)^{\theta-1}\big)^2 d\tau\Big)\Big(\int_0^{\tau_1}E\big|h(\xi,\eta)\big|^2\\
&+\sigma_1^2 E\big|\frac{\partial z(\xi,\eta)}{\partial\xi}\big|^2+\sigma_2^2 E\big|\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}\big|^2 d\eta\Big)\\
&+12\Big(\int_{\tau_1}^{\tau_2}(\tau_2-\eta)^{2\theta-2}d\eta\Big)\Big(\int_{\tau_1}^{\tau_2}E\big|h(\xi,\eta)\big|^2+\sigma_1^2 E\big|\frac{\partial z(\xi,\eta)}{\partial\xi}\big|^2+\sigma_2^2\\
&\times E\big|\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}\big|^2 d\eta\Big)+4\sigma_3^2 m_2^2\Big(\int_0^{\tau_1}\big((\tau_2-\eta)^{\theta-1}-(\tau_1-\eta)^{\theta-1}\big)^2\, d\eta\Big)\\
&+4\sigma_3^2 m_2^2\Big(\int_{\tau_1}^{\tau_2}(\tau_2-\eta)^{2\theta-2}d\eta\Big),
\end{aligned}
$$

using of the assumption A1, we obtain

$$
\begin{aligned}
&E\big|\mathcal{A}z(\xi,\tau_2)-\mathcal{A}z(\xi,\tau_1)\big|^2 \leqslant 12(m_3^2+\sigma_1^2\ m_1^2+\sigma_2^2 m_2^2)\tau_1\\
&\times\Big(\int_0^{\tau_1}\Big((\tau_1-\eta)^{2\theta-2}-(\tau_2-\eta)^{2\theta-2}\Big)d\eta\Big)\\
&+12(m_3^2+\sigma_1^2\ m_1^2+\sigma_2^2 m_2^2)(\tau_2-\tau_1)\Big(\int_{\tau_1}^{\tau_2}(\tau_2-\tau)^{2\theta-2}d\eta\Big)\\
&+4\sigma_3^2 m_2^2\Big(\int_0^{\tau_1}\Big((\tau_1-\eta)^{2\theta-2}-(\tau_2-\eta)^{2\theta-2}\Big)d\eta\Big)\\
&+4\sigma_3^2 m_2^2\Big(\int_{\tau_1}^{\tau_2}(\tau_2-\eta)^{2\theta-2}d\eta\Big)\\
&\leqslant 12(m_3^2+\sigma_1^2\ m_1^2+\sigma_2^2 m_2^2)\Big(\frac{1}{2\theta-1}\big((\tau_2-\tau_1)^{2\theta-1}-\tau_2^{2\theta-1}+\tau_1^{2\theta-1}\big)\Big)\\
&+12(m_3^2+\sigma_1^2\ m_1^2+\sigma_2^2 m_2^2)(\tau_2-\tau_1)\frac{(\tau_2-\tau_1)^{2\theta-1}}{2\theta-1}\\
&+4\sigma_3^2 m_2^2\Big(\frac{1}{2\theta-1}\big((\tau_2-\tau_1)^{2\theta-1}-\tau_2^{2\theta-1}+\tau_1^{2\theta-1}\big)\Big)\\
&+4\sigma_3^2 m_2^2\frac{(\tau_2-\tau_1)^{2\theta-1}}{2\theta-1},
\end{aligned}
\tag{6.15}
$$

It is evident that, as $\tau_2 \to \tau_1$, the right-hand side of Eq. (6.15) approaches zero, thereby fulfilling the necessary condition for equicontinuity. Consequently, invoking the Arzela-Ascoli Theorem permits the deduction of the compactness of the closure of $\mathcal{A}(U)$. Using the Schauder's Fixed Point Theorem, the operator $\mathcal{A}$ has a fixed point which is a solution of the stochastic equation (1.1). Thus, the stochastic equation (1.1) under the conditions (1.2) and (1.3) has at least one solution.

Lemma 6.3. *Under the assumptions H1-H2 of theorem 5.3, the problem (1.1) under the conditions (1.2) and (1.3) has a unique solution if*

$$3\left(\sigma_1^2\varpi_1 K_1^2+\sigma_2^2\varpi_1 K_2^2+\sigma_3^2\varpi_2 K_2^2\right)<1.$$

Proof: We consider the operator $\mathcal{A}:\hat{X}\to\hat{X}$ as defined in the Eq. (6.3). The demonstration of a unique fixed point is established through the utilization of the Banach contraction theorem in the Banach space $\hat{X}$.
By using from the relation $\left(\sum_{i=0}^{n}\varsigma_i\right)^2\leqslant n\left(\sum_{i=0}^{n}\varsigma_i^2\right)$, for any $z,w\in\hat{X}$, we have

$$\begin{aligned}|\mathcal{A}z(\xi,\tau)-\mathcal{A}w(\xi,\tau)|^2\leqslant 3\Bigg(&\left|\frac{\sigma_1}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\left(\frac{\partial z(\xi,\eta)}{\partial\xi}-\frac{\partial w(\xi,\eta)}{\partial\xi}\right)d\eta\right|^2\\&+\left|\frac{\sigma_2}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\left(\frac{\partial^2 z(\xi,\tau)}{\partial\xi^2}-\frac{\partial^2 w(\xi,\tau)}{\partial\xi^2}\right)d\eta\right|^2\\&+\left|\frac{\sigma_3}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\left(\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}-\frac{\partial^2 w(\xi,\eta)}{\partial\xi^2}\right)dB(\eta)\right|^2\Bigg).\end{aligned}\tag{6.16}$$

By taking expectation of the Eq. (6.16), we obtain

$$\begin{aligned}&E|\mathcal{A}z(\xi,\tau)-\mathcal{A}w(\xi,\tau)|^2\\&\leqslant 3\Bigg(E\left|\frac{\sigma_1}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\left(\frac{\partial z(\xi,\eta)}{\partial\xi}-\frac{\partial w(\xi,\eta)}{\partial\xi}\right)d\eta\right|^2\\&+E\left|\frac{\sigma_2}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\left(\frac{\partial^2 z(\xi,\tau)}{\partial\xi^2}-\frac{\partial^2 w(\xi,\tau)}{\partial\xi^2}\right)d\eta\right|^2\\&+E\left|\frac{\sigma_3}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\left(\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}-\frac{\partial^2 w(\xi,\eta)}{\partial\xi^2}\right)dB(\eta)\right|^2\Bigg).\end{aligned}\tag{6.17}$$

Using Cauchy-Schwartz inequality and the assumption H1 of theorem 5.3, we get

$$\begin{aligned}E\Big|\frac{\sigma_1}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\left(\frac{\partial z(\xi,\eta)}{\partial\xi}-\frac{\partial w(\xi,\eta)}{\partial\xi}\right)d\eta\Big|^2&\leqslant\frac{\sigma_1^2}{\Gamma^2(\theta)}\\&\times\Big(\int_0^\tau(\tau-\eta)^{2(\theta-1)}d\eta\Big)E\Big(\int_0^\tau\Big|\frac{\partial z(\xi,\eta)}{\partial\xi}-\frac{\partial w(\xi,\eta)}{\partial\xi}\Big|^2d\eta\Big)\\&\leqslant\sigma_1^2\varpi_1\left(\int_0^\tau E\Big|\frac{\partial z(\xi,\eta)}{\partial\xi}-\frac{\partial w(\xi,\eta)}{\partial\xi}\Big|^2d\eta\right)\\&\leqslant\sigma_1^2\varpi_1K_1^2\|z-w\|_{\hat{X}},\end{aligned}\tag{6.18}$$

where ϖ_1 is a positive constant depended on θ.
Similarly, using the assumption H2 of theorem 5.3, the second term of Eq. (6.17) can be written as follows

$$E\Big|\frac{\sigma_2}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\left(\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}-\frac{\partial^2 w(\xi,\eta)}{\partial\xi^2}\right)d\eta\Big|^2\leqslant\sigma_2^2\varpi_1K_2^2\|z-w\|_{\hat{X}}.\tag{6.19}$$

using from the Itô isometry property and the assumption (H2) of theorem (5.3), we obtain

$$\begin{aligned}&E\Big|\frac{\sigma_3}{\Gamma(\theta)}\int_0^\tau(\tau-\eta)^{\theta-1}\left(\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}-\frac{\partial^2 w(\xi,\eta)}{\partial\xi^2}\right)dB(\eta)\Big|^2\\&\leqslant\frac{\sigma_3^2}{\Gamma^2(\theta)}\varpi_2E\Big|\int_0^\tau\left(\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}-\frac{\partial^2 w(\xi,\eta)}{\partial\xi^2}\right)dB(\eta)\Big|^2\\&\leqslant\frac{\sigma_3^2}{\Gamma^2(\theta)}\varpi_2\int_0^\tau E\Big|\frac{\partial^2 z(\xi,\eta)}{\partial\xi^2}-\frac{\partial^2 w(\xi,\eta)}{\partial\xi^2}\Big|^2d\eta\\&\leqslant\sigma_3^2\varpi_2K_2^2\|z-w\|_{\hat{X}},\end{aligned}\tag{6.20}$$

where ϖ_2 is a positive constant depended on θ. By substituting Eqs. (6.18), (6.19) and (6.20) in Eq. (6.17). we obtain

$$\|\mathcal{A}z(\xi,\tau)-\mathcal{A}w(\xi,\tau)\|^2 \leqslant 3\left(\sigma_1^2\varpi_1K_1^2+\sigma_2^2\varpi_1K_2^2+\sigma_3^2\varpi_2K_2^2\right)\|z-w\|_{\hat{X}} \tag{6.21}$$

Hence,

$$\begin{aligned}\|\mathcal{A}z(\xi,\tau)-\mathcal{A}w(\xi,\tau)\|_{\hat{X}} &= \sup_{(\xi,\tau)\in\Omega}\|\mathcal{A}z(\xi,\tau)-\mathcal{A}w(\xi,\tau)\|^2\\ &\leqslant 3\left(\sigma_1^2\varpi_1K_1^2+\sigma_2^2\varpi_1K_2^2+\sigma_3^2\varpi_2K_2^2\right)\|z-w\|_{\hat{X}}\end{aligned}$$

since $3\left(\sigma_1^2\varpi_1K_1^2+\sigma_2^2\varpi_1K_2^2+\sigma_3^2\varpi_2K_2^2\right) < 1$; therefore, the operator $\mathcal{A}$ is a contraction. In Banach contraction principle, $\mathcal{A}$ has a unique fixed point which proves the uniqueness of the stochastic equation (1.1) under the conditions (1.2) and (1.3).

7. Illustrative examples

Here, we present the computational outcomes derived from the application of the aforementioned method to a selection of problems

Example 7.1. Initially, We examine the subsequent time-fractional stochastic Advection-Diffusion type equation presented as follows [4]:

$$D_\tau^\theta z(\xi,\tau)+\sigma_1\frac{\partial z(\xi,\tau)}{\partial\xi}=\left(\sigma_2+\sigma_3\dot{B}(\tau)\right)\frac{\partial^2 z(\xi,\tau)}{\partial\xi^2}+h(\xi,\tau), \tag{7.1}$$

subjected to the following the initial and boundary conditions

$$z(\xi,0)=0,\quad 0\leqslant\xi\leqslant1, \tag{7.2}$$
$$z(0,\tau)=0,\quad z(1,\tau)=0,\quad 0\leqslant\tau\leqslant1, \tag{7.3}$$

here, $\sigma_1=\sigma_3=1$, $\sigma_2=\frac{1}{\pi^2}$ and

$$h(\xi,\tau)=\frac{2\tau^{2-\theta}\sin(\pi\xi)}{\Gamma(3-\theta)}+\left(\frac{1}{\pi^2}+\dot{B}(\tau)\right)\pi^2\tau^2\sin(\pi\xi)+\pi\tau^2\cos(\pi\xi)$$

which the exact solution of Eq. (7.1) is $z(\xi,\tau)=\tau^2\sin(\pi\xi)$.

The suggested approach given in section 4 is applied for Eq. (7.1). Numerical results are surveyed for the different case of θ and n. The absolute error of the proposed method for different quantities of θ with $n=8$ have been reported in Table 1. Table 2 provides the absolute errors of our scheme in the three values, $n=2,4,6$ with $\theta=0.25$. The results, as shown in Table 2, indicate that the by increasing n in the number of the Vieta-Lucas basis, the accuracy of the results have improved. In Table 3, the approximate solutions of Eq. (7.1) are compared with the obtained results in [4]. The curves of the absolute errors for different amount of θ at $n=8$ are depicted in Fig. 1. Graphical comparisons of our approximate solution with $n=4,\theta=0.75$ at $\tau=0.4$ and the exact solution are shown in Fig. 2. It is apparent of these figures that the suggested method has satisfactory efficiency and accuracy with small terms.

Table 1: Absolute errors of $z(\xi, \tau)$ for different values of θ with $n = 8$ for Ex 7.1.

(ξ_i, τ_i)	$\theta = 0.55$	$\theta = 0.75$	$\theta = 0.95$
(0.0,0.0)	2.64938×10^{-14}	5.86131×10^{-14}	5.31474×10^{-14}
(0.1,0.1)	1.4964×10^{-8}	1.08718×10^{-7}	4.63208×10^{-8}
(0.2,0.2)	8.83294×10^{-7}	1.32365×10^{-6}	2.32081×10^{-7}
(0.3,0.3)	4.25383×10^{-7}	2.54493×10^{-7}	2.29556×10^{-7}
(0.4,0.4)	1.78798×10^{-6}	4.41676×10^{-6}	1.99018×10^{-6}
(0.5,0.5)	8.13074×10^{-7}	2.73507×10^{-6}	1.57564×10^{-6}
(0.6,0.6)	8.78908×10^{-7}	1.48224×10^{-6}	3.23874×10^{-7}
(0.7,0.7)	2.17724×10^{-6}	2.84203×10^{-6}	4.74647×10^{-7}
(0.8,0.8)	3.58211×10^{-6}	5.04978×10^{-6}	1.87516×10^{-6}
(0.9,0.9)	1.10681×10^{-6}	2.03120×10^{-6}	2.18297×10^{-6}
(1.0,1.0)	3.21625×10^{-16}	6.66134×10^{-16}	4.44089×10^{-16}

Table 2: Absolute errors of $z(\xi, \tau)$ for various values of n with $\theta = 0.75$ for Ex 7.1.

(ξ_i, τ_i)	$n = 2$	$n = 4$	$n = 6$
(0.0,0.0)	3.79732×10^{-3}	7.0892×10^{-8}	2.88143×10^{-12}
(0.1,0.1)	4.59348×10^{-3}	4.31838×10^{-6}	7.45113×10^{-7}
(0.2,0.2)	2.3169×10^{-2}	7.6630×10^{-5}	2.12189×10^{-6}
(0.3,0.3)	3.86192×10^{-2}	1.27503×10^{-4}	1.45214×10^{-6}
(0.4,0.4)	4.27661×10^{-2}	1.05026×10^{-5}	3.67179×10^{-6}
(0.5,0.5)	3.0171×10^{-2}	2.33827×10^{-4}	2.58880×10^{-7}
(0.6,0.6)	1.44345×10^{-3}	1.61977×10^{-4}	1.71616×10^{-6}
(0.7,0.7)	4.78207×10^{-2}	3.98977×10^{-4}	4.09985×10^{-6}
(0.8,0.8)	9.22421×10^{-2}	9.25146×10^{-4}	1.97627×10^{-6}
(0.9,0.9)	9.79256×10^{-2}	4.64086×10^{-4}	1.00417×10^{-5}
(1.0,1.0)	6.66134×10^{-16}	1.33227×10^{-16}	4.44089×10^{-16}

Table 3: Comparison of $z(\xi, \tau)$ for $\theta = 0.75$ and $n = 8$ with method in [4] for Ex 7.1.

(ξ_i, τ_i)	Exact solution	Method in [4]	Present method
(0.0,0.0)	0.000000000	0.000000000	0.000000000
(0.1,0.1)	0.003090169	0.002911341	0.003090061
(0.2,0.2)	0.023511410	0.023332581	0.023510087
(0.3,0.3)	0.072811529	0.072632700	0.072811275
(0.4,0.4)	0.152169042	0.151990213	0.152164625
(0.5,0.5)	0.250000000	0.249821171	0.249997265
(0.6,0.6)	0.342380345	0.342201517	0.342378863
(0.7,0.7)	0.396418327	0.396239498	0.396415485
(0.8,0.8)	0.376182561	0.376003732	0.376177512
(0.9,0.9)	0.250303765	0.250124936	0.250301734
(1.0,1.0)	0.000000000	0.0000000000	0.000000000

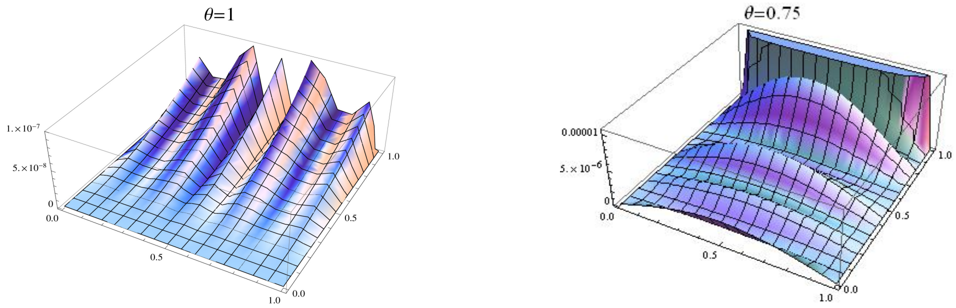

Figure 1: Absolute errors of $z(\xi, \tau)$ with $n = 8$ for Ex 7.1

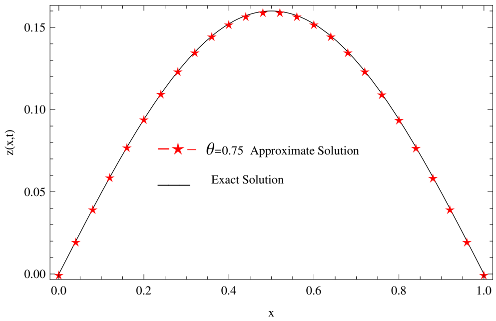

Figure 2: Exact and numerical solutions with $\theta = 0.75$ and $n = 4$ at $\tau = 0.4$ for Ex 7.1 .

Example 7.2. Suppose, we have another time-fractional stochastic Advection-Diffusion type equation as the following form [4]:

$$D_\tau^\theta z(\xi,\tau) + \sigma_1 \frac{\partial z(\xi,\tau)}{\partial \xi} = \left(\sigma_2 + \sigma_3 \dot{B}(\tau)\right) \frac{\partial^2 z(\xi,\tau)}{\partial \xi^2} + h(\xi,\tau), \tag{7.4}$$

with having the initial and boundary conditions

$$z(\xi,0) = e^{2\xi},\ 0 \leqslant \xi \leqslant 1, \tag{7.5}$$
$$z(0,\tau) = e^{-2\tau},\ z(1,\tau) = e^{2(1-\tau)},\ 0 \leqslant \tau \leqslant 1, \tag{7.6}$$

here, $\sigma_1 = 3$, $\sigma_2 = \sigma_3 = 1$ and

$$h(\xi,\tau) = -2e^{2\xi-2\tau}[1 + 2(1 + \dot{B}(\tau)) - 3].$$

The analytical solution for $\theta = 1$ is $z(\xi,\tau) = e^{2\xi-2\tau}$.

Here, we employed the given scheme for Eq. (7.4) at the various amount of θ and n. The absolute error of the suggested method for various amounts of n with $\theta = 1$ have been reported in Table 4. The results presented in Table 4 demonstrate that increasing the parameter n in the Vieta-Lucas basis leads to improved accuracy of the outcomes. A comparison between the exact and approximate solutions at time $\tau = 1$ for $\theta = 1$ and $n = 8$, as well as a method in [22], is displayed in Table 5. These findings indicate that our proposed approach is effective and yields high accuracy. Furthermore, Table 6 compares the absolute error values obtained using our method for $\theta = 1$ and $n = 8$ with those from methods introduced in [11, 22] and, underscoring the superior precision of our recommended approach. The graphical representations of the absolute errors for $\theta = 1$ with $n = 8$ at $\tau = 0.4$ are depicted in Fig. 3, while Fig. 4 illustrates comparable data for $\theta = 1$ and $n = 8$. Finally, Fig. 5 presents visual comparisons between our approximate solution with $n = 8$ at $\tau = 0.4$ and the exact solution.

Table 4: Absolute errors of results for various values of n with $\theta = 1$ for Ex 7.2.

(ξ_i, τ_i)	$n = 4$	$n = 6$	$n = 8$
(0.0,0.0)	1.28913×10^{-3}	7.53232×10^{-6}	2.57303×10^{-8}
(0.1,0.1)	9.88784×10^{-4}	9.36103×10^{-7}	1.68829×10^{-8}
(0.2,0.2)	1.42179×10^{-4}	1.15718×10^{-5}	3.32687×10^{-9}
(0.3,0.3)	4.52839×10^{-4}	1.00942×10^{-5}	2.24709×10^{-8}
(0.4,0.4)	3.8995×10^{-4}	7.09569×10^{-6}	4.01963×10^{-8}
(0.5,0.5)	2.09617×10^{-4}	4.9171×10^{-6}	2.43891×10^{-8}
(0.6,0.6)	3.21537×10^{-4}	1.20699×10^{-6}	7.16726×10^{-9}
(0.7,0.7)	5.52236×10^{-4}	2.55684×10^{-8}	4.04672×10^{-9}
(0.8,0.8)	2.95614×10^{-4}	3.57679×10^{-6}	8.81017×10^{-9}
(0.9,0.9)	4.57707×10^{-4}	9.40445×10^{-7}	2.32519×10^{-8}
(1.0,1.0)	1.36175×10^{-3}	8.18131×10^{-6}	2.71638×10^{-8}

Table 5: Comparison of results at $\tau = 1$ and $\theta = 1$ with method in [22] for Ex 7.2.

ξ_i	Exact solution	Method in [22]	Presented method
0.0	0.1353352	0.1353352	0.1353352
0.1	0.1652988	0.1653104	0.1652988
0.2	0.2018965	0.2019021	0.2018965
0.3	0.2465969	0.2465873	0.2465969
0.4	0.3011942	0.3011704	0.3011942
0.5	0.3678794	0.3678614	0.3678793
0.6	0.4493289	0.4493096	0.4493288
0.7	0.5488116	0.5487896	0.5488115
0.8	0.6703200	0.6702804	0.6703199
0.9	0.8187307	0.8187188	0.8187307
1.0	1.0000000	0.9999675	1.0000000

Table 6: Comparison of the absolute errors of Ex. 7.2 extracted by three methods for $\theta = 1$ and $n = 8$.

(ξ_i, τ_i)	Method in [11]	Method in [22]	Present method
$(0.2, 0.2)$	1.5377×10^{-5}	1.5182×10^{-6}	2.85555×10^{-8}
$(0.4, 0.4)$	3.7708×10^{-4}	8.7889×10^{-7}	5.47901×10^{-8}
$(0.6, 0.6)$	4.4558×10^{-4}	2.4369×10^{-6}	5.7085×10^{-8}
$(0.8, 0.8)$	3.0031×10^{-4}	2.3604×10^{-6}	5.79067×10^{-8}

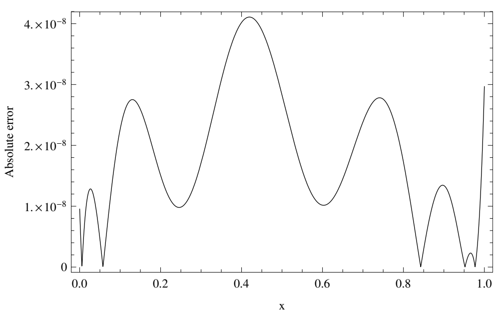

Figure 3: Absolute errors of $z(\xi, \tau)$ with $n = 8$ and $\theta = 1$ at $\tau = 0.4$ for Ex 7.2

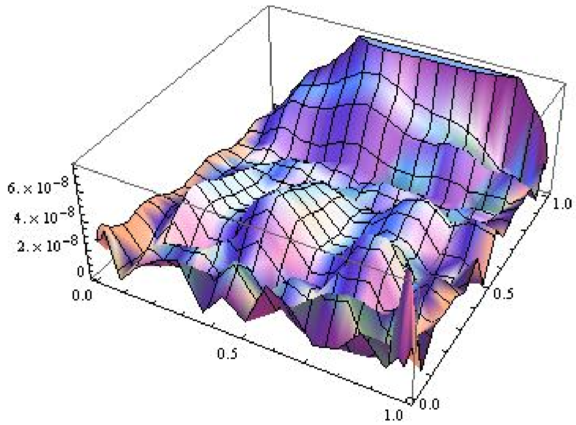

Figure 4: Absolute errors of $z(\xi,\tau)$ with $n=8$ and $\theta=1$ for Ex 7.2

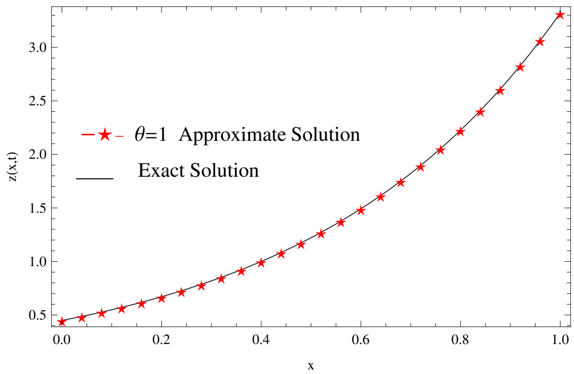

Figure 5: Exact and numerical solutions with $\theta=1$ and $n=8$ at $\tau=0.4$ for Ex 7.2

Example 7.3. Finaly, we take into account the following time-fractional stochastic Advection-Diffusion type equation as [22]

$$D_\tau^\theta z(\xi,\tau)+\sigma_1\frac{\partial z(\xi,\tau)}{\partial \xi}=\left(\sigma_2+\sigma_3\dot{B}(\tau)\right)\frac{\partial^2 z(\xi,\tau)}{\partial \xi^2}+h(\xi,\tau), \tag{7.7}$$

with having the initial and boundary conditions

$$z(\xi,0)=0, \quad 0\leqslant\xi\leqslant 1, \tag{7.8}$$

$$z(0,\tau)=\int_0^\tau B(s)ds, \qquad 0\leqslant\tau\leqslant 1, \tag{7.9}$$

$$z(1,\tau)=\tau^2+\int_0^\tau B(s)ds, \tag{7.10}$$

here, $\sigma_1=-1$, $\sigma_2=\sigma_3=1$ and

$$h(\xi,\tau)=\frac{2\xi\tau^{2-\theta}}{\Gamma(3-\theta)}+\frac{1}{\Gamma(1-\theta)}\int_0^\tau(\tau-s)^{-\theta}ds+\sigma_1\tau^2,$$

The exact solution is $z(\xi,\tau)=\xi\tau^2+\int_0^\tau B(s)ds$.

The computational answers to the issue at hand were determined by employing the technique suggested in this document. The resultant absolute errors for diverse values of θ with $n=6$ are tabulated in Table 7. Figure 6 illustrates the absolute error corresponding to $\theta=1$ and $n=6$ at $\tau=1$. Further, Figure 7 displays the absolute errors for assorted values of θ with $n = 6$. Additionally, Figure 8 juxtaposes the exact solution with the numerical estimation for $\theta = 0.25$ and $n = 6$ when $\tau = 0.4$.

Table 7: Absolute errors of $z(\xi, \tau)$ for different values of θ with $n = 6$ for Ex 7.3.

(ξ_i, τ_i)	$\theta = 0.55$	$\theta = 0.75$	$\theta = 0.95$
(0.0,0.0)	2.71721×10^{-10}	4.36082×10^{-11}	3.52945×10^{-10}
(0.1,0.1)	2.6729×10^{-6}	4.38497×10^{-6}	2.10631×10^{-6}
(0.2,0.2)	3.05545×10^{-6}	2.76435×10^{-5}	5.69975×10^{-6}
(0.3,0.3)	7.58557×10^{-5}	5.13495×10^{-5}	5.86225×10^{-6}
(0.4,0.4)	9.07661×10^{-5}	5.35281×10^{-5}	5.72575×10^{-6}
(0.5,0.5)	5.6178×10^{-5}	3.10229×10^{-5}	2.64885×10^{-6}
(0.6,0.6)	4.87282×10^{-6}	4.56696×10^{-6}	2.67329×10^{-6}
(0.7,0.7)	2.42829×10^{-5}	5.7808×10^{-6}	4.53066×10^{-7}
(0.8,0.8)	3.29032×10^{-5}	4.01169×10^{-6}	1.86241×10^{-5}
(0.9,0.9)	3.77882×10^{-5}	4.48311×10^{-6}	3.20634×10^{-5}
(1.0,1.0)	6.05418×10^{-15}	9.28597×10^{-15}	6.70248×10^{-15}

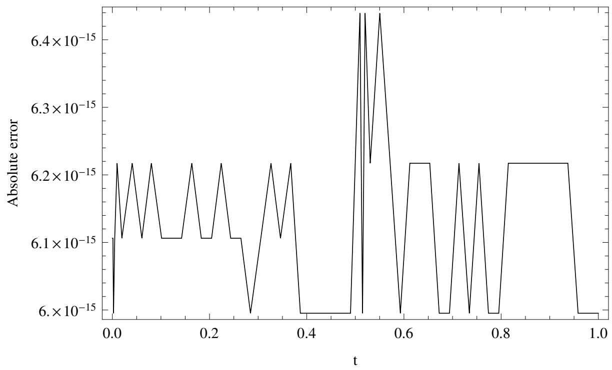

Figure 6: Absolute errors of $z(\xi, \tau)$ with $n = 6$ and $\theta = 1$ at $\tau = 1$ for Ex 7.3

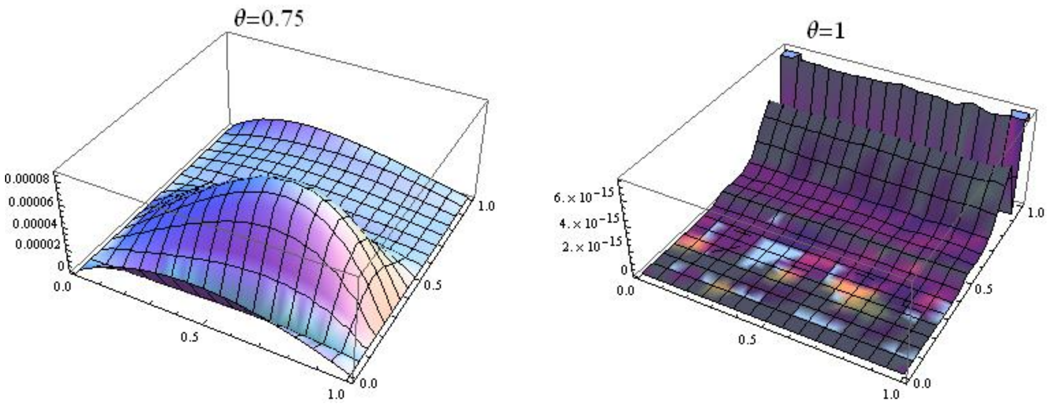

Figure 7: Absolute errors of $z(\xi, \tau)$ accompanied by the different values for θ and $n = 6$ for Ex 7.3

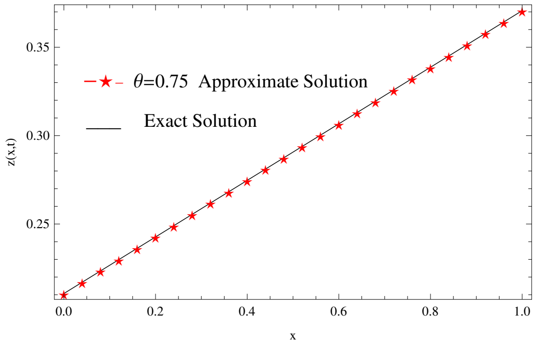

Figure 8: Exact and numerical solutions with $\theta = 0.25$ and $n = 6$ at $\tau = 0.4$ for Ex 7.3

8. Conclusion

This research examined the numerical solutions of Advection-Diffusion equations with a fractional time component. The Caputo derivative is utilized as the fractional derivative in this work. A numerical approach employing Vieta-Lucas polynomials is implemented to solve the aforementioned equations. The core concept of this technique involves approximating Brownian motion using the Gauss-Legendre quadrature, thus simplifying the computational process. The operational matrices of Vieta-Lucas polynomials facilitate the application of the suggested approach, transforming the stochastic model into a more tractable form. Additionally, the error estimation and convergence analysis of the proposed scheme are substantiated through theoretical proofs. The existence and uniqueness of the solutions are validated using the Banach fixed point theorem. The validity and efficacy of the suggested method have been demonstrated through numerical illustrations. The curves of the absolute errors for different amount of θ and n are depicted in Figs. 1, 3, 4, 6 and 7. Graphical comparisons of the approximate solutions and the exact solution for different values of θ, n and $\tau = 0.4$ are shown in Figs. 2, 5 and 8. It is apparent of these figures that the suggested method has satisfactory efficiency and accuracy with small terms. The numerical calculations indicate that the proposed technique is easy to implement and it can give accurate solutions with no complexity. Also, by increasing n in the number of the Vieta-Lucas basis, the accuracy of the results have improved.
Calculations in this paper were performed using *Mathematica*.

Author contributions

M. Zaboli and H. Tajadodi contributed equally to the conceptualization, methodology, formal analysis, investigation, Validation, software, Writing-original draft, Writing-review & editing of this manuscript. All authors have read and approved the final version of the manuscript.

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