Available online at http://jprm.sms.edu.pk/
Journal of Prime Research in Mathematics, 2026, 161–177

Analysis and Discrete Approximation of an Inverse Source Problem for a Time-Fractional Sobolev–Galpern Equation

Abdeldjalil Chattouh[a,*], Djamila Chergui[a], Maroua Nouar[a]

[a] *Department of Mathematics, Khenchela University, Khenchela, Algeria.*

Abstract

We address an inverse source problem for a time-fractional Sobolev–Galpern-type equation endowed with Neumann boundary conditions. The goal is to reconstruct an unknown time-dependent source term from integral observations of the state variable over the spatial domain. The analysis is carried out within a variational framework, where Rothe method is employed to establish the existence and uniqueness of weak solutions under suitable assumptions on the data. A time-discrete approximation scheme is then introduced for the simultaneous computation of the state variable and the unknown source term. Rigorous convergence of the approximations is derived through energy estimates and discrete Gronwall-type arguments.

Keywords: Sobolev equation, inverse source problem, Rothe's method, fractional derivative, uniqueness and existence
2010 MSC: 35R11, 35R30, 65M12.

1. Introduction

In the mathematical modelling, when the characteristics of the material or its surrounding environment are not always known a priori , it is possible that, in addition to the solution $u(x,t)$, one or more coefficients of the governing equation, and/or the external forcing terms, remain unspecified. In the mathematical framework, problems in which both the solution of a partial differential equation and one or more of its coefficients, or the source term, are unknown are referred to as inverse problems [6]. Typically, in such problems, alongside the boundary conditions associated with the corresponding direct problem, where all coefficients and the right-hand side are known, supplementary information is provided to account for the presence of the additional unknown functions. In this work, the unknown is the a source term which depends only on the time variable.

*Corresponding author
Email addresses: chattouh.abdeljalil@univ-khenchela.dz (Abdeldjalil Chattouh), chergui_djamila@univ-khenchela.dz (Djamila Chergui), nouar.maroua@univ-khenchela.dz (Maroua Nouar)

Received : 31 August 2025*; Accepted:* 29 March 2026*; Published Online:* 10 May 2026.

Fractional differential equations extend ordinary and partial differential equations by allowing the differentiation order to take arbitrary fractional values. In this context, a linear time-fractional parabolic equation refers to a parabolic-type PDE whose evolution is governed by master equations involving fractional derivatives in time. Thanks to their ability to model memory and hereditary effects, these equations have become central in both theoretical analysis and practical applications.Consequently, a wide range of analytical and semi-analytical techniques has been developed to investigate their qualitative behaviour. Among the most commonly employed approaches, we mention fixed-point methods [7], Laplace–based decomposition techniques [8, 9], and the variational iteration transform method [10, 11], among others.

In the current work, we aim to investigate a special class of time-fractional pseudo-parabolic equations. Let $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}$) be a bounded open set with Lipschitz boundary $\Gamma = \partial\Omega$. We denote by ν the unit outward normal vector on Γ. The final time is denoted by T, and we introduce the space–time cylinder $Q_T := \Omega \times (0, T]$ together with its lateral boundary $\Gamma \times (0, T]$. Let us consider the time-fractional Sobolev–Galpern-type problem

$$\partial_t^\alpha u - \partial_t^\alpha(\Delta u) - \Delta u = F(u) + p(t)f(x), \quad (x,t) \in Q_T, \tag{1.1}$$

supplemented with Neumann boundary condition and initial data

$$\begin{cases} \nabla u(x,t) \cdot \nu = b(x,t), & (x,t) \in \Gamma \times (0,T), \\ u(x,0) = u^0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

where $u^0 \in L^2(\Omega)$ is the initial data and $b \in W^{1,\infty}(0,T; L^2(\Gamma))$ is a boundary term. Here, ∂_t^α denotes the Caputo fractional derivative of order $\alpha \in (0,1)$, defined for $(x,t) \in Q_T$ by

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}\, \partial_t u(x,s)\, ds.$$

Equivalently, it can be expressed in convolution form as

$$\partial_t^\alpha u(x,t) = \partial_t(\omega^\alpha * (u - u_0))(x,t), \quad (x,t) \in Q_T,$$

where the kernel is given by $\omega^\alpha(t) = \dfrac{t^{-\alpha}}{\Gamma(1-\alpha)}$ for $t > 0$, and the symbol $*$ denotes the standard convolution product

$$(w * z)(t) = \int_0^t w(t-s)z(s)\, ds.$$

The kernel ω^α belongs to $L^1(0,T)$, is non-negative, singular at $t = 0$, and satisfies the monotonicity properties (see [19])

$$\partial_t \omega^\alpha(t) \le 0, \quad \partial_{tt}\omega^\alpha(t) \ge 0, \quad \text{for all } t > 0.$$

We interpret the term $\partial_t\big(\omega^\alpha * (\partial_t u - u_0)\big)$ as the Caputo fractional derivative, in accordance with the definition presented in [5, Definition 3.2]. We assume that $F : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous, i.e., there exists a constant $L > 0$ such that

$$|F(s) - F(r)| \le L\,|s - r|, \quad \forall s, r \in \mathbb{R}.$$

As a direct consequence, F satisfies for some constant $C > 0$. the following linear growth condition:

$$|F(s)| \le |F(0)| + L|s| \le C(1 + |s|), \quad \forall s \in \mathbb{R}, \tag{1.3}$$

Inverse problems occupy a central role in applied mathematics, providing essential tools for extracting information from indirect measurements in a wide range of real-world applications. They naturally arise in medical imaging, including computed tomography [17] and magnetic resonance imaging [31], in geophysical exploration [23], in non-destructive testing of materials [16], and in financial modelling [2]. In such contexts, direct access to the internal properties or dynamics of a system is often unavailable; instead, one must infer these characteristics from observable external effects.

The inverse problem addressed in this work involves the simultaneous reconstruction of the pair (u, p), where u denotes the state variable and $p(t)$ represents the temporally dependent source term in the problem (1.1)-(1.2), from supplementary measurement data:

$$\int_\Omega u(x,t)\,dx = \theta(t), \quad t \in [0,T]. \tag{1.4}$$

For the sake of simplifying subsequent computations, we shall consider the linear shift $F(u) - u$ instead of $F(u)$. For notational convenience, we keep writing $F(u)$. Note that F still satisfies the global Lipschitz continuity assumption.

The solution of the inverse problem (1.1)–(1.2)–(1.4) will be understood in the following sense .

Definition 1.1. Assume that

$$\left|\int_\Omega f(x)\,dx\right| \geq C_f > 0, \qquad u^0 \in H^1(\Omega), \qquad \theta \in C^1([0,T]), \qquad b \in W^{1,\infty}(0,T;L^2(\Gamma)).$$

A pair (u, p) with

$$u \in L^\infty(0,T;H^1(\Omega)), \qquad p \in L^2(0,T),$$

and such that

$$\partial_t\big(\omega^\alpha * (u - u_0)\big) \in L^2(0,T;H^1(\Omega)),$$

is called a weak solution of the inverse problem (1.1)–(1.2)–(1.4) if the following two conditions hold : For every $\varphi \in H^1(\Omega)$ and almost every $t \in (0,T)$,

$$\big\langle \partial_t\big(\omega^\alpha * (u - u_0)\big)(t), \varphi\big\rangle + \big\langle \partial_t\big(\omega^\alpha * \nabla(u - u_0)\big)(t), \nabla\varphi\big\rangle + \big(\nabla u(t), \nabla\varphi\big) + \big(u(t), \varphi\big)$$
$$= p(t)\big(f, \varphi\big) + \big(F(u(t)), \varphi\big) - \big(B(t), \varphi\big)_\Gamma. \tag{1.5}$$

For almost every $t \in (0,T)$,

$$p(t) = \frac{(\omega^\alpha * \theta')(t) + \theta(t) - \displaystyle\int_\Omega F(u(x,t))\,dx + \int_\Gamma B(x,t)\,dx}{\displaystyle\int_\Omega f(x)\,dx}, \tag{1.6}$$

with $B(x,t) = b(x,t) + \partial_t^\alpha b(x,t)$ for all $(x,t) \in \Gamma \times (0,T)$.

More precisely, this study focuses on an inverse source problem for a time-fractional Sobolev–Galpern-type equation. The principal objective is to determine both the solution $u(x,t)$ and the temporal source intensity $p(t)$ by leveraging the Neumann boundary conditions together with the integral measurement given in (1.4).

In applied mathematics, pseudo-parabolic equations including Sobolev-type equations, have emerged as powerful models for capturing processes that simultaneously display diffusive and wave-like behavior. They appear in diverse applications, such as two-phase flow in porous media with dynamic capillary pressure effects [26], energy transfer phenomena in isotropic materials [29], certain classes of wave propagation problems [4], and the rheological modeling of dilute polymeric solutions of Oldroyd-B type [25]. These examples highlight the versatility of pseudo-parabolic equations in bridging diffusion-dominated and wave-driven dynamics.

Due to their applicability across different scientific contexts, pseudo-parabolic equations are frequently investigated in the literature from many aspects (see e.g. [3, 14, 15]). This growing interest has not been limited to direct problems but has also extended to inverse problems for this class of equations, encompassing both linear and nonlinear models, with either classical or fractional derivative.

In the context of inverse problems, various methods have been proposed to tackle inverse problems, ranging from classical regularization strategies, such as Tikhonov or Lavrentiev methods [32, 36], to optimization-based approaches [21], and advanced numerical discretization schemes. Each of these techniques aims to overcome the intrinsic ill-posedness of inverse problems by ensuring stability of the reconstruction with respect to noisy data and by providing convergence guarantees for the approximations.

Among these approaches, Rothe method, also known as the time semi-discretization technique, has emerged as a powerful analytical and numerical tool for the study of both integer-order and fractional-order partial differential equations . The key idea lies in discretizing the temporal variable while keeping the spatial domain continuous, thereby reducing the original evolutionary system to a sequence of stationary elliptic subproblems that can be treated using variational methods. In recent years, Rothe method has also been successfully adapted to inverse problems, particularly those concerned with the identification of unknown coefficients or time-dependent sources in time-fractional parabolic-type equations.

For instance, Slodička in [27] analyzed a time-fractional diffusion model, where the inverse recovery of a time-dependent source was carried out via Rothe discretization. In a related direction, Ma and Sun [18] investigated an inverse potential problem for a semilinear generalized time-fractional diffusion equation with spatio-temporal dependent coefficients, deriving uniqueness and stability results within a variational framework and supporting the theory with numerical reconstructions. Most recently, Nouar et al. [20] applied Rothe's method to a fractional degenerate semi-linear parabolic equation, proving unique solvability of the inverse source problem and validating their analysis through numerical simulations.

A survey of the related literature reveals that most existing studies on inverse problems for fractional pseudo-parabolic equations have concentrated on the identification of parameters depending on the spatial variable. We briefly outline the following contributions, among others.

Soudani et al. [28] investigated the inverse problem of recovering a space-dependent source term in a Sobolev-type equation, establishing stability and local uniqueness results within an optimal control framework. In [24] the authors studied the inverse problem of identifying an unknown source in a time-fractional pseudo-parabolic equation with non-local in time conditions, in which they proposed a regularization approach based on the Modified Fractional Landweber method. An inverse problem for a linear third-order pseudo-parabolic equation has been studied in [12], where the existence, uniqueness, and stability results for a space-dependent source term have been established. Recently, Zhang and Zhang in [35] addressed an inverse problem for a fractional pseudo-parabolic equation, establishing conditional Hölder-type stability and proposing a novel iterative regularization method based on the fractional Landweber iteration and Fourier truncation for the numerical reconstruction.

It must be emphasized that most research on inverse problems for time-fractional pseudo-parabolic equations has primarily focused on the reconstruction of spatially dependent source terms. In contrast, studies addressing time-dependent source identification have been largely restricted to classical fractional diffusion models. To the best of the authors' knowledge, there are currently no results dealing with the identification of a time-dependent source term in a time-fractional pseudo-parabolic equation, even in the integer-order setting. Consequently, the present work aims to extend and improve existing results by establishing the unique solvability of such inverse problems and by developing a constructive numerical scheme that rigorously converges for this class of fractional models.

This contribution focuses on a specific class of inverse problems associated with time-fractional semilinear pseudo-parabolic equations, namely a time-fractional semilinear Sobolev-Galpern-type equation, distinguished by the presence of a mixed term that couples the Laplacian with a fractional derivative. The primary contribution of this paper is to establish the unique solvability of the considered inverse problem under reasonable regularity assumptions using the Rothe method. To this end, the inverse problem is first reformulated, incorporating the measurement data into an appropriate direct formulation as a decoupled two-forward problem. Following a time discretization based on the Rothe method, we demonstrate the existence of approximate solutions at each discrete time level and derive corresponding a priori estimates. These estimates are then employed to establish convergence results, which in turn enable the proof of existence and uniqueness of the solution to the inverse problem.

2. Time discretization and a priori estimates

To establish the existence of a weak solution, we adopt the classical Rothe method, which relies on a semi-discretization of the time variable. We begin by partitioning the interval $[0, T]$ into n equal subintervals of length $\tau = T/n$, and we denote the discrete time nodes by $t_i = i\tau$ for $i = 0, \ldots, n$. For a given function z we write $z_i \approx z(t_i)$ to indicate its time-discrete approximation at t_i. Within this framework, the temporal derivative is represented by the backward difference quotient :

$$\delta z_i := \frac{z_i - z_{i-1}}{\tau}, \qquad i = 1, \ldots, n.$$

We define the time discrete convolution as follows:

$$(\omega^\alpha * z)_i := \sum_{k=1}^{i} \omega^\alpha_{i+1-k}\, z_k\, \tau, \qquad (\omega^\alpha * z)_0 := 0.$$

Furthermore, we consider the identity known as

$$\delta(\omega^\alpha * z)_i = \omega^\alpha_i z_0 + (\omega^\alpha * \delta z)_i, \qquad i \geq 1. \tag{2.1}$$

The weak formulation (1.5)–(1.6) is approximated at each time level by the following semi-discrete problem. For $i = 1, \ldots, n$, find $u_i \in H^1(\Omega)$ and $p_i \in \mathbb{R}$ such that, for all $\varphi \in H^1(\Omega)$,

$$\big((\omega^\alpha * \delta u)_i, \varphi\big) + \big((\omega^\alpha * \nabla \delta u)_i, \nabla \varphi\big) + (\nabla u_i, \nabla \varphi) + (u_i, \varphi) = p_i(f, \varphi) + \big(F(u_{i-1}), \varphi\big) - (B_i, \varphi)_\Gamma. \tag{2.2}$$

Choosing $\varphi \equiv 1$ in (2.2) and using the above definition of θ_i, we obtain the explicit formula

$$p_i = \frac{(\omega^\alpha * \theta')_i + \theta_i - \displaystyle\int_\Omega F(u_{i-1})\, dx + \int_\Gamma B_i\, dx}{\displaystyle\int_\Omega f\, dx}, \qquad i = 1, \ldots, n. \tag{2.3}$$

with

$$\theta_i := \theta(t_i), \quad b_i := b(\cdot, t_i), \quad B_i = b_i + (\omega^\alpha * b')_i, \quad i = 0, \ldots, n.$$

Notice that, due to the explicit choice of $F(u_{i-1})$, the source term p_i is explicitly determined from known quantities at time level t_i and the previous state u_{i-1}, so that the problem at level t_i is linear in u_i Consequently, p_i can be evaluated first using (2.3), and then substituted into (2.2) to determine u_i.

Before proceeding further, we establish a solvability result for the discrete problem (2.2)–(2.3). Its proof is standard and follows directly from an application of the Lax–Milgram theorem.

Lemma 2.1. *Assume that $u^0 \in H^1(\Omega)$, $f \in L^2(\Omega)$ satisfies*

$$\Big| \int_\Omega f(x)\, dx \Big| \;\geq\; C_f > 0.$$

for a positive constant $C_f > 0$, F is globally Lipschitz, and $B_i \in L^2(\Gamma)$ for each $i = 1, \ldots, n$. Then, for any $i = 1, \ldots, n$, given u_{i-1}, there exists a unique pair $(u_i, p_i) \in H^1(\Omega) \times \mathbb{R}$ solving the semi-discrete system (2.2)–(2.3).

In what follows, we establish several auxiliary lemmas that will play a crucial role in deriving the a priori estimates.

Lemma 2.2. *Let $(z_i)_{i\in\mathbb{N}}$ be a sequence of $H^1(\Omega)$ and $(W_i)_{i\in\mathbb{N}}$ be sequence of real numbers. Assume that $(W_i)_{i\in\mathbb{N}}$ is non-negative, bounded, and decreasing. It then holds that*

$$2\sum_{i=1}^{j}\big(\delta(W*z)_i\,,z_i\big)_{H^1(\Omega)}\tau \;\geq\; \delta(W*\|z\|^2_{H^1(\Omega)})_i+\sum_{i=1}^{j}W_i\|z_i\|^2_{H^1(\Omega)}\tau, \qquad 1\leq i\leq j\leq n.$$

Proof. The proof of this result can be established by adopting the same approach as in the proof of [22, Lemma 3.2.]. □

Lemma 2.3. *Let $m\in C([0,T])$. Assume that $C_\alpha>0$ and $\tilde{\alpha}\in(\alpha,1)$ exist such that*

$$|m(t)|\leq C_\alpha\, t^{\tilde{\alpha}-1}, \qquad 0<t\leq T. \tag{2.4}$$

Then the discrete convolution satisfies the uniform bound

$$\big|(\omega^\alpha*m)_i\big|\leq C(\alpha,\tilde{\alpha},T), \qquad i=1,\dots,n, \tag{2.5}$$

where the constant $C(\alpha,\tilde{\alpha},T)>0$ is independent of τ.

Proof. For $s\in[t_{k-1},t_k]$ with $1\leq k\leq i$, we have

$$t_{i-k}=t_i-t_k\leq t_i-s\leq t_i-t_{k-1}=t_{i+1-k}.$$

Since the kernel $t\mapsto\omega^\alpha(t)$ is positive and decreasing on $(0,T]$, it follows that

$$\omega^\alpha_{i+1-k}\leq\omega^\alpha(t_i-s)\leq\omega^\alpha_{i-k}.$$

Integrating over $[t_{k-1},t_k]$ yields

$$\tau\,\omega^\alpha_{i+1-k}\leq\int_{t_{k-1}}^{t_k}\omega^\alpha(t_i-s)\,ds.$$

Invoking (2.4), we obtain

$$\begin{aligned}|(\omega^\alpha*m)_i|&=\left|\sum_{k=1}^{i}\int_{t_{k-1}}^{t_k}\omega^\alpha(t_i-s)\,m(s)\,ds\right|\\&\leq C_\alpha\sum_{k=1}^{i}\int_{t_{k-1}}^{t_k}\omega^\alpha(t_i-s)\,s^{\tilde{\alpha}-1}\,ds.\end{aligned}$$

As $\tilde{\alpha}-1<0$, the function $s\mapsto s^{\tilde{\alpha}-1}$ is decreasing, and hence $s^{\tilde{\alpha}-1}\leq t_{k-1}^{\tilde{\alpha}-1}$ for all $s\in[t_{k-1},t_k]$. Therefore,

$$|(\omega^\alpha*m)_i|\leq\frac{C_\alpha}{\Gamma(1-\alpha)}\int_0^{t_i}(t_i-s)^{-\alpha}s^{\tilde{\alpha}-1}\,ds.$$

The integral above is a Beta integral and can be evaluated explicitly:

$$\int_0^{t_i}(t_i-s)^{-\alpha}s^{\tilde{\alpha}-1}\,ds=t_i^{\tilde{\alpha}-\alpha}\frac{\Gamma(\tilde{\alpha})\Gamma(1-\alpha)}{\Gamma(\tilde{\alpha}+1-\alpha)}.$$

Consequently,

$$|(\omega^\alpha*m)_i|\leq\frac{C_\alpha\,\Gamma(\tilde{\alpha})}{\Gamma(\tilde{\alpha}+1-\alpha)}\,t_i^{\tilde{\alpha}-\alpha}\leq C\,T^{\tilde{\alpha}-\alpha}, \qquad i=1,\dots,n.$$

which is the desired uniform bound. □

We now establish a priori estimates, which play a crucial role in the arguments of the next section.

Lemma 2.4. *Let the assumptions of Lemma (2.1) hold. Furthermore, assume $\theta \in C^1((0,T])$ and satisfies*

$$|\theta(t)| \le Ct^{\tilde{\alpha}}, \quad |\theta'(t)| \le Ct^{\tilde{\alpha}-1}, \quad \forall t \in (0,T], \tag{2.6}$$

with $\tilde{\alpha} \in (\alpha, 1)$, and $b \in W^{1,\infty}(0,T; L^2(\Gamma))$. Then, there exists a positive constant $C > 0$ independent of n, such that

$$\max_{1\le i\le n} \left(\omega^\alpha * \|u\|^2\right)_i \;+\; \sum_{i=1}^{n} \omega_i^\alpha \|u_i\|^2\tau \;+\; \sum_{i=1}^{n} \|u_i\|^2_{H^1(\Omega)}\tau \;\le\; C. \tag{2.7}$$

Proof. Starting from the relation (2.3), we obtain

$$|p_i| \le \frac{|(\omega^\alpha * \theta')_i| + |\theta_i| + \int_\Omega |F(u_{i-1})|dx + \int_\Gamma |b_i|dx + \int_\Gamma |(\omega^\alpha * b')_i|dx}{\left|\int_\Omega f\,dx\right|}. \tag{2.8}$$

First, according to Lemma (2.3) we have that $(\omega^\alpha * \theta')_i$ is uniformly bounded under the assumption (2.6). Next, we use the linear growth of F, a constant $C_F > 0$ exists such that

$$|F(s)| \le C_F\,(1+|s|), \qquad \forall s \in \mathbb{R}.$$

Therefore, applying the Cauchy–Schwartz inequality yields the following result

$$\int_\Omega |F(u_{i-1})|\,dx \;\le\; C\int_\Omega \big(1+|u_{i-1}|\big)\,dx \le C\,(1+\|u_{i-1}\|)\,.$$

We also assumed that $b \in W^{1,\infty}(0,T; L^2(\Gamma))$. Hence, there exists $M > 0$ such that: $\|\partial_t b(t_i)\|_\Gamma + \|b(t_i)\|_\Gamma \le M$ for all $i = 0,\dots,n$. Moreover, the convolution kernel ω^α is positive and has a uniformly bounded discrete sum, that is

$$\omega_i^\alpha \ge 0, \qquad \text{and} \qquad \sum_{k=1}^{n} \omega_k^\alpha\,\tau \le C_\omega,$$

for some constant $C_\omega > 0$ independent of the time step τ. Consequently, we obtain the following uniform estimate

$$\|B_i\|_\Gamma \le \|b_i\|_\Gamma + \|(\omega^\alpha * b')_i\|_\Gamma \le M + M\sum_{k=1}^{i} \omega_{i+1-k}^\alpha\,\tau \le M(C_\omega+1), \quad i = 0,\dots,n.$$

Using the lower bound of $\int_\Omega f\,dx$, we deduce from (2.8) that

$$|p_i| \le C\,(1+\|u_{i-1}\|)\,, \quad i = 1,\dots,n. \tag{2.9}$$

Now, if we choose $\varphi = \tau u_i$ in (2.2) and sum over $i = 1,\dots,j$ with $1 \le j \le n$, we obtain

$$\tau\sum_{i=1}^{j} \big((\omega^\alpha * \delta u)_i, u_i\big) \;+\; \tau\sum_{i=1}^{j} \big((\omega^\alpha * \nabla\delta u)_i, \nabla u_i\big) \;+\; \tau\sum_{i=1}^{j} \|u_i\|^2_{H^1(\Omega)} = \tau\sum_{i=1}^{j} p_i\,(f,u_i) \;+\; \tau\sum_{i=1}^{j} \big(F(u_{i-1}), u_i\big) \;-\; \tau\sum_{i=1}^{j} (B_i,u_i)_{L^2(\Gamma)}. \tag{2.10}$$

Applying the discrete convolution identity (2.1) to the first two terms gives

$$\tau\sum_{i=1}^{j} \big((\omega^\alpha * \delta u)_i, u_i\big) + \tau\sum_{i=1}^{j} \big((\omega^\alpha * \nabla\delta u)_i, \nabla u_i\big) = \tau\sum_{i=1}^{j} \big(\delta(\omega^\alpha * u)_i, u_i)_{H^1(\Omega)} - \tau\sum_{i=1}^{j} \omega_i^\alpha (u_0,u_i\big)_{H^1(\Omega)}.$$

Therefore, by Lemma (2.2), it follows that

$$\sum_{i=1}^{j}\tau\left((\omega^{\alpha}*\delta u)_i,u_i\right)_{H^1(\Omega)}\geq\frac{1}{2}(\omega^{\alpha}*\|u\|^2_{H^1(\Omega)})_j+\frac{1}{2}\sum_{i=1}^{j}\tau\,\omega_i^{\alpha}\,\|u_i\|^2_{H^1(\Omega)}-\frac{1}{2}\sum_{i=1}^{j}\tau\,\omega_i^{\alpha}\,(u_0,u_i)_{H^1(\Omega)}.$$

Using the Cauchy–Schwartz and Young inequalities, we further deduce

$$\sum_{i=1}^{j}\tau\left((\omega^{\alpha}*\delta u)_i,u_i\right)_{H^1(\Omega)}\geq\frac{1}{2}(\omega^{\alpha}*\|u\|^2_{H^1(\Omega)})_j+\left(\frac{1}{2}-\varepsilon\right)\sum_{i=1}^{j}\tau\,\omega_i^{\alpha}\,\|u_i\|^2_{H^1(\Omega)}-C_\varepsilon\sum_{i=1}^{j}\tau\,\omega_i^{\alpha}\,\|u_0\|^2_{H^1(\Omega)}.$$

For the first term on the right-hand side of (2.10), using the bound (2.9) and Young's inequality, we obtain

$$\begin{aligned}\sum_{i=1}^{j}\tau|p_i(f,u_i)|&\leq\sum_{i=1}^{j}\tau\,|p_i|\,\|f\|\,\|u_i\|\\&\leq C\sum_{i=1}^{j}\tau\left(1+\|u_{i-1}\|\right)\|u_i\|\\&\leq C_\varepsilon+\varepsilon\sum_{i=1}^{j}\tau\,\|u_i\|^2_{H^1(\Omega)}+C_\varepsilon\sum_{i=1}^{j}\tau\,\|u_{i-1}\|^2_{H^1(\Omega)}.\end{aligned}$$

For the second term on the right-hand side of (2.10), using the Lipschitz growth of F, we obtain

$$\begin{aligned}\sum_{i=1}^{j}\tau\left(F(u_{i-1}),u_i\right)&\leq\sum_{i=1}^{j}\tau\,\|F(u_{i-1})\|\,\|u_i\|\\&\leq C\sum_{i=1}^{j}\tau\,(1+\|u_{i-1}\|)\,\|u_i\|\\&\leq C_\varepsilon+\varepsilon\sum_{i=1}^{j}\tau\,\|u_i\|^2+C_\varepsilon\sum_{i=1}^{j}\tau\,\|u_{i-1}\|^2.\\&\leq C_\varepsilon+\varepsilon\sum_{i=1}^{j}\tau\,\|u_i\|^2_{H^1(\Omega)}+C_\varepsilon\sum_{i=1}^{j}\tau\,\|u_{i-1}\|^2_{H^1(\Omega)}.\end{aligned}$$

For the boundary terms, we use the Cauchy–Schwartz and Young inequalities together trace theorem to obtain

$$\begin{aligned}\sum_{i=1}^{j}\tau|(B_i,u_i)_\Gamma|&\leq C_\varepsilon\sum_{i=1}^{j}\tau\|B_i\|^2_\Gamma+\varepsilon\sum_{i=1}^{j}\tau\|u_i\|^2_{H^1(\Omega)}\\&\leq C_\varepsilon+\varepsilon\sum_{i=1}^{j}\tau\,\|u_i\|^2_{H^1(\Omega)}.\end{aligned}$$

Collecting all the above estimates, we arrive at the discrete energy inequality

$$(\omega^{\alpha}*\|u\|^2_{H^1(\Omega)})_j+(1-\varepsilon)\sum_{i=1}^{j}\tau\,\omega_i^{\alpha}\,\|u_i\|^2_{H^1(\Omega)}+(1-\varepsilon)\sum_{i=1}^{j}\tau\,\|u_i\|^2_{H^1(\Omega)}\;\leq\;C_\varepsilon\left(1+\sum_{i=1}^{j}\tau\,\|u_{i-1}\|^2_{H^1(\Omega)}\right)$$

Choosing $\varepsilon>0$ sufficiently small, and applying the discrete Gronwall lemma, yields the uniform bound

$$\left(\omega^{\alpha}*\|u\|^2_{H^1(\Omega)}\right)_j+\sum_{i=1}^{j}\omega_i^{\alpha}\|u_i\|^2_{H^1(\Omega)}\tau+\sum_{i=1}^{j}\|u_i\|^2_{H^1(\Omega)}\tau\;\leq\;C. \tag{2.11}$$

Finally, by combining (2.11) with (2.9), we obtain the desired uniform boundedness result. □

Lemma 2.5. *Let the assumptions of Lemma 2.1 hold. Moreover, assume that* $u_0 \in H^1(\Omega), \theta \in C^1((0,T])$, *and* $B \in C^1([0,T], L^2(\Gamma))$ *satisfying*

$$|(\omega^\alpha * \theta')'(t)| \leq C(1 + t^{\tilde{\alpha}-\alpha-1}), \quad \forall t \in (0,T], \tag{2.12}$$

$$|B(t)| \leq C(1 + t^{\tilde{\alpha}}), \quad |B'(t)| \leq C(1 + t^{\tilde{\alpha}-1}), \quad \forall t \in (0,T]. \tag{2.13}$$

with $\tilde{\alpha} \in (\alpha, 1)$. *Then, there exists a constant* $C > 0$, *independent of* n *and* τ, *such that*

$$\max_{1\leq j\leq n} \|u_j\|^2_{H^1(\Omega)} + \sum_{i=1}^{n} \|u_i - u_{i-1}\|^2_{H^1(\Omega)} \ \leq \ C, \quad \max_{1\leq j\leq n} |p_j| \ \leq \ C, \quad \sum_{i=1}^{n} |\delta p_i|\tau \leq C. \tag{2.14}$$

Proof. Setting $\varphi = \delta u_i \tau$ in (2.2) and summing over $i = 1, \ldots, j$, we obtain the following discrete energy identity:

$$\sum_{i=1}^{j} \tau \big((\omega^\alpha * \delta u)_i, \delta u_i\big) + \sum_{i=1}^{j} \tau \big((\omega^\alpha * \nabla \delta u)_i, \nabla \delta u_i\big) + \sum_{i=1}^{j} (\nabla u_i, \nabla \delta u_i)\tau +$$
$$\sum_{i=1}^{j} (u_i, \delta u_i)\tau = \sum_{i=1}^{j} \tau p_i\big(f, \delta u_i\big) + \sum_{i=1}^{j} \tau\big(F(u_{i-1}), \delta u_i)\big) - \sum_{i=1}^{j} \tau\, (B_i, \delta u_i)_\Gamma. \tag{2.15}$$

The first two terms are non-negative due to the positive definiteness of the convolution quadrature induced by the kernel $\omega^\alpha(t)$. Using the elementary identity

$$2\sum_{i=1}^{j} a_i(a_i - a_{i-1}) = a_j^2 - a_0^2 + \sum_{i=1}^{j} (a_i - a_{i-1})^2, \quad \forall a_i \in \mathbb{R}, \ i = 0, \ldots, j,$$

with $a_i = \|u_i\|$ and $a_i = \|\nabla u_i\|$ separately, we obtain

$$\sum_{i=1}^{j} \tau\, (u_i, \delta u_i) + \sum_{i=1}^{j} \tau\, (\nabla u_i, \nabla \delta u_i) = \frac{1}{2}\big(\|u_j\|^2_{H^1(\Omega)} - \|u_0\|^2_{H^1(\Omega)}\big) + \frac{1}{2}\sum_{i=1}^{j} \|u_i - u_{i-1}\|^2_{H^1(\Omega)}. \tag{2.16}$$

We turn to seeking for an upper bound for the right hand side (2.15). For the first term, a summation by parts gives

$$\sum_{i=1}^{j} p_i(f, \delta u_i)\tau = p_j(f, u_j) - p_0(f, u_0) - \sum_{i=1}^{j} \delta p_i(f, u_{i-1})\tau, \tag{2.17}$$

Now, applying the δ-operation on (2.3) for $i \geq 2$ yields

$$\delta p_i = \frac{\delta(\omega^\alpha * \theta')_i + \displaystyle\int_\Gamma \delta B_i \, dx + \delta\theta_i - \int_\Omega \delta F(u_{i-1}) \, dx}{\displaystyle\int_\Omega f \, dx}, \quad i \geq 2.$$

The mean value theorem is applied to the time variable together with assumptions (2.6), (2.12) and (2.13), therefore, by taking into consideration, $\tau < 1, \tilde{\alpha} \in (\alpha, 1)$ it results for $i \geq 2$ that

$$\|\delta(\omega^\alpha * \theta')_i\| \leq C(1 + t_{i-1}^{\tilde{\alpha}-\alpha-1}) \leq C(1 + \tau^{-1}),$$
$$\|\delta B_i\|_{L^2(\Gamma)} \leq C(1 + t_{i-1}^{\tilde{\alpha}-\alpha-1}) \leq C(1 + \tau^{-1}),$$
$$|\delta\theta_i| \leq C(1 + t_{i-1}^{\tilde{\alpha}-1}) \leq C(1 + \tau^{-1}).$$

On the other hand, by using Cauchy–Schwartz and Young inequalities, together with the growth assumption (1.3), we obtain

$$\int_\Omega |\delta F(u_{i-1})|\,dx = \frac{1}{\tau}\int_\Omega |F(u_{i-1}) - F(u_{i-2})|\,dx \le C\|\delta u_{i-1}\|.$$

Therefore, we have

$$|\delta p_i| \le C\big(1 + t_{i-1}^{\tilde{\alpha}-\alpha-1} + t_{i-1}^{\tilde{\alpha}-\alpha-1} + \|\delta u_{i-1}\|\big) \le C\big(1 + \tau^{-1} + \|\delta u_{i-1}\|\big), \quad \forall i \ge 2. \tag{2.18}$$

Now, we treat the case $i = 1$, and for doing so, we assume the compatibility of the measurement equation (1.6) at $t = 0$, which allows to defining p_0 as follows

$$p_0 = \frac{(\omega^\alpha * \theta')(0) + \displaystyle\int_\Gamma B_0\,dx + \theta_0 - \int_\Omega F(u_0)\,dx}{\displaystyle\int_\Omega f\,dx}.$$

Consequently, taking the difference of the identity above with (2.3) for $i = 1$ gives

$$\delta p_1 = \frac{\delta(\omega^\alpha * \theta')_1 + \displaystyle\int_\Gamma \delta B_1\,dx + \delta\theta_1}{\displaystyle\int_\Omega f\,dx}.$$

Taking into consideration, $\tau < 1, \tilde{\alpha} \in (\alpha, 1)$ and $(\omega^\alpha * \theta')(0) = 0$, we obtain

$$\begin{aligned}
&|\delta(\omega^\alpha * \theta')_1| = |\omega^\alpha(\tau)\theta'(\tau)| \le C(\tau^{-1} + \tau^{\tilde{\alpha}-\alpha}) \le C\tau^{-1},\\
&\|\delta B_1\|_{L^2(\Gamma)} \le C\tau^{-1},\\
&|\delta\theta_1| \le C\tau^{-1}.
\end{aligned}$$

Hence, it follows that $|\delta p_1| \le C(1 + \tau^{-1})$. Keeping in mind that $\tau \le \tau_0$, we have

$$\begin{aligned}
\sum_{i=2}^{j-1} \delta p_i(f, u_{i-1})\tau &\le \varepsilon \sum_{i=2}^{j-1} \tau^2\,|\delta p_i|^2 + C_\varepsilon \sum_{i=2}^{j-1} \|u_{i-1}\|^2\\
&\le \varepsilon \sum_{i=2}^{j-1} \big(1 + \tau^2 + \tau^2\|\delta u_{i-1}\|^2\big) + C_\varepsilon \sum_{i=2}^{j-1} \|u_{i-1}\|^2\\
&\le C_\varepsilon + \varepsilon \sum_{i=1}^{j-1} \|u_i - u_{i-1}\|^2 + C_\varepsilon \sum_{i=1}^{j-2} \|u_i\|^2.
\end{aligned}$$

from which it follows

$$\begin{aligned}
\sum_{i=1}^{j-1} \delta p_i(f, u_{i-1})\tau &\le \delta p_1(f, u_0)\tau + \sum_{i=2}^{j-1} \delta p_i(f, u_{i-1})\tau\\
&\le \varepsilon \sum_{i=1}^{j-1} \|u_i - u_{i-1}\|^2 + C_\varepsilon \left(1 + \sum_{i=1}^{j-2} \|u_i\|^2\right).
\end{aligned}$$

On the other hand, using Cauchy–Schwartz and Young inequalities together with (2.9), we obtain

$$|p_j(f, u_j)| \le C_\varepsilon(1 + \|u_{j-1}\|^2) + \varepsilon\|u_j\|^2.$$

Combining all the findings above with (2.17), we find

$$\sum_{i=1}^{j} p_i(f, \delta u_i)\tau \le \varepsilon \|u_j\|^2 + \varepsilon \sum_{i=1}^{j-1} \|u_i - u_{i-1}\|^2 + C_\varepsilon \left(1 + \sum_{i=1}^{j-1} \|u_i\|^2\right).$$

Next, we estimate the nonlinear term using Cauchy–Schwartz and Young inequalities, together with the growth assumption (1.3) and the bound (2.7)

$$\begin{aligned}\sum_{i=1}^{j}(F(u_{i-1}), \delta u_i)\tau &= \sum_{i=1}^{j}(F(u_{i-1}), u_i)\tau - \sum_{i=1}^{j}(F(u_{i-1}), u_{i-1})\tau \\ &\le C\sum_{i=1}^{j}(1 + \|u_{i-1}\|)\|u_i\|\tau + C\sum_{i=1}^{j}(1 + \|u_{i-1}\|)\|u_{i-1}\|\tau \\ &\le C\left(1 + \sum_{i=1}^{j} \|u_i\|^2\tau + \sum_{i=1}^{j} \|u_{i-1}\|^2\tau\right) \le C\end{aligned} \tag{2.19}$$

Using an analogous arguments for estimating the boundary terms, one can easily obtain

$$\begin{aligned}\sum_{i=1}^{j}(B_i, \delta u_i)_\Gamma\tau &= \sum_{i=1}^{j}(B_i, u_i - u_{i-1})_\Gamma \\ &\le C_\varepsilon \sum_{i=1}^{j} \tau \|B_i\|_\Gamma^2 + \varepsilon \sum_{i=1}^{j} \tau \|u_i - u_{i-1}\|_\Gamma^2 \\ &\le C_\varepsilon + \varepsilon \sum_{i=1}^{j} \tau \|u_i - u_{i-1}\|_{H^1(\Omega)}^2.\end{aligned} \tag{2.20}$$

Combining all the above estimates, and absorbing the terms with ε into the left-hand side for ε sufficiently small, we obtain

$$\|u_j\|_{H^1(\Omega)}^2 + \sum_{i=1}^{j} \|u_i - u_{i-1}\|_{H^1(\Omega)}^2 \le C\Big(1 + \sum_{i=0}^{j-1} \tau\|u_i\|_{H^1(\Omega)}^2\Big).$$

Applying the discrete Gronwall lemma then yields the first uniform bound. This last one can be used with (2.9) to obtain $|p_i| \le C$ for all $i = 1, \dots, n$ which leads to the second estimate. Finally, the third estimate follows from (2.18) and (2.14). □

3. Existence and uniqueness

In this section, we aim to establish the existence and uniqueness of solutions for the considered inverse problem. To this end, we begin by introducing the Rothe functions, which allow the extension of the discrete problem defined at selected time points to a continuous formulation over the entire temporal domain

Let us introduce the following Rothe functions associated with the discrete solution $\{u_i\}_{i=0}^n$:

$$u_n : [0,T] \to L^2(\Omega), \quad u_n(t) := \begin{cases} u_0, & t = 0, \\ u_{i-1} + (t - t_{i-1})\,\delta u_i, & t \in (t_{i-1}, t_i],\ 1 \le i \le n, \end{cases}$$

$$\bar{u}_n : [0,T] \to L^2(\Omega), \quad \bar{u}_n(t) := \begin{cases} u_0, & t = 0, \\ u_i, & t \in (t_{i-1}, t_i],\ 1 \le i \le n, \end{cases}$$

$$\tilde{u}_n : [0,T] \to L^2(\Omega), \quad \tilde{u}_n(t) := \begin{cases} u_0, & t \in [0,\tau], \\ u_n(t-\tau), & t \in (t_{i-1}, t_i],\ 2 \le i \le n. \end{cases}$$

Similarly, we define $\overline{\varpi}_n^\alpha(t), \overline{\theta}_n(t), \overline{\theta}'_n(t)$ and $\overline{B}_n(t)$. Moreover, we define

$$(\omega^\alpha * u)_n : [0,T] \to L^2(\Omega), \quad (\omega^\alpha * u)_n(t) := \begin{cases} 0, & t = 0, \\ (\omega^\alpha * u)_{i-1} + (t - t_{i-1})\, \delta(\omega^\alpha * u)_i, & t \in (t_{i-1}, t_i],\ 1 \le i \le n, \end{cases}$$

Using the Rothe functions, the semi-discrete system (2.2)-(2.3) can be rewritten on the whole time interval $[0,T]$ as

$$\begin{aligned} \big\langle \partial_t(\omega^\alpha * u)_n(t) - \overline{\varpi}_n^\alpha u_0, \varphi \big\rangle_{H^1} + \big\langle \partial_t(\omega^\alpha * \nabla u)_n(t) - \overline{\varpi}_n^\alpha \nabla u_0, \nabla\varphi \big\rangle_{L^2} + (\nabla \overline{u}_n(t), \nabla\varphi) + \\ (\overline{u}_n(t), \varphi) = \overline{p}_n(t)\,(f,\varphi) + \big(F(\tilde{u}_n(t)), \varphi\big) - (\overline{B}_n(t), \varphi)_\Gamma, \end{aligned} \tag{3.1}$$

for all $\varphi \in H^1(\Omega)$, with the time-dependent source

$$\overline{p}_n(t) = \frac{(\overline{\varpi}^\alpha * \overline{\theta}')_n(t) + \overline{\theta}_n(t) - \displaystyle\int_\Omega F(\tilde{u}_n(t))\,dx - \int_\Gamma \overline{B}_n(t)\,dx}{\displaystyle\int_\Omega f\,dx}. \tag{3.2}$$

We are now in position to state the main results concerning the existence and uniqueness of weak solutions for the inverse problem (1.1)–(1.4) as defined in Definition 1.1.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary Γ, and let $T > 0$. Assume that $u_0 \in H^1(\Omega)$, $b \in W^{1,\infty}(0,T; L^2(\Gamma))$ is such that the induced B satisfies* (2.13)*, $f \in L^2(\Omega)$ with $\left|\int_\Omega f(x)\,dx\right| \ge C_f > 0$, $\theta \in C^1([0,T])$ satisfies* (2.12)*, and $F : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous. Then there exists a pair (u,p) with the regularity*

$$u \in L^\infty(0,T; H^1(\Omega)), \qquad p \in L^2(0,T),$$

and such that

$$\partial_t\big(\omega^\alpha * (u - u_0)\big) \in L^2(0,T; H^1(\Omega)),$$

which is a weak solution of the inverse problem (1.1)–(1.4) *in the sense of Definition 1.1.*

Proof. The estimate (2.14) implies that the sequence $(\overline{u}_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T; H^1(\Omega))$ and, thanks to the bound

$$\sum_{i=1}^n \|u_i - u_{i-1}\|^2 \le C,$$

it is also mean-square equicontinuous in $L^2(0,T; L^2(\Omega))$. By the Riesz–Fréchet–Kolmogorov compactness theorem (see e.g. [13, Theorem 2.13.1]), there exist a function $u^\dagger \in L^2(0,T; L^2(\Omega))$ and a subsequence $(\overline{u}_{n_l})_{l\in\mathbb{N}}$ such that

$$\overline{u}_{n_l} \longrightarrow u^\dagger \quad \text{strongly in } L^2(0,T; L^2(\Omega)). \tag{3.3}$$

Lemma 2.4 moreover yields that $(\overline{u}_{n_l})$ is bounded in the reflexive space $L^2(0,T; H^1(\Omega))$, hence there exists subsequence, still denoted by the same symbol, such that

$$\overline{u}_{n_l} \rightharpoonup u^\dagger \quad \text{weakly in } L^2(0,T; H^1(\Omega)). \tag{3.4}$$

Since $(\bar{u}_n)$ is bounded in $L^\infty(0,T; H^1(\Omega))$, there exist a subsequence (still denoted by $(\bar{u}_n)$) and a function

$$u^\dagger \in L^\infty(0,T; H^1(\Omega))$$

such that

$$\bar{u}_n \rightharpoonup^* u^\dagger \quad \text{in } L^\infty(0,T;H^1(\Omega)).$$

By uniqueness of the limit in $L^2(0,T;L^2(\Omega))$, the whole sequence converges to the same $u^\dagger$. Consequently, by weak-$*$ lower semicontinuity, we obtain

$$\|u^\dagger\|_{L^\infty(0,T;H^1(\Omega))} \le \liminf_{l\to\infty} \|\overline{u}_{n_l}\|_{L^\infty(0,T;H^1(\Omega))} \le C,$$

so that $u^\dagger \in L^\infty(0,T;H^1(\Omega))$. Furthermore, the piecewise delayed interpolant $\tilde{u}_{n_l}$ satisfies

$$\int_0^T \|\overline{u}_{n_l}(t) - \tilde{u}_{n_l}(t)\|^2\,dt \le \tau \sum_{i=1}^{n_l} \|u_i - u_{i-1}\|^2 \le C\tau \to 0 \quad (l\to\infty),$$

hence $\tilde{u}_{n_l}$ converges to the same limit $u^\dagger$ strongly in $L^2(0,T;L^2(\Omega))$.

Lemma 2.4 also provides a uniform bound for $(\overline{p}_n)$ in $L^2(0,T)$. Since $L^2(0,T)$ is reflexive, there exist $p^\dagger \in L^2(0,T)$ and a subsequence (again not relabelled) such that

$$\overline{p}_{n_l} \rightharpoonup p^\dagger \quad \text{weakly in } L^2(0,T).$$

Consequently, for every $\xi \in (0,T)$,

$$\int_0^\xi \overline{p}_{n_l}(t)\,dt \longrightarrow \int_0^\xi p^\dagger(t)\,dt \qquad (l\to\infty).$$

Under the assumptions on $\omega^\alpha(t)$, $\theta(t)$, and $b(x,t)$, the following convergences hold

$$\overline{\omega}^\alpha_{n_l} \to \omega^\alpha \text{ in } L^1(0,T), \qquad \overline{\theta}_{n_l} \to \theta, \quad \overline{\theta}'_{n_l} \to \theta' \text{ in } L^\infty(0,T), \qquad \overline{B}_{n_l} \to B \text{ in } L^\infty(0,T;L^2(\Gamma)).$$

Because F is globally Lipschitz, the strong convergence $\overline{u}_{n_l} \to u^\dagger$ in $L^2(0,T;L^2(\Omega))$ implies

$$F(\tilde{u}_{n_l}) \to F(u^\dagger) \quad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

We now integrate the discrete weak formulation (2.2) in time over $(0,\eta)$ for the convergent subsequence. Using the identity (2.1) and the definition of the Rothe functions, we obtain for any $\varphi \in H^1(\Omega)$

$$\begin{aligned}\big((\omega^\alpha * u)_{n_l}(\eta),\varphi\big) &- \int_0^\eta \big(\overline{\omega}^\alpha_{n_l}(t)u_0,\varphi\big)\,dt \\ &+ \big((\omega^\alpha * \nabla u)_{n_l}(\eta),\nabla\varphi\big) - \int_0^\eta \big(\overline{\omega}^\alpha_{n_l}(t)\nabla u_0,\nabla\varphi\big)\,dt \\ &+ \int_0^\eta \Big[\big(\nabla\overline{u}_{n_l}(t),\nabla\varphi\big) + \big(\overline{u}_{n_l}(t),\varphi\big)\Big]\,dt \\ &= \int_0^\eta \overline{p}_{n_l}(t)(f,\varphi)\,dt + \int_0^\eta \big(F(\tilde{u}_{n_l}(t)),\varphi\big)\,dt - \int_0^\eta \big(\overline{B}_{n_l}(t),\varphi\big)_\Gamma\,dt, \end{aligned} \tag{3.5}$$

To facilitate the passage to the limit, we integrate once more over $(0,\xi)$ with $0<\xi<T$. This yields

$$\begin{aligned}\int_0^\xi \big((\omega^\alpha * u)_{n_l}(\eta),\varphi\big)\,d\eta &- \int_0^\xi\int_0^\eta \big(\overline{\omega}^\alpha_{n_l}(t)u_0,\varphi\big)\,dt\,d\eta \\ &+ \int_0^\xi \big((\omega^\alpha * \nabla u)_{n_l}(\eta),\nabla\varphi\big)\,d\eta - \int_0^\xi\int_0^\eta \big(\overline{\omega}^\alpha_{n_l}(t)\nabla u_0,\nabla\varphi\big)\,dt\,d\eta \\ &+ \int_0^\xi\int_0^\eta \Big[\big(\nabla\overline{u}_{n_l}(t),\nabla\varphi\big) + \big(\overline{u}_{n_l}(t),\varphi\big)\Big]\,dt\,d\eta \\ &= \int_0^\xi\int_0^\eta \overline{p}_{n_l}(t)(f,\varphi)\,dt\,d\eta + \int_0^\xi\int_0^\eta \big(F(\tilde{u}_{n_l}(t)),\varphi\big)\,dt\,d\eta - \int_0^\xi\int_0^\eta \big(\overline{B}_{n_l}(t),\varphi\big)_\Gamma\,dt\,d\eta. \end{aligned} \tag{3.6}$$

Thanks to the convergence properties listed above and the argument of [30, Eq. (13)], the convolution terms pass to the limit in the sense that

$$\lim_{l\to\infty}\int_0^T \big((\omega^\alpha * u)_{n_l}(t),\varphi\big)\,dt = \int_0^T \big((\omega^\alpha * u^\dagger)(t),\varphi\big)\,dt,$$

and similarly for the gradient convolution. Letting $l\to\infty$ in (3.6) we therefore obtain

$$\begin{aligned}
&\int_0^\xi \big((\omega^\alpha * u^\dagger)(\eta),\varphi\big)\,d\eta - \int_0^\xi\int_0^\eta \big(\omega^\alpha(t)u_0,\varphi\big)\,dt\,d\eta \\
&\qquad + \int_0^\xi \big((\omega^\alpha * \nabla u^\dagger)(\eta),\nabla\varphi\big)\,d\eta - \int_0^\xi\int_0^\eta \big(\omega^\alpha(t)\nabla u_0,\nabla\varphi\big)\,dt\,d\eta \\
&\qquad + \int_0^\xi\int_0^\eta \Big[\big(\nabla u^\dagger(t),\nabla\varphi\big) + \big(u^\dagger(t),\varphi\big)\Big]\,dt\,d\eta \\
&\quad = \int_0^\xi\int_0^\eta p^\dagger(t)(f,\varphi)\,dt\,d\eta + \int_0^\xi\int_0^\eta \big(F(u^\dagger(t)),\varphi\big)\,dt\,d\eta - \int_0^\xi\int_0^\eta \big(B(t),\varphi\big)_\Gamma\,dt\,d\eta. \qquad (3.7)
\end{aligned}$$

Differentiating (3.7) with respect to ξ gives, for almost every $\xi\in(0,T)$,

$$\begin{aligned}
&\big((\omega^\alpha * u^\dagger)(\xi),\varphi\big) - \int_0^\xi \big(\omega^\alpha(t)u_0,\varphi\big)\,dt \\
&\qquad + \big((\omega^\alpha * \nabla u^\dagger)(\xi),\nabla\varphi\big) - \int_0^\xi \big(\omega^\alpha(t)\nabla u_0,\nabla\varphi\big)\,dt \\
&\qquad + \int_0^\xi \Big[\big(\nabla u^\dagger(t),\nabla\varphi\big) + \big(u^\dagger(t),\varphi\big)\Big]\,dt \\
&\quad = \int_0^\xi p^\dagger(t)(f,\varphi)\,dt + \int_0^\xi \big(F(u^\dagger(t)),\varphi\big)\,dt - \int_0^\xi \big(B(t),\varphi\big)_\Gamma\,dt. \qquad (3.8)
\end{aligned}$$

Since $u^\dagger\in L^\infty(0,T;H^1(\Omega))$, letting $\xi\to 0^+$ in (3.8) shows that

$$(\omega^\alpha * u^\dagger)(0)=0, \qquad (\omega^\alpha * \nabla u^\dagger)(0)=0$$

in the sense of $H^1(\Omega)^*$. Differentiating (3.8) once more with respect to ξ (and replacing ξ by t) yields the weak formulation

$$\begin{aligned}
&\big(\partial_t(\omega^\alpha * (u^\dagger - u_0))(t),\varphi\big) + \big(\partial_t(\omega^\alpha * \nabla(u^\dagger-u_0))(t),\nabla\varphi\big) \\
&\qquad + \big(\nabla u^\dagger(t),\nabla\varphi\big) + \big(u^\dagger(t),\varphi\big) \\
&\quad = p^\dagger(t)(f,\varphi) + \big(F(u^\dagger(t)),\varphi\big) - \big(B(t),\varphi\big)_\Gamma, \qquad \text{a.e. } t\in(0,T). \qquad (3.9)
\end{aligned}$$

Concerning the regularity of the time derivative, let us denote $U=\omega^\alpha * (u^\dagger - u_0)$. The weak formulation (3.9) can be rewritten in the abstract form

$$\partial_t(\mathcal{A}U) + \mathcal{A}u^\dagger = g,$$

where $\mathcal{A}: H^1(\Omega)\to H^1(\Omega)^*$ is the standard isomorphism defined by

$$\langle \mathcal{A}v,\varphi\rangle = (\nabla v,\nabla\varphi) + (v,\varphi).$$

Since $p^\dagger\in L^2(0,T)$ and the boundary/source terms are sufficiently regular, the right-hand side g belongs to $L^2(0,T;H^1(\Omega)^*)$. Furthermore, since

$$u^\dagger\in L^\infty(0,T;H^1(\Omega))\subset L^2(0,T;H^1(\Omega)),$$

it follows that $\mathcal{A}u^\dagger \in L^2(0,T;H^1(\Omega)^*)$. Consequently, the distributional time derivative satisfies

$$\partial_t(\mathcal{A}U) = g - \mathcal{A}u^\dagger \in L^2(0,T;H^1(\Omega)^*).$$

Using the isomorphism property of $\mathcal{A}$, we deduce that

$$\partial_t\big(\omega^\alpha * (u^\dagger - u_0)\big) \in L^2(0,T;H^1(\Omega)).$$

This matches the exact regularity required in Definition 1.1. Now, we pass to show that $p^\dagger$ satisfies (1.6) . Integrating the discrete measurement equation (2.3) in time over $(0,\xi)$ for the subsequence $(\overline{p}_{n_l})$ yields

$$\int_0^\xi \overline{p}_{n_l}(t)\,dt = \frac{1}{\displaystyle\int_\Omega f\,dx}\left(\int_0^\xi \big((\overline{\omega}^\alpha * \overline{\theta}')_{n_l}(t)\big)\,dt + \int_0^\xi \overline{\theta}_{n_l}(t)\,dt - \int_0^\xi\int_\Omega F\big(\tilde{u}_{n_l}(t)\big)\,dx\,dt + \int_0^\xi\int_\Gamma \overline{B}_{n_l}(t)\,d\Gamma\,dt\right). \tag{3.10}$$

Passing to the limit $l \to \infty$ and using the convergence results established previously, we deduce that for every $\xi \in (0,T)$,

$$\int_0^\xi p^\dagger(t)\,dt = \frac{1}{\int_\Omega f\,dx}\left(\int_0^\xi (\omega^\alpha * \theta')(t)\,dt + \int_0^\xi \theta(t)\,dt - \int_0^\xi\int_\Omega F\big(u^\dagger(t)\big)\,dx\,dt + \int_0^\xi\int_\Gamma B(t)\,d\Gamma\,dt\right).$$

Both sides of this identity are absolutely continuous in ξ. Differentiating with respect to ξ and setting $\xi = t$ gives, for almost every $t \in (0,T)$,

$$p^\dagger(t) = \frac{1}{\int_\Omega f\,dx}\left((\omega^\alpha * \theta')(t) + \theta(t) - \int_\Omega F\big(u^\dagger(t)\big)\,dx + \int_\Gamma B(t)\,d\Gamma\right).$$

Thus, the pair $(u^\dagger, p^\dagger)$ satisfies both the weak equation (3.9) and the measurement equation; i.e., it is a weak solution of the inverse problem in the sense of Definition 1.1.

□

Theorem 3.2. *Under the assumptions of Theorem 3.1, the weak solution (u,p) of the inverse problem (1.1)–(1.4) is unique. That is, if (u_1,p_1) and (u_2,p_2) are two weak solutions in the sense of Definition 1.1, then*

$$u_1 = u_2 \quad \text{in } C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \quad p_1 = p_2 \quad \text{in } L^2(0,T).$$

Proof. Starting from the weak formulations corresponding to the solution pairs (u_1,p_1) and (u_2,p_2), we consider their difference. Denoting $u := u_1 - u_2$ and $p := p_1 - p_2$, it follows that (u,p) satisfies

$$\big(\partial_t(\omega^\alpha * u)(t),\varphi\big)_{H^1(\Omega)} + (\nabla u(t),\nabla\varphi) + (u(t),\varphi) = p(t)(f,\varphi) + \big(F(u_1(t)) - F(u_2(t)),\varphi\big), \tag{3.11}$$

for all $\varphi \in H^1(\Omega)$, with homogeneous initial and boundary conditions. The corresponding measured problem reads

$$p(t) = -\frac{\displaystyle\int_\Omega \big(F(u_1(t)) - F(u_2(t))\big)\,dx}{\displaystyle\int_\Omega f\,dx}, \qquad t \in (0,T). \tag{3.12}$$

Testing (3.11) with $\varphi = u(t)$ and integrating over $(0,\xi) \subset (0,T)$, we obtain

$$\big(\partial_t^\alpha u(t),u(t)\big)_{H^1(\Omega)} + \|u(t)\|^2_{H^1(\Omega)} = p(t)(f,u(t)) + \big(F(u_1) - F(u_2),u(t)\big).$$

Invoking the Alikhanov inequality (cf. [1]), the first term on the left-hand side satisfies the lower bound

$$\big(\partial_t^\alpha u(t),u(t)\big)_{H^1(\Omega)} \geq \frac{1}{2}\,\partial_t^\alpha \|u(t)\|^2_{H^1(\Omega)}.$$

Using the Cauchy–Schwartz inequality, the Lipschitz continuity of F, and the non-degeneracy of f, estimate (3.12) yields

$$|p(t)(f, u(t))| \le C\|u(t)\|^2_{H^1(\Omega)},$$
$$(F(u_1) - F(u_2)\, u(t)) \le C_F\|u(t)\|^2_{H^1(\Omega)}.$$

Collecting the above estimates, we arrive at

$$\frac{1}{2}\,\partial_t^\alpha\|u(t)\|^2_{H^1(\Omega)} + \|u(t)\|^2_{H^1(\Omega)} \le C\,\|u(t)\|^2_{H^1(\Omega)}.$$

Applying the fractional integral operator I_t^α to both sides

$$\|u(t)\|^2_{H^1(\Omega)} \le C\, I_t^\alpha \left(\|u(\cdot)\|^2_{H^1(\Omega)}\right)(t).$$

as $u(0) = 0$, By the fractional Grönwall lemma (cf. [33]) it follows that $\|u(t)\|^2_{H^1(\Omega)} = 0$ for all $t \in [0, T]$; hence $u \equiv 0$. Substituting this into (3.12) gives $p(t) = 0$, $p(t) \equiv 0$ almost everywhere $t \in (0, T)$. which completes the proof of uniqueness. □

References

[1] A.A. Alikhanov, *A new difference scheme for the time fractional diffusion equation*, J. Comput. Phys. **280** (2015), 424–438. 3

[2] J. Baumeister, *Inverse problems in finance*, in: Recent developments in computational finance: Foundations, algorithms and applications, World Scientific, 2013, 81–157. 1

[3] A. Chattouh, K. Saoudi and M. Nouar, *Rothe–Legendre pseudospectral method for a semilinear pseudoparabolic equation with nonclassical boundary condition*, Nonlinear Anal. Model. Control **27** (2022), 38–53. 1

[4] B.D. Coleman and M.E. Gurtin, *Thermodynamics with internal state variables*, J. Chem. Phys. **47** (1967), 597–613. 1

[5] K. Diethelm, *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type*, Springer, Berlin, 2010. 1

[6] V. Isakov, *Inverse problems for partial differential equations*, Springer, 2006. 1

[7] H. Jafari, H. K. Jassim and D. Baleanu, *On the Existence and Uniqueness of Solutions for Local differential equations*, Entropy, **18** (2016), 1–9. 1

[8] H. Jafari, M.Y. Zayir and H. K. Jassim, *Analysis of fractional Navier-Stokes equations*, Heat Transf. **52** (2023), 2859-2877. 1

[9] H. Jafari, H. K. Jassim, C. Ünlü and V.T. Nguyen, *Laplace Decomposition Method for Solving the Two-Dimensional Diffusion Problem in Fractal Heat Transfer*, Fractals, **32** (2024), 1–6. 1

[10] H. Jafari, H. K. Jassim, A. Ansari and V. T. Nguyen, *Local Fractional Variational Iteration Transform Method: A Tool For Solving Local Fractional Partial Differential Equations*, Fractals, **32** (2024), 1–8. 1

[11] H. Jafari, H. K. Jassim and J. Vahidi, *Reduced Differential Transform and Variational Iteration Methods for 3D Diffusion Model in Fractal Heat Transfer within Local Fractional Operators*, Therm. Sci. **22** (2018), S301-S307. 1

[12] K. Khompysh, M. J. Huntul, M. K. Shazyndayeva, and M. K. Iqbal, *An inverse problem for pseudoparabolic equation: Existence, uniqueness, stability, and numerical analysis*, Quaestiones Mathematicae, **47** (2024), 1979–2001. 1

[13] A. Kufner, O. John and S. Fucík, *Function Spaces*, Monographs and Textbooks on Mechanics of Solids and Fluids, Noordhoff International Publishing, Leyden, 1977. 3

[14] W. Lian, J. Wang and R. Xu, *Global existence and blow-up of solutions for pseudo-parabolic equations with singular potential*, J. Differential Equations **269** (2020), 4914–4959. 1

[15] N.H. Luc, H. Jafari, P. Kumam and N.H. Tuan, *On an initial value problem for time fractional pseudo-parabolic equation with Caputo derivative*, Math. Methods Appl. Sci. **47** (2024), 11245–11267. 1

[16] M. Lu, *Forward and inverse analysis for non-destructive testing based on electromagnetic computation methods*, The University of Manchester, 2018. 1

[17] F. Natterer, *The mathematics of computerized tomography*, SIAM, Philadelphia, 2001. 1

[18] W. Ma and L. Sun, *Inverse potential problem for a semilinear generalized fractional diffusion equation with spatio-temporal dependent coefficients*, Inverse Problems **39** (2022), 015005. 1

[19] J.A. Nohel and D.F. Shea, *Frequency domain methods for Volterra equations*, Adv. Math. **22** (1976), 278–304. 1

[20] M. Nouar, A. Chattouh, O. Mossa Alsalhi and H. Ould Sidi, *Inverse problem of identifying a time-dependent source term in a fractional degenerate semi-linear parabolic equation*, Mathematics **13** (2025), 1486. 1

[21] M. Nouar and A. Chattouh, *On recovering space-dependent source term in a degenerate nonlocal parabolic equation*, J. Math. Modeling **13** (2025), 645–662. 1

[22] M. Nouar, M. Z. Youssef, H. Ould Sidi, A. Chattouh, *Recovering of dual time-varying source terms in a system of coupled time-fractional diffusion equations*, AIMS Math. , **10** (10): 29285-29318. doi: 10.3934/math.20251287 2

[23] W. Menke, *Geophysical data analysis: Discrete inverse theory*, 3rd ed., Academic Press, 2012. 1

[24] N. D. Phuong, L. D. Long, A. T. Nguyen and D. Baleanu, *Regularization of the inverse problem for time fractional pseudo-parabolic equation with non-local in time conditions*, Acta Mathematica Sinica, English Series, **38** (2022), 2199–2219. 1

[25] M. Renardy, *Mathematical analysis of viscoelastic flows*, SIAM Rev. **33** (1991), 1–68. 1

[26] R.E. Showalter, *Singular quasi-linear evolution equations*, SIAM J. Math. Anal. **1** (1970), 69–93. 1

[27] M. Slodička, *On an inverse source problem for semilinear time-fractional diffusion equations via Rothe's method*, J. Inverse Ill-Posed Probl. **24** (2016), 687–704. 1

[28] A. Soudani, K. Saoudi, A. Chattouh and A. Menasri, *Identifying the source term in a Sobolev-type equation by optimization method*, J. Math. Anal. **14** (2023), 1–17. 1

[29] S.L. Sobolev, *Applications of functional analysis in mathematical physics*, American Mathematical Society, Providence, 1964. 1

[30] K. Van Bockstal, *Existence and uniqueness of a weak solution to a non-autonomous time-fractional diffusion equation (of distributed order)*, Appl. Math. Lett. **109** (2020), 106540. 3

[31] C.R. Vogel and M.E. Oman, *Iterative methods for total variation denoising in magnetic resonance imaging*, SIAM J. Sci. Comput. **17** (1996), 227–238. 1

[32] F. Yang, C.-L. Fu and X.-X. Li, *The inverse source problem for time-fractional diffusion equation: stability analysis and regularization*, Inverse Problems Sci. Eng. **23** (2015), 969–996. 1

[33] H. Ye, J. Gao and Y. Ding, *A generalized Grönwall inequality and its application to a fractional differential equation*, J. Math. Anal. Appl. **328** (2007), 1075–1081. 3

[34] Y. Luchko, *Operational calculus for the general fractional derivative and its applications*, Fract. Calc. Appl. Anal. **24** (2021), 338–375.

[35] H. Zhang and P. Zhang, *An iteration regularization method for the inverse initial value problem of fractional pseudo-parabolic equation*, Int. J. Comput. Math. **102** (2025), 1–26. 1

[36] G.H. Zheng and T. Wei, *A new regularization method for solving a time-fractional inverse diffusion problem*, J. Math. Anal. Appl. **378** (2011), 418–431. 1