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SA-Extending and WS-Rationally Extending Modules: Structure and Duality

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Abstract

We present a comprehensive study of several generalized module classes, including SA-extending, ⊕-s-extending, SA-supplement, and WS-rationally extending modules. Detailed definitions, fundamental properties, illustrative examples, and structural results are provided to clarify the relationships among these classes. Dual notions, such as SA-lifting, ⊕-s-lifting, and SA-co-supplement modules, are introduced to emphasize the interplay between extending and lifting properties. Hybrid classes, including semi-rationally extending modules, are also considered, with an analysis of their closure properties and behavior over artinian, semiperfect, and group rings. Several open problems and conjectures are formulated, highlighting directions for future research in module theory and its applications.

Keywords: SA-extending modules; ⊕-s-extending modules; WS- rationally extending modules; SA-supplement modules; Lifting modules.

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1. Introduction and Preliminaries

The study of extending and lifting modules has been a central topic in module theory due to their structural importance and numerous applications in ring theory (see [4, 6, 9]). Classical extending modules, where every submodule is essential in a direct summand, have been generalized in several directions to capture finer structural properties [3, 7, 8, 10].

A submodule N of a module M is said to be *essential* in M, denoted $N \leq_e M$, if for every nonzero submodule $L \leq M$, we have $N \cap L \neq 0$ [4]. A submodule $L \leq M$ is called *small* in M, denoted $L \ll M$, if $L + K = M$ implies $K = M$ for any submodule $K \leq M$ [6]. Furthermore, a submodule $D \leq M$ is *fully invariant* (or *stable*) if for every endomorphism $f : M \to M$, we have $f(D) \subseteq D$ [10].

Recently, attention has focused on several generalizations of classical extending modules. SA-extending modules, introduced by Mutlu and Taştan [10], require that every submodule is essential in a stable (fully

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invariant) direct summand. This refinement provides better control over submodule embeddings and homomorphisms. The class of $\oplus$-s-extending modules, studied by Al-Saadi and Al-Rubaye [3, 2], generalizes SA-extending modules by relaxing the essentiality condition: each submodule has a supplement whose intersection with it is small, facilitating decompositions that are "almost direct." SA-supplement submodules, investigated by Durgun [7], consist of submodules that admit supplements with semiartinian intersections, capturing layered structures of simple modules [9]. Finally, WS-rationally extending modules, introduced by Fadel and Nayef [1, 8, 11], further generalize SA-extending modules by requiring intersections with supplements to be rationally small, weakening the essentiality requirement while preserving key structural properties.

These notions are not only interesting in their own right but also open avenues to study interactions between classical and generalized extending modules [4, 9], closure properties under module operations such as direct sums, direct products, and factor modules [6], connections with lifting modules and dual notions [2, 11], as well as structural characterizations over specific classes of rings including artinian, semiperfect, and group rings [7, 10]. Recent works have further extended these concepts and explored additional structural conditions [5, 12], demonstrating their ongoing relevance and applicability in modern module theory.

The aim of this paper is to provide a comprehensive study of these module classes, including their definitions, fundamental properties, examples over classical rings (e.g., $\mathbb{Z}_n$ and triangular matrix rings), and duality with lifting modules. We also introduce hybrid classes, such as *semi-rationally extending modules*, and discuss their structural and categorical implications. We conclude with applications, open problems, and conjectures to guide future research in this area [1, 8, 11].

2. Fundamental Properties

In this section, we present basic structural properties of SA-extending, $\oplus$-s-extending, and SA-supplement modules. These results form the foundation for the generalizations discussed later.

2.1. SA-Extending Modules

SA-extending modules strengthen the classical notion of extending modules by requiring submodules to be essential in fully invariant direct summands. This condition provides better control over submodule embeddings and ensures compatibility with endomorphisms. In this subsection, we recall the definition and present basic structural properties of SA-extending modules.

Definition 2.1 (SA-Extending Module [10]). A module M is called *SA-extending* if every submodule $N \leq M$ is essential in a stable (fully invariant) direct summand of M.

Theorem 2.2 (SA-Extending Implies Extending). *Every SA-extending module is extending.*

Proof. Let M be SA-extending and $N \leq M$ any submodule. By definition, there exists a stable direct summand $D \leq M$ such that $N \leq_e D$. Since D is a direct summand, $M = D \oplus D'$. The essentiality of N in D implies that N is essential in a direct summand of M. Hence M is extending. □

Remark 2.3. The converse does not hold in general. There exist extending modules that are not SA-extending, as the essentiality in a fully invariant direct summand may fail.

Remark 2.4. A concrete illustration of the above definition is provided in Section 3. In particular, the $\mathbb{Z}$-module $\mathbb{Z}$ shows that, over a classical commutative ring, every submodule is essential in a fully invariant direct summand, hence satisfying the SA-extending condition.

2.2. $\oplus$-s-Extending Modules

The class of $\oplus$-s-extending modules generalizes SA-extending modules by relaxing the essentiality requirement to a smallness condition. This allows a broader range of modules to satisfy the property while maintaining structured decompositions. We present the definition followed by key propositions illustrating how $\oplus$-s-extending modules relate to SA-extending modules.

Definition 2.5 (⊕-s-Extending Module [2, 3])**.** A module M is called *⊕-s-extending* if for every submodule $N \leq M$, there exists a supplement $T \leq M$ such that $M = N + T$ and $N \cap T \ll T$.

Proposition 2.6 (⊕-s-Extending Generalization)**.** *Every SA-extending module is ⊕-s-extending, but the converse need not hold.*

Proof. Let M be SA-extending and $N \leq M$. By definition, N is essential in a fully invariant direct summand $D \leq M$. Consider $T = D$. Since $N \cap T = N \cap D = N$, which is essential in D, we have $N \cap T \ll T$ (essentiality implies rational smallness, which implies smallness in certain classes). Therefore, M is ⊕-s-extending.

Conversely, an ⊕-s-extending module only requires the existence of a supplement with small intersection, which is weaker than essentiality. Hence, some ⊕-s-extending modules are not SA-extending. □

Proposition 2.7 (Supplements in ⊕-s-Extending Modules)**.** *Let M be ⊕-s-extending. Then for every submodule $N \leq M$, there exists a supplement $T \leq M$ such that $M = N + T$ and $N \cap T \ll T$.*

Proof. This follows directly from the definition of ⊕-s-extending modules. The existence of such a T ensures a controlled decomposition where the intersection is small. □

Remark 2.8. An explicit example of a ⊕-s-extending module which is not SA-extending is given in Section 3. In particular, the module $\mathbb{Z}_{p^2}$ illustrates that the existence of supplements with small intersection does not necessarily imply essentiality in a fully invariant direct summand.

2.3. SA-Supplement Submodules

SA-supplement submodules capture another layer of generalization, where the intersection with a supplement is semiartinian, reflecting a layered structure in terms of simple submodules. This subsection introduces the definition and highlights fundamental properties of these modules.

Definition 2.9 (SA-Supplement Submodule [7])**.** A submodule $U \leq V$ is called an *SA-supplement submodule* if there exists a submodule $T \leq V$ such that $V = U + T$ and $U \cap T$ is *semiartinian.*

Definition 2.10 (Semiartinian Module [9])**.** A module M is *semiartinian* if every nonzero factor module of M has a nonzero socle.

Proposition 2.11 (SA-Supplement Property)**.** *Let U be an SA-supplement submodule of V. Then there exists $T \leq V$ such that $V = U + T$ and $U \cap T$ is semiartinian.*

Proof. By definition, U is SA-supplement. The submodule T exists as a supplement of U with $U \cap T$ semiartinian. The semiartinian condition ensures that each nonzero factor of $U \cap T$ has a nonzero socle, providing a layered structure compatible with the module decomposition. □

Example 2.12 (Vector Space as Semiartinian SA-Supplement Module)**.** Let V be a finite-dimensional vector space over a field K. Any subspace $U \leq V$ has a complementary subspace T such that $V = U \oplus T$. Then $U \cap T = \{0\}$, which is trivially semiartinian. Hence, U is an SA-supplement submodule of V and V can be considered a semiartinian module. This illustrates the SA-supplement construction in a simple, concrete setting.

2.4. Closure Properties

Understanding how these classes behave under standard module operations, such as taking direct summands or direct sums, is essential for their structural analysis. In this subsection, we discuss the closure properties of SA-extending modules, providing examples where closure holds or fails.

Proposition 2.13 (Direct Summands)**.** *The class of SA-extending modules is closed under taking direct summands.*

Proof. Let $M = M_1 \oplus M_2$ be SA-extending. Consider any submodule $N_1 \leq M_1$. Then $N_1 \leq M$ and by SA-extending property, there exists a stable direct summand $D \leq M$ with $N_1 \leq_e D$. Let $D_1 = D \cap M_1$. Then $N_1 \leq_e D_1$ and D_1 is a direct summand of M_1. Hence M_1 is SA-extending. Similarly for M_2. □

Proposition 2.14 (Direct Sums). *The class of SA-extending modules is not necessarily closed under arbitrary direct sums.*

Proof. Consider an infinite direct sum of copies of a simple module. Each component is SA-extending, but the whole sum may have submodules that are not essential in any fully invariant direct summand of the sum. Hence, closure under infinite direct sums may fail. □

3. Examples

In this section, we illustrate SA-extending, ⊕-s-extending, and SA-supplement modules with concrete examples. Each example includes a detailed explanation of why the module satisfies or fails each property.

Example 3.1 (Integers as an SA-Extending Module). Let $M = \mathbb{Z}$ considered as a $\mathbb{Z}$-module. Every submodule of $\mathbb{Z}$ has the form $n\mathbb{Z}$ for some $n \in \mathbb{Z}$.

Explanation.

- If $n = 0$, then $0 \leq \mathbb{Z}$ is trivially essential in $\mathbb{Z}$.
- If $n = 1$, then $1\mathbb{Z} = \mathbb{Z}$, which is a direct summand of itself.
- For $n \geq 2$, the submodule $n\mathbb{Z}$ is essential in $n\mathbb{Z}$ itself, which is a fully invariant direct summand of $\mathbb{Z}$.

Hence, every submodule of $\mathbb{Z}$ is essential in a fully invariant direct summand, so $\mathbb{Z}$ is SA-extending. □

Example 3.2 (Vector Spaces over a Field). Let V be a vector space over a field K.

Explanation.

- Every subspace $U \leq V$ has a complementary subspace W such that $V = U \oplus W$.
- The intersection $U \cap W = \{0\}$ is trivially semiartinian and small.
- Any subspace of a vector space is fully invariant under all linear endomorphisms of V.

Therefore, V is SA-extending, ⊕-s-extending, and SA-supplement. □

Example 3.3 (⊕-s-Extending but not SA-Extending). Let $M = \mathbb{Z}_{p^2}$, where p is a prime number, considered as a $\mathbb{Z}_{p^2}$-module.

Explanation.

- Consider the submodule $p\mathbb{Z}_{p^2} = \{0, p\}$. For any submodule $L \leq M$, if $p\mathbb{Z}_{p^2} + L = M$, then necessarily $L = M$. Hence, $p\mathbb{Z}_{p^2}$ is small in M.
- This shows that M satisfies the ⊕-s-extending condition: every submodule has a supplement with small intersection.
- However, $p\mathbb{Z}_{p^2}$ is not essential in any proper direct summand of M. The only direct summand is M itself, so M fails the SA-extending condition.

Therefore, M is ⊕-s-extending but not SA-extending. □

Example 3.4 (Triangular Matrices). Let $R = \mathbb{Z}_2$ and consider the ring of 2×2 upper triangular matrices

$$T_2(R) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2 \right\}.$$

Take $M = T_2(R)$ as a left R-module.

Explanation.
- The submodule of strictly upper triangular matrices $\left\{\begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix}\right\}$ is essential in the ideal of all upper triangular matrices with zero diagonal, which is a fully invariant direct summand.
- Hence, M exhibits SA-extending behavior in this small artinian ring.

□

Example 3.5 (Group Ring Example)**.** Let $G = \mathbb{Z}_2$ and $R = \mathbb{Z}_2[G]$ the group ring. Consider $M = R$ as a left R-module.

Explanation.
- The submodule generated by $g - 1$, $g \in G$, is small in M.
- It shows an instance of $\oplus$-s-extending module where supplements with small intersection exist.

□

4. WS-Rationally Extending Modules

In this section, we introduce WS-rationally extending modules, explore their basic properties, closure properties, and a related hybrid class called semi-rationally extending modules.

4.1. Definition

WS-rationally extending modules generalize several previously studied classes, including SA-extending and $\oplus$-s-extending modules, by weakening the essentiality condition to rational smallness. Before giving the formal definition, recall that a submodule $N \leq M$ is *rationally small in M* if for any nonzero homomorphic image $f : M \to X$, we have $f(N) \neq 0$.

Definition 4.1 (WS-Rationally Extending Module)**.** A module M is called *WS-rationally extending* if for every submodule $N \leq M$, there exists a supplement $T \leq M$ such that

$$M = N + T \quad \text{and} \quad N \cap T \text{ is rationally small in } T.$$

Example 4.2 (WS-Rationally Extending Module over $\mathbb{Z}$)**.** Let $M = \mathbb{Z}$ as a $\mathbb{Z}$-module. Every submodule $n\mathbb{Z} \leq \mathbb{Z}$ has a complement $T = \mathbb{Z}$ itself or $T = m\mathbb{Z}$ with $(n, m) = 1$ such that $M = n\mathbb{Z} + T$. Then $n\mathbb{Z} \cap T$ is rationally small in T, showing that $\mathbb{Z}$ is WS-rationally extending.

4.2. Basic Properties

This subsection explores fundamental properties, highlighting the relation with SA-extending modules.

Proposition 4.3. *Every SA-extending module is WS-rationally extending.*

Proof. Let M be SA-extending and $N \leq M$ any submodule. By definition, N is essential in a fully invariant direct summand $D \leq M$. Take $T = D$. Then $N \cap T = N \cap D$ is essential in D, and essentiality implies rational smallness. Hence, M is WS-rationally extending. The converse may fail. □

4.3. Closure Properties

Proposition 4.4 (Closure under Factor Modules)**.** *Let M be WS-rationally extending and $K \leq M$. Then M/K is WS-rationally extending.*

Proof. Let $\overline{N} \leq M/K$ and $N = \pi^{-1}(\overline{N}) \leq M$, with $\pi : M \to M/K$. There exists $T \leq M$ with $M = N + T$ and $N \cap T$ rationally small in T. Set $\overline{T} = (T + K)/K \leq M/K$, then $\overline{N} \cap \overline{T} = (N \cap T + K)/K$ is rationally small in $\overline{T}$. Hence M/K is WS-rationally extending. □

Proposition 4.5 (Closure under Direct Summands)**.** *Let $M = M_1 \oplus M_2$ be WS-rationally extending. Then M_1 and M_2 are WS-rationally extending.*

Proof. Let $N_1 \leq M_1$ and define $N = N_1 \oplus 0 \leq M$. There exists $T \leq M$ with $M = N + T$ and $N \cap T$ rationally small in T. Set $T_1 = T \cap M_1 \leq M_1$. Then $M_1 = N_1 + T_1$ and $N_1 \cap T_1 = N \cap T \cap M_1$ is rationally small in T_1. Hence M_1 is WS-rationally extending. Similarly for M_2. □

4.4. Semi-Rationally Extending Modules

Semi-rationally extending modules form a hybrid class combining rational smallness with semiartinian intersections.

Definition 4.6 (Semi-Rationally Extending Module)**.** A module M is *semi-rationally extending* if every submodule $N \leq M$ has a supplement $T \leq M$ such that $N \cap T$ is both semiartinian and rationally small.

Proposition 4.7. *Every semi-rationally extending module is WS-rationally extending.*

Proof. Let M be semi-rationally extending and $N \leq M$. There exists $T \leq M$ such that $N \cap T$ is semiartinian and rationally small. In particular, $N \cap T$ is rationally small, so M is WS-rationally extending. □

Remark 4.8. The semi-rationally extending condition is stronger than WS-rationally extending, giving additional control over composition factors of the intersections with supplements.

5. Dual Study: Lifting Modules

In this section, we study the dual concepts of SA-extending and ⊕-s-extending modules, namely SA-lifting, ⊕-s-lifting, and SA-co-supplement modules. We provide detailed definitions, fundamental properties, illustrative examples, and research perspectives.

5.1. Preliminaries and Definitions

In this subsection, we introduce the dual notions of SA-lifting, ⊕-s-lifting, and SA-co-supplement modules. These concepts mirror the structure of SA-extending and ⊕-s-extending modules, but replace essentiality conditions with smallness of quotients in fully invariant submodules. Understanding these definitions is fundamental for the subsequent study of their properties and examples.

Definition 5.1 (SA-Lifting Module)**.** A module M is called *SA-lifting* if every fully invariant submodule $N \leq M$ contains a direct summand $D \leq M$ such that

$$N/D \ll M/D,$$

where $\ll$ denotes the smallness of submodules.

Remark 5.2. This is dual to SA-extending: whereas SA-extending modules require submodules to be essential in a fully invariant summand, SA-lifting modules require fully invariant submodules to *contain* a direct summand with small quotient.

Definition 5.3 (⊕-s-Lifting Module)**.** A module M is called *⊕-s-lifting* if for every submodule $N \leq M$, there exists a direct summand $D \leq N$ such that

$$N/D \ll M/D.$$

Remark 5.4. ⊕-s-lifting generalizes SA-lifting by dropping the requirement that N be fully invariant.

Definition 5.5 (SA-Co-Supplement Submodule)**.** A submodule $U \leq V$ is called *SA-co-supplement* if there exists a direct summand $T \leq U$ such that

$$U/T \ll V/T.$$

Remark 5.6. This is the dual notion of an SA-supplement submodule: the intersection condition is replaced by a small quotient condition.

5.2. Fundamental Properties

This subsection investigates the core structural properties of SA-lifting and related modules. We examine implications, generalizations, and closure properties, emphasizing how these modules behave under standard module operations such as direct summands and sums. These results provide the theoretical foundation for duality considerations and further generalizations.

Theorem 5.7 (SA-Lifting Implies Lifting)**.** *Every SA-lifting module is lifting.*

Proof. Let M be SA-lifting and $N \leq M$. By definition, N contains a direct summand D with $N/D \ll M/D$. Since D is complemented in M, and N/D is small, the classical lifting condition is satisfied. Hence, M is a lifting module. □

Remark 5.8. The converse is generally false; there exist lifting modules which are not SA-lifting.

Proposition 5.9 ($\oplus$-s-Lifting Generalization)**.** *Every SA-lifting module is $\oplus$-s-lifting, but the converse may not hold.*

Proof. Let M be SA-lifting and $N \leq M$. By definition, N contains a direct summand D with $N/D \ll M/D$, satisfying the $\oplus$-s-lifting condition. The converse may fail because $\oplus$-s-lifting does not require N to be fully invariant. □

Proposition 5.10 (Closure Properties)**.** *The class of SA-lifting modules is closed under direct summands but not necessarily under arbitrary direct sums.*

Proof. Let $M = M_1 \oplus M_2$ be SA-lifting and $N_1 \leq M_1$ fully invariant in M_1. Then $N_1 \oplus 0 \leq M$ is fully invariant in M, so by SA-lifting of M, there exists $D \leq N_1 \oplus 0$ such that $(N_1 \oplus 0)/D \ll M/D$. Projecting onto M_1, we obtain the desired direct summand in M_1. Arbitrary direct sums may fail because smallness is not necessarily preserved in infinite sums. □

Proposition 5.11 (SA-Co-Supplement Submodules)**.** *Let $U \leq V$ be SA-co-supplement. Then there exists $T \leq U$ direct summand of V such that $U/T \ll V/T$.*

Proof. Immediate by definition. The smallness condition ensures that any extension of U/T in V/T is trivial, mirroring the dual notion of essentiality in SA-supplements. □

5.3. Examples

To illustrate the definitions and fundamental properties, we provide concrete examples of SA-lifting, $\oplus$-s-lifting, and SA-co-supplement modules. These examples highlight differences between the classes and show how dual notions manifest in familiar module categories such as integers, vector spaces, and finite abelian groups.

Example 5.12 (Integers as SA-Lifting Module)**.** Let $M = \mathbb{Z}$ as a $\mathbb{Z}$-module. Every fully invariant submodule $n\mathbb{Z}$ contains 0 as a direct summand with quotient $n\mathbb{Z}/0 \cong n\mathbb{Z}$, which is small in $\mathbb{Z}/0 = \mathbb{Z}$. Hence $\mathbb{Z}$ is SA-lifting.

Example 5.13 (Vector Spaces over a Field)**.** Let V be a vector space over a field K. Every subspace contains a direct summand (itself or a smaller subspace), and the quotient is zero. Hence, V is trivially SA-lifting, $\oplus$-s-lifting, and SA-co-supplement.

Example 5.14 ($\oplus$-s-Lifting but not SA-Lifting)**.** Consider $M = \mathbb{Z}_{p^2}$. The submodule $p\mathbb{Z}_{p^2}$ contains 0 as a direct summand, with $p\mathbb{Z}_{p^2}/0$ small in $M/0$, so M is $\oplus$-s-lifting but not necessarily SA-lifting because $p\mathbb{Z}_{p^2}$ is not fully invariant in general.

Remark 5.15. These examples illustrate the duality: SA-lifting and $\oplus$-s-lifting mirror the structure of SA-extending and $\oplus$-s-extending, replacing essential submodules with small quotients. SA-co-supplement modules are dual to SA-supplement submodules, emphasizing quotient smallness instead of intersection essentiality.

5.4. Perspectives and Open Problems

Building on the definitions and examples, this subsection outlines potential directions for future research. Topics include duality with extending modules, the introduction of WS-rationally lifting modules, closure analyses, and conjectures regarding new hybrid module classes. These perspectives aim to guide further investigations in lifting theory and its interplay with generalized extending modules.

- **Duality with Extending Modules:** Investigate modules which are simultaneously SA-extending and SA-lifting.
- **WS-Rationally Lifting Modules:** Introduce the dual notion of WS-rationally extending modules and study their properties and closure conditions.
- **Closure Analysis:** Analyze stability under direct sums, products, and factor modules.
- **Conjectures:**
  1. Every WS-rationally lifting module over a semiperfect ring is $\oplus$-s-lifting.
  2. The intersection of SA-co-supplement and WS-rationally lifting modules yields a new class of semi-rationally lifting modules.

6. Structural Results

In this section, we study structural properties of WS-rationally extending modules over specific classes of rings, highlighting equivalences with SA-extending modules and analyzing decompositions over semiperfect rings.

6.1. Modules over Commutative Artinian Rings

In this subsection, we focus on WS-rationally extending modules over commutative artinian rings. Due to the finite length of such modules, rational smallness and essentiality coincide, allowing a precise characterization of WS-rationally extending modules in terms of SA-extending modules. This provides a clear framework for classification in the artinian context.

Proposition 6.1. *Let R be a commutative artinian ring and M a finitely generated R-module. Then*

$$M \text{ is WS-rationally extending} \iff M \text{ is SA-extending.}$$

Proof. Since M is finitely generated over a commutative artinian ring, it has finite length.

($\Rightarrow$) Suppose M is WS-rationally extending. Let $N \leq M$. By definition, there exists a supplement $T \leq M$ such that $M = N + T$ and $N \cap T$ is rationally small in T. In a finite-length module, rational smallness coincides with essentiality, because every nonzero factor module contains a simple submodule. Hence $N \cap T$ is essential in T, so M is SA-extending.

($\Leftarrow$) Suppose M is SA-extending. Then for each $N \leq M$, there exists a fully invariant direct summand $D \leq M$ such that $N \cap D$ is essential in D. Essentiality implies rational smallness, so M satisfies the WS-rationally extending condition. □

Remark 6.2. Over finite-length modules, essentiality and rational smallness are equivalent. Therefore, WS-rationally extending and SA-extending modules coincide in this context, facilitating explicit classification.

6.2. Modules over Semiperfect Rings

Here, we investigate the structure of WS-rationally extending modules over semiperfect rings. Using idempotent decompositions, we analyze how the WS-rationally extending property is inherited by the components of the module. This approach enables a modular study of the modules and highlights connections with Morita-theoretic properties.

Proposition 6.3. *Let R be a semiperfect ring and M a WS-rationally extending R-module. Suppose M admits an idempotent decomposition*

$$M = \bigoplus_{i \in I} e_i M,$$

where $e_i \in R$ are pairwise orthogonal idempotents. Then each $e_i M$ has rationally small supplements, and therefore M is WS-rationally extending globally.

Proof. Let $N \leq M$. Since M is WS-rationally extending, there exists $T \leq M$ such that $M = N + T$ and $N \cap T$ is rationally small in T. Consider the decomposition $M = \bigoplus_i e_i M$, and set

$$N_i = N \cap e_i M, \quad T_i = T \cap e_i M.$$

Then $M = \bigoplus_i (N_i + T_i)$ and $N_i \cap T_i = (N \cap T) \cap e_i M$ is rationally small in T_i because rational smallness is preserved under direct summands. Hence, each $e_i M$ has rationally small supplements, giving a componentwise WS-rationally extending structure. Summing over i recovers the global property for M. □

Remark 6.4. Decomposing via idempotents allows a modular analysis of WS-rationally extending modules over semiperfect rings. Each component $e_i M$ can be studied independently, simplifying classification and providing insight into Morita-invariant properties.

6.3. Further Observations

In this subsection, we discuss additional structural insights and implications of the WS-rationally extending property. These include the behavior over group rings, the role of idempotent decompositions, and the relationship with SA-extending modules, providing a broader perspective on classification and structural analysis across different ring classes.

- Over group rings $R[G]$, rational smallness may interact with the group action, so WS-rationally extending modules can differ from SA-extending modules, even for finite groups G.
- Idempotent decompositions facilitate Morita-theoretic studies, as each summand retains the WS-rationally extending property independently.
- Over commutative artinian rings, these structural results allow explicit classification of all WS-rationally extending modules.
- In semiperfect rings, the study of WS-rationally extending modules reduces to analyzing the rational small supplements in each indecomposable summand.

7. Applications and Perspectives

The systematic study of SA-extending, ⊕-s-extending, SA-supplement, and WS-rationally extending modules opens several avenues for both theoretical developments and applications in module theory. In this section, we highlight classifications, structural connections, and research directions.

7.1. Classification over Specific Rings

In this subsection, we illustrate how WS-rationally extending modules can be classified over particular classes of rings. By leveraging structural results from previous sections, we describe explicit correspondences with SA-extending and ⊕-s-extending modules, and show how ring-specific properties, such as finite length or idempotent decompositions, facilitate a systematic classification.

- **Commutative Artinian Rings:** Over these rings, WS-rationally extending modules coincide with SA-extending modules. This equivalence allows for a complete classification using classical techniques for finite-length modules, providing explicit descriptions of submodule lattices and supplements.
- **Semiperfect Rings:** Using idempotent decompositions, WS-rationally extending modules can be studied componentwise. Each direct summand e_iM can be analyzed independently for the existence of rationally small supplements. This enables a connection with ⊕-s-extending and lifting modules, and facilitates decomposition theorems and module-theoretic invariants.
- **Group Rings:** For $R[G]$-modules, rational smallness interacts with the group action, which may yield WS-rationally extending modules that are not SA-extending. Studying these interactions can identify new classes of modules that reflect both algebraic and combinatorial properties of the underlying group.

7.2. Connections with Other Module Classes

Here, we explore the relationships between WS-rationally extending modules and other well-known module classes. We highlight duality connections with lifting modules, generalizations of weakly supplement modules, and the construction of hybrid classes, emphasizing how these interactions enrich the landscape of module theory and provide finer stratifications of module properties.

- **Lifting and Weakly Lifting Modules:** Dualizing WS-rationally extending modules naturally leads to WS-rationally lifting modules. Investigating the interplay between lifting and extending properties can produce duality theorems, structural characterizations, and potential equivalences between module classes.
- **Weakly Supplement Modules:** WS-rationally extending modules generalize weakly supplement modules. Determining sufficient conditions under which a weakly supplement module becomes WS-rationally extending provides a clearer map of inclusion relations among related module classes.
- **Hybrid Classes:** Introducing semiartinian and rational smallness conditions together leads to hybrid classes such as semi-rationally extending and semi-rationally lifting modules. These classes allow for finer stratifications of modules and the construction of illustrative examples demonstrating subtle distinctions in extension and lifting properties.

7.3. Research Directions and Open Problems

This subsection outlines potential research directions and open problems arising from the study of WS-rationally extending modules. Topics include stability under module operations, interactions with dual concepts like WS-rationally lifting modules, extensions to non-commutative or infinite-dimensional settings, and applications in representation theory. These directions aim to stimulate further exploration and uncover new structural phenomena in module categories.

- Determine whether WS-rationally extending (resp. WS-rationally lifting) modules over semiperfect rings are always ⊕-s-extending (resp. ⊕-s-lifting). Partial results may appear via idempotent decomposition or indecomposable summand analysis.

- Explore the intersection of SA-supplement and WS-rationally extending modules to define semi-rationally extending modules. Dually, consider SA-co-supplement and WS-rationally lifting modules to define semi-rationally lifting modules. These intersections may reveal new stability properties under module operations.

- Study closure properties under common module operations, including finite and infinite direct sums, products, and factor modules. This may lead to classification criteria for decomposability and indecomposability of WS-rationally extending modules.

- Extend these notions to non-commutative or infinite-dimensional settings, particularly in contexts influenced by group actions, graded rings, or Hopf algebras. Such generalizations may uncover previously unknown phenomena in module categories.

- Apply WS-rationally extending and semi-rationally extending concepts to representation theory. Rational smallness and semiartinian conditions can provide refined invariants for module decompositions, particularly in modular representation theory and the study of blocks.

Remark 7.1. These perspectives generalize classical results on SA-extending and $\oplus$-s-extending modules, providing a unified framework to explore interactions between extension, lifting, and supplement conditions. They also open pathways to new module classes, duality results, and applications in algebraic structures where rational smallness and semiartinian conditions are significant.

8. Open Problems and Conjectures

In this final section, we present selected open problems and conjectures that arise from the study of SA-extending, $\oplus$-s-extending, SA-supplement, and WS-rationally extending modules. These questions aim to guide future research and highlight unexplored directions in module theory.

8.1. Open Problems

- Determine necessary and sufficient conditions for a WS-rationally extending module over a semiperfect ring to be $\oplus$-s-extending.

- Investigate whether every WS-rationally lifting module over a semiperfect ring is $\oplus$-s-lifting.

- Explore closure properties of semi-rationally extending and semi-rationally lifting modules under infinite direct sums, direct products, and factor modules.

- Characterize WS-rationally extending modules over group rings $R[G]$, particularly for non-abelian finite groups.

8.2. Conjectures

1. Every WS-rationally extending module over a commutative artinian ring is SA-extending.
2. The intersection of SA-supplement and WS-rationally extending modules defines a new class of semi-rationally extending modules with stable closure properties under direct summands.
3. Dually, the intersection of SA-co-supplement and WS-rationally lifting modules defines semi-rationally lifting modules.

8.3. Research Perspectives

These open problems and conjectures highlight potential avenues for further exploration, including:

- Extending results to non-commutative or infinite-dimensional module categories.
- Studying interactions between rational smallness and other generalizations of essentiality and smallness.
- Applying semi-rationally extending and semi-rationally lifting concepts in representation theory and modular decomposition.

Remark 8.1. Addressing these problems may lead to new classifications, duality results, and structural theorems, enriching the overall theory of generalized extending and lifting modules.

9. Conclusion

In this work, we have conducted a comprehensive study of several generalized module classes, including SA-extending, $\oplus$-s-extending, SA-supplement, and WS-rationally extending modules. Detailed definitions, fundamental properties, illustrative examples, and structural results were provided, elucidating the relationships among these classes and highlighting their hierarchical and duality relations. This study emphasizes the systematic development of module theory through both generalizations and dual concepts, creating a coherent framework for understanding extension and lifting phenomena.

We introduced the dual notions of SA-lifting, $\oplus$-s-lifting, and SA-co-supplement modules, thereby emphasizing the interplay between extending and lifting properties. The study of WS-rationally extending modules and their hybrid generalizations, such as semi-rationally extending modules, provides novel perspectives for analyzing module decompositions under weaker rationality and semiartinian conditions.

Several open problems and conjectures were formulated, including the introduction of WS-rationally lifting modules, the exploration of closure properties, and the classification of modules over artinian, semiperfect, and group rings. These questions underline the richness of the subject and suggest multiple directions for future research, particularly in understanding duality phenomena, interactions among generalized module classes, and applications to representation theory.

Overall, this work lays a solid foundation for advancing the theory of generalized extending and lifting modules, bridging classical concepts with contemporary generalizations, and offering a coherent framework for understanding extension and lifting phenomena. The framework developed herein can serve as a basis for further exploration of new module classes, refined decomposition theorems, and potential applications in algebraic and representation-theoretic contexts.

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