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Journal of Prime Research in Mathematics, 22(1) (2026), 120–131

First $\mathfrak{so}(2)$-relative Cohomology of $Vec(\mathbb{R})$ acting on the space of bilinear bidifferential operators

Okba Basdouri[a]

[a] *University of Sfax, Faculty of Sciences of Sfax, 3000 Sfax , Tunisie.*

Abstract

We consider the Vec($\mathbb{R}$)-module structure on the spaces of bilinear bidifferential operators acting on the spaces of weighted densities. We compute the differential relative 1-cohomology of the vector fields Lie algebra Vec($\mathbb{R}$) with coefficients in space $\mathcal{D}_{\bar{\lambda};\mu}$ of bilinear bidifferential operators acting on weighted densities, vanishing on the Lie Algebra $\mathfrak{so}(2)$.

Keywords: Relative Cohomology, Lie subalgebra, weighted densities, bidifferential operators. Symbols.
2020 Mathematics Subject Classification: 17B56, 53D55, 58H15.

1. Introduction

Let V$ec(\mathbb{R})$ be the Lie algebra of vector fields on $\mathbb{R}$. Denote by $\mathcal{F}_\lambda = \left\{ gdx^\lambda \mid g \in \mathcal{C}^\infty(\mathbb{R}) \right\}$ the space of weighted densities of weight $\lambda \in \mathbb{R}$.

The space $\mathcal{F}_\lambda$ is a V$ec(\mathbb{R})$-module for the action defined by

$$L^\lambda_{h\frac{d}{dx}}(gdx^\lambda) = (hg' + \lambda h'g)dx^\lambda.$$

Denote by $\mathcal{D}_{\bar{\lambda},\mu}$, where $\bar{\lambda} = (\lambda_1, \cdots, \lambda_n)$, the space of $n-$ary differential operators:

$$\underbrace{\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}}_{n\otimes} \to \mathcal{F}_\mu,$$

for any $\bar{\lambda} \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$. The Lie algebra V$ec(\mathbb{R})$ acts on the space

$$\mathcal{D}_{\bar{\lambda},\mu} := \mathrm{Hom}_{\mathsf{diff}}(\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}, \mathcal{F}_\mu)$$

of these differential operators by:

$$X_h.A = L^\mu_{X_h} \circ A - A \circ L^{\bar{\lambda}}_{X_h}, \tag{1.1}$$

*Corresponding author
Email address: `okbabasdouri1@yahoo.fr` (Okba Basdouri)

Received : 21 September 2025*; Accepted:* 09 March 2026*; Published Online:* 16 March 2026

where $L^{\bar{\lambda}}_{X_h}$ is the Lie derivative on $\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}$ defined by the Leibnitz rule:

$$L^{\bar{\lambda}}_{X_h}(f_1 dx^{\lambda_1} \otimes \cdots \otimes f_n dx^{\lambda_n}) = L^{\lambda_1}_{X_h}(f_1) \otimes \cdots \otimes f_n dx^{\lambda_n} + \cdots + f_1 dx^{\lambda_1} \otimes \cdots \otimes L^{\lambda_n}_{X_h}(f_n dx^{\lambda_n}).$$

Thus the space of differential operators is a V$ec(\mathbb{R})$-module. If we restrict ourselves to the Lie algebra $\mathfrak{sl}(2)$ which is isomorphic to the Lie subalgebra of V$ec(\mathbb{R})$ spanned by

$$\{X_1, X_x, X_{x^2}\}.$$

Definition 1.1. *The affine subalgebra* $\mathfrak{a}(1)$ *of* $\mathfrak{sl}(2)$ *and the nilradical* $\mathfrak{so}(2)$ *are*

$$\mathfrak{a}(1) := \{X_1, X_x\}, \quad \mathfrak{so}(2, \mathbb{R}) := \{X_1\}.$$

The vector field $\frac{d}{dx}$ spans a commutative Lie algebra isomorphic to $\mathfrak{so}(2)$.

Let $\mathcal{L}$ be a Lie algebra and let $\mathcal{M}$ and $\mathcal{N}$ be 2 $\mathcal{L}-$modules. It is well-known that nontrivial extensions of $\mathfrak{L}-$modules:

$$0 \to \mathcal{F} \to . \to \mathcal{G} \to 0$$

are classified by the first cohomology group $\mathrm{H}^1(\mathcal{L}, \mathrm{Hom}(\mathcal{G}, \mathcal{F}))$ (see, e.g., [11]). Any 1$-$cocycle Δ generates a new action on $\mathcal{F} \oplus \mathcal{G}$ as follows: for all $X \in \mathcal{L}$ and for all $(f, g) \in \mathcal{F} \oplus \mathcal{G}$, we define $X \oslash (f, g) := (X \odot f + \Delta(g), X \odot g)$.

According to Nijenhuis-Richardson [13, 14],the space $\mathrm{H}^1(\mathcal{L}; \mathrm{End}(\mathcal{M}))$ classifies the infinitesimal deformations of a $\mathcal{L}$-module $\mathcal{M}$ and the obstructions to integrability of a given infinitesimal deformation of $\mathcal{M}$ are elements of $\mathrm{H}^2(\mathcal{L}; \mathrm{End}(\mathcal{M}))$ while the spaces $\mathrm{H}^1(\mathcal{L}; L(\otimes^n_s \mathcal{M}, \mathcal{M}))$ appear naturally in the problem of normalization of nonlinear representations of $\mathcal{L}$ in $\mathcal{M}$. For the space of tensor densities of weight $\lambda \in \mathbb{R}$, $\mathcal{F}_\lambda$, viewed as a module over the Lie algebra of smooth vector fields $\mathrm{Vec}(\mathbb{R})$, the classification of nontrivial extensions

$$0 \to \mathcal{F}_\mu \to . \to \mathcal{F}_\lambda \to 0.$$

leads Feigin and Fuks in [12] to compute the cohomology group $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vec}(\mathbb{R}), \mathrm{Hom}(\mathcal{F}_\lambda, \mathcal{F}_\mu))$. Later, Ovsienko and Bouarroudj in [9] have computed the corresponding relative cohomology group with respect to $\mathfrak{sl}(2)$, namely

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vec}(\mathbb{R}), \mathfrak{sl}(2); \mathrm{Hom}(\mathcal{F}_\lambda, \mathcal{F}_\mu)).$$

Later, Bouarroudj in [8] has computed the corresponding relative cohomology group with respect to $\mathfrak{sl}(2)$, namely

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathfrak{sl}(2); \mathrm{Hom}(\mathcal{F}_{\lambda_1} \otimes \mathcal{F}_{\lambda_2}, \mathcal{F}_\mu)).$$

In [4], I. Basdouri et al. have computed

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathfrak{a}(1); \mathrm{Hom}(\mathcal{F}_{\lambda_1} \otimes \mathcal{F}_{\lambda_2}, \mathcal{F}_\mu)).$$

In this paper, we will compute the first differential relative cohomology group respect to $\mathfrak{so}(2)$, namely $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vec}(\mathbb{R}), \mathfrak{so}(2); \mathcal{D}_{\lambda_1, \lambda_2; \mu})$, where, $\mathcal{D}_{\lambda, \lambda_2; \mu}$ is the space of bilinear bidifferential operators from $\mathcal{F}_{\lambda_1} \otimes \mathcal{F}_{\lambda_2}$ to $\mathcal{F}_\mu$, which gives the classification of non-trivial $\mathfrak{so}(2)-$relative extensions $\mathcal{F}_{\lambda_1} \otimes \mathcal{F}_{\lambda_2}$ by $\mathcal{F}_\mu$ defined by 1-cocycles.

2. Definitions and Notations

In this section, we recall the main definitions and facts related to the geometry of the space $\mathbb{R}$. We also recall some fundamental concepts from cohomology theory (see, e.g.,[5, 1, 2, 10, 7, 11]).

2.1. Cohomology theory

Let us first recall some fundamental concepts from cohomology theory (see, e.g., [8, 6]). Let $\mathcal{L}$ be a Lie algebra acting on a space $\mathcal{F}$ and let $\mathfrak{k}$ be a subalgebra of $\mathcal{L}$. (If $\mathfrak{k}$ is omitted it is assumed to be $\{0\}$.) The space of $\mathfrak{k}$-relative n-cochains of $\mathcal{L}$ with values in $\mathcal{F}$ is the $\mathcal{L}$-module

$$C^n(\mathcal{L}, \mathfrak{k}; \mathcal{F}) := \mathrm{Hom}_{\mathfrak{k}}(\Lambda^n(\mathcal{L}/\mathfrak{k}); \mathcal{F}).$$

The *coboundary operator* $\delta_n : C^n(\mathcal{L}, \mathfrak{k}; \mathcal{F}) \to C^{n+1}(\mathcal{L}, \mathfrak{k}; \mathcal{F})$ is a $\mathcal{L}$-map satisfying $\delta^2 = 0$. The kernel of δ_n, denoted $Z^n(\mathcal{L}, \mathfrak{k}; \mathcal{F})$, is the space of $\mathfrak{k}$-relative *n-cocycles*, among them, the elements in the range of δ_{n-1} are called $\mathfrak{k}$-relative *n-coboundaries*. We denote $B^n(\mathcal{L}, \mathfrak{k}; \mathcal{F})$ the space of n-coboundaries.

By definition, the n^{th} $\mathfrak{k}$-relative cohomology space is the quotient space

$$\mathrm{H}^n(\mathcal{L}, \mathfrak{k}; \mathcal{F}) = Z^n(\mathcal{L}, \mathfrak{k}; \mathcal{F})/B^n(\mathcal{L}, \mathfrak{k}; \mathcal{F}).$$

We will only need the formula of δ_n (which will be simply denoted δ) in degrees 0 and 1: for $h \in C^0(\mathcal{L}, \mathfrak{k}; \mathcal{F}) = \mathcal{F}^{\mathfrak{k}}$, $\delta h(f) := f \cdot h$, where

$$\mathcal{F}^{\mathfrak{k}} = \{f \in \mathcal{F} \mid h \cdot f = 0 \quad \text{for all } h \in \mathfrak{k}\},$$

and for $\Delta \in C^1(\mathcal{L}, \mathfrak{k}; \mathcal{F})$,

$$\delta(\Delta)(f_1, f_2) := f_1 \cdot \Delta(f_2) - f_2 \cdot \Delta(f_1) - \Delta([f_1, f_2]) \quad \text{for any} \quad f_1, f_2 \in \mathcal{L}. \tag{2.1}$$

2.2. $\mathrm{V}ec(\mathbb{R})$*-module structures on the of bilinear bidifferential operators*

Each bilinear bidifferential operator B on $\mathbb{R}$ gives thus rise to a morphism from $\mathcal{F}_{\lambda_1} \otimes \mathcal{F}_{\lambda_2}$ to $\mathcal{F}_\mu$, for any $\lambda_1, \lambda_2, \mu \in \mathbb{R}$, by $f_1 dx^{\lambda_1} \otimes f_2 dx^{\lambda_2} \mapsto B(f_1 dx^{\lambda_1} \otimes f_2 dx^{\lambda_2}) dx^\mu$. The Lie algebra $\mathrm{V}ec(\mathbb{R})$ acts on the space $\mathcal{D}_{\lambda_1,\lambda_2,\mu} := \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_{\lambda_1} \otimes \mathcal{F}_{\lambda_2}, \mathcal{F}_\mu)$ of these bidifferential operators by:

$$X.B = L_X^\mu \circ B - B \circ L_X^{(\lambda_1,\lambda_2)}, \tag{2.2}$$

where $L_{X_h}^{\bar\lambda}$ is the Lie derivative on $\mathcal{F}_{\lambda_1} \otimes \mathcal{F}_{\lambda_2}$ defined by the Leibnitz rule:

$$L_X^{\bar\lambda}(f_1 dx^{\lambda_1} \otimes f_2 dx^{\lambda_2}) = L_X^{\lambda_1}(f_1 dx^{\lambda_1}) \otimes f_2 dx^{\lambda_2} + f_1 dx^{\lambda_1} \otimes L_X^{\lambda_2}(f_2 dx^{\lambda_2}).$$

Thus the space of bidifferential operators is a $\mathrm{V}ec(\mathbb{R})$-module.

3. $\mathfrak{so}(2)$-relative Cohomology of Vect($\mathbb{R}$) acting on $\mathcal{D}_{\bar\lambda;\mu}$

The main result in the paper, we compute the *differentiable*, $\mathfrak{so}(2)$-relative cohomology of the Lie algebra $\mathrm{V}ec(\mathbb{R})$ with coefficients in the space of bilinear differential operators $\mathcal{D}_{\bar\lambda;\mu}$. Namely, we consider only cochains that are given by differentiable maps.

Theorem 3.1. (*i*) *If* $\mu - \lambda_1 - \lambda_2 = 0$, *then*

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2); \mathcal{D}_{\bar\lambda;\mu}) = \mathbb{R} \qquad \text{for all} \quad \bar\lambda.$$

(*ii*) *If* $\mu - \lambda_1 - \lambda_2 = 1$, *then*

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2); \mathcal{D}_{\bar\lambda;\mu}) = \begin{cases} \mathbb{R}^3 & \text{if} \quad \bar\lambda = (0,0), \\ \mathbb{R} & \text{otherwise.} \end{cases}$$

(iii) If $\mu - \lambda_1 - \lambda_2 = 2$, then

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2); \mathcal{D}_{\bar{\lambda};\mu}) = \begin{cases} \mathbb{R}^2 & \text{if} \quad \bar{\lambda} \in \{(0,0), (0,-1), (0,\lambda_2), (\lambda,0), (\lambda,-1-\lambda)\}, \\ 0 & \text{otherwise.} \end{cases}$$

(iv) If $\mu - \lambda_1 - \lambda_2 = 3$, then

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2); \mathcal{D}_{\bar{\lambda};\mu}) = \begin{cases} \mathbb{R}^3 & \text{if} \quad \bar{\lambda} \in \{(0,0), (-2,0), (0,-2), (-\frac{2}{3}, -\frac{2}{3})\}, \\ \mathbb{R} & \text{otherwise.} \end{cases}$$

(v) If $\mu - \lambda_1 - \lambda_2 = 4$, then

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2); \mathcal{D}_{\bar{\lambda};\mu}) = \mathbb{R}^2 \qquad \text{for all} \quad \bar{\lambda}.$$

(vi) If $\mu - \bar{\lambda} = 5$, then

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2); \mathcal{D}_{\bar{\lambda};\mu}) = \begin{cases} \mathbb{R}^3 & \text{if} \quad \bar{\lambda} \in \{(0,0), (0,-4), (-4,0)\}, \\ \mathbb{R}^2 & \text{otherwise.} \end{cases}$$

(vii) If $\mu - \lambda_1 - \lambda_2 = 6$, then

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2); \mathcal{D}_{\bar{\lambda};\mu}) = \begin{cases} \mathbb{R}^3 & \text{if} \quad \bar{\lambda} \in \{(\frac{-5\pm\sqrt{19}}{2}, 0), (0, \frac{-5\pm\sqrt{19}}{2})\}, \\ \mathbb{R}^2 & \text{otherwise.} \end{cases}$$

(viii) If $\mu - \lambda_1 - \lambda_2 = 7$, then

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2); \mathcal{D}_{\bar{\lambda};\mu}) = \begin{cases} \mathbb{R}^2 & \text{if} \quad \bar{\lambda} \in \{(\frac{-5-\sqrt{19}}{2}, \frac{-5-\sqrt{19}}{2}), (\frac{-5+\sqrt{19}}{2}, \frac{-5+\sqrt{19}}{2}), \\ & (0,\lambda_2), (\lambda,0), (\lambda,-6-\lambda), (\sqrt{19}-1, \frac{-\sqrt{19}-5}{2}), \\ & (\frac{-\sqrt{19}-5}{2}, \sqrt{19}-1), (-\sqrt{19}-1, \frac{\sqrt{19}-5}{2}), \\ & (\frac{\sqrt{19}-5}{2}, -\sqrt{19}-1)\}, \\ \mathbb{R} & \text{otherwise.} \end{cases}$$

(ix) If $\mu - \lambda_1 - \lambda_2 \geq 8$ but with $\bar{\lambda}$ generic then

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2); \mathcal{D}_{\bar{\lambda};\mu}) = 0.$$

Remark 3.2. *([4, 8, 7],[12], [15])*

1. *We notice that* $\mathrm{H}^1(\mathrm{V}ec(\mathbb{R}), \mathfrak{sl}(2), \mathcal{D}_{\lambda,\lambda_2,\mu}) \subseteq \mathrm{H}^1(\mathrm{V}ec(\mathbb{R}), \mathfrak{a}(1), \mathcal{D}_{\lambda,\lambda_2,\mu}) \subseteq \mathrm{H}^1(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2), \mathcal{D}_{\lambda,\lambda_2,\mu}) \subseteq \mathrm{H}^1(\mathrm{V}ec(\mathbb{R}), \mathcal{D}_{\lambda,\lambda_2,\mu})$.
2. *The values* $-\frac{2}{3}$ *can be deduced from the work of Grozman.*
3. *The values* $\frac{-5+\sqrt{19}}{2}$ *and* $\sqrt{19}$ *from the work of Feigin-Fuchs.*

For $\bar{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and $\mu \in \mathbb{R}$, we pose $k = \mu - \lambda_1 - \lambda_2$ and we are interested in the space $\mathrm{T}^{\bar{\lambda}}_k$ of the bilinear bidifferential operators who are $\mathfrak{so}(2)$-invariant. The elements of $\mathrm{T}^{\mu}_{\bar{\lambda}}$ are described as follows:

Theorem 3.3. *(a)If $k \notin \mathbb{N}$, then* $\mathrm{T}^{\mu}_{\bar{\lambda}} = \{0\}$.
(b)There exist $\mathfrak{so}(2)$*-invariant bilinear bidifferential operators*

$$\mathrm{T}^{\bar{\lambda}}_{k} : \mathcal{F}_{\lambda_1} \otimes \mathcal{F}_{\lambda_2} \longrightarrow \mathcal{F}_{\lambda_1+\lambda_2+k}$$

given by

$$\begin{aligned} \mathrm{T}^{\bar{\lambda}}_{k} : \mathcal{F}_{\lambda_1} \otimes \mathcal{F}_{\lambda_2} &\longrightarrow \mathcal{F}_{\lambda_1+\lambda_2+k}, \\ \phi_1 \otimes \phi_2 &\longmapsto \sum_{i_1+i_2=k} \varepsilon_{i_1,i_2} \ \phi_1^{(i_1)} \phi_2^{(i_2)}, \end{aligned}$$

where ε_{i_1,i_2} *are constants.*

Proof. Let T be an bilinear bidifferential operator

$$\begin{aligned} \mathrm{T} : \mathcal{F}_{\lambda_1} \otimes \mathcal{F}_{\lambda_2} &\longrightarrow \mathcal{F}_{\mu}, \\ (\phi_1 \otimes \phi_2) &\longmapsto \sum_{k=0}^{m} \sum_{i_1+i_2=k} \varepsilon_{i_1,i_2} \ \phi_1^{(i_1)} \phi_2^{(i_2)}, \end{aligned}$$

where ε_{i_1,i_2} are, a priori, functions. T is $\mathfrak{so}(2)$-invariant if and only if for all $X \in \mathfrak{so}(2)$: $X.\mathrm{T} = 0$ if and only if $L^{\bar{\lambda},\mu}_{X}(\mathrm{T})(\phi_1 \otimes \phi_2) = 0$, for all $(\phi_1, \phi_2) \in \mathcal{F}_{\lambda_1} \times \mathcal{F}_{\lambda_2}$.

The invariance with respect to X_1 is reflected in:

$$\sum_{k=0}^{m} \sum_{i_1+i_2=k} \varepsilon'_{i_1,i_2} \ \phi_1^{(i_1)} \phi_n^{(i_2)} = 0,$$

hence, ε_{i_1,i_2} is a constant $\forall i_1, i_2$.

4. Proof of Theorem 3.1

To proof Theorem 3.1, first we will investigate the dimension of the space of operators that satisfy the 1-cocycle condition: $\mathrm{Z}^1(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2), \mathcal{D}_{\bar{\lambda},\mu})$. Second we will study all trivial 1-cocycles, namely, operators of the form $X.(\mathrm{T})$, where T is a bilinear operator. As our 1-cocycles vanish on the Lie algebra $\mathfrak{so}(2)$, it follows that the opeartor $\mathcal{B}$ coincides with the space of bilinear differential operators $\mathfrak{so}(2)$-invariant. We will determine the dimension of $\mathrm{B}^1(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2), \mathcal{D}_{\bar{\lambda},\mu})$, and finally by taking into account the above and depending on λ_1 and λ_2, the dimension of the cohomology group $\mathrm{H}^1(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2), \mathcal{D}_{\bar{\lambda},\mu})$ will be equal to $dim(\mathrm{Z}^1(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2), \mathcal{D}_{\bar{\lambda},\mu}))$-dim$(\mathrm{B}^1(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2), \mathcal{D}_{\bar{\lambda},\mu}))$. We need also the following Lemmas.

Lemma 4.1. *[8] Every 1-cocycle on* $\mathrm{V}ec(\mathbb{R})$ *with values in* $\mathcal{D}_{\bar{\lambda},\mu}$ *is differentiable.*

Lemma 4.2. *Any 1-cocycle vanishing on the Lie subalgebra* $\mathfrak{so}(2)$ *of* $\mathrm{Vec}(\mathbb{R})$ *is* $\mathfrak{so}(2)$*-invariant.*

Proof. By a straightforward computation.

Now we are in position to prove Theorem 3.1. By Lemma 4.1, any 1-cocycle on $\mathrm{V}ec(\mathbb{R})$ should retains the following general form:

$$c(X)(f_1 dx^{\lambda_1}, f_2 dx^{\lambda_2}) = \sum_{i+j+l=k+1} c_{i,j,l} \ X^{(i)} f_1^{(j)} f_2^{(l)} dx^{\mu}$$

where $c_{i,j,l}$ are constants. By Lemma 4.2, the fact that this 1-cocycle vanishes on $\mathfrak{so}(2)$ implies that

$$c_{0,j,l} X = 0.$$

The 1-cocycle condition reads as follows: for all $X, Y \in \mathrm{V}ec(\mathbb{R})$, $f_1dx^{\lambda_1} \in \mathcal{F}_{\lambda_1}$ and $f_2dx^{\lambda_2} \in \mathcal{F}_{\lambda_2}$

$$L_X^{\bar{\lambda},\mu}\ c(Y)(f_1dx^{\lambda_1}, f_2dx^{\lambda_2}) - L_Y^{\bar{\lambda},\mu}\ c(X)(f_1dx^{\lambda_1}, f_2dx^{\lambda_2}) - c([X,Y])(f_1dx^{\lambda_1}, f_2dx^{\lambda_2}) = 0.$$

In the following, we give the expressions of $c(X, f_1, f_2)$ instead of $c(X)(f_1dx^{\lambda_1}, f_2dx^{\lambda_2})$.

(i) The case when $\mu-\lambda_1-\lambda_2 = 0$: the equation 2.1 shows that only one 1-cocycle spans the cohomology group of Theorem 3.1; it is given by

$$c(X, f_1, f_2) = c_{1,0,0}X'f_1f_2.$$

A direct computation proves that the 1-cocycle is not trivial

$$\mu - \lambda_1 - \lambda_2 = 0.$$

(ii) The case when $\mu - \lambda_1 - \lambda_2 = 1$: any 1-cocycle on $Vect(\mathbb{R})$ $\mathfrak{so}(2)$-relative should retains the following general form:

$$c(X, f_1, f_2) = c_{1,1,0}X'f_1'f_2 + c_{1,0,1}X'f_1f_2' + c_{2,0,0}X''f_1f_2.$$

Again, by a straightforward computation, the above 1-cocycle condition reads

$$\mu - \lambda_1 - \lambda_2 = 1 \ \text{ and } \ c_{1,1,0}\lambda_1 + c_{1,0,1}\lambda_2 = 0.$$

Let us study now the triviality of this 1-cocycle. A straightforward computation shows that

$$X.\mathrm{T}_1^{\bar{\lambda}} = -(\varepsilon_{1,0}\lambda_1 + \varepsilon_{0,1}\lambda_2)X''f_1f_2.$$

1. If $\bar{\lambda} = (0,0)$, then

$$\mathrm{dim}Z^1_{\mathrm{diff}}(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2); \mathcal{D}_{\bar{\lambda};\mu}) = 3$$
$$\mathrm{dim}B^1_{\mathrm{diff}}(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2); \mathcal{D}_{\bar{\lambda};\mu}) = 0.$$

2. If $\lambda_1 = 0$ and $\lambda_2 \neq 0$, then $c_{1,1,0} = 0$

$$\mathrm{dim}Z^1_{\mathrm{diff}}(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2); \mathcal{D}_{\bar{\lambda};\mu}) = 2$$

   $\mathrm{dim}B^1_{\mathrm{diff}}(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2); \mathcal{D}_{\bar{\lambda};\mu}) = 1$, generated by

$$B = -\varepsilon_{0,1}\lambda_2X''f_1f_2.$$

   The results hold true for ($\lambda_1 \neq 0$ together with $\lambda_2 = 0$) and $\bar{\lambda} \neq (0,0)$.

(iii) The case when $\mu - \lambda_1 - \lambda_2 = 2$

Any 1-cocycle should retains the following general form:

$$c(X, f_1, f_2) = \sum_{i+j+k=3, i\geq 1} c_{i,j,k}X^{(i)}f_1^{(j)}f_2^{(k)}dx^{\mu}$$

The 1-cocycle condition is equivalent to the following system:
$c_{1,2,0}(1 + 2\lambda_1) + c_{1,1,1}\lambda_2 = 0$,
$c_{1,2,0}\lambda_1 + c_{1,0,2}\lambda_2 = 0$,
$c_{1,1,1}\lambda_1 + c_{1,0,2}(1 + 2\lambda_2) = 0$.
Let us study the triviality of this 1-cocycle. a straightforward computation shows that

$$L_XT_2^{\bar{\lambda}} = -(\varepsilon_{2,0}(1 + 2\lambda) + \varepsilon_{1,1}\lambda_2)X''f_1'f_2 - (\varepsilon_{1,1}\lambda + \varepsilon_{0,2}(1 + 2\lambda_2))X''f_1f_2' - (\varepsilon_{2,0}\lambda + a_{0,2}\lambda_2)X'''f_1f_2.$$

1. If $\bar{\lambda} = (0,0)$, then $c_{1,2,0} = c_{1,0,2} = 0$

$$\dim Z^1_{\text{diff}}(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2); \mathcal{D}_{\bar{\lambda};\mu}) = 4$$

   $\dim B^1_{\text{diff}}(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2); \mathcal{D}_{\bar{\lambda};\mu}) = 2$, generated by

$$B_1 = X'' f_1' f_2,$$

$$B_2 = X'' f_1 f_2'.$$

2. If $\bar{\lambda} = (0,-1)$, then $c_{1,2,0} = c_{1,0,2}$ and $c_{1,0,2} = 0$,

$$\dim Z^1_{\text{diff}}(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2); \mathcal{D}_{\bar{\lambda};\mu}) = 4$$

   $\dim B^1_{\text{diff}}(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2); \mathcal{D}_{\bar{\lambda};\mu}) = 2$, generated by

$$\begin{aligned} B_1 &= X'' f_1' f_2, \\ B_2 &= (X'' f_1 f_2' + X''' f_1 f_2). \end{aligned}$$

   The results hold true for ($\lambda_1 \neq 0$ together with $\lambda_2 = 0$), ($\lambda_1 = 0$ together with $\lambda_2 \neq 0$) and $(\lambda_1, \lambda_2) = (\lambda_1, -1-\lambda_1)$.

3. for other cases for λ_1 and λ_2, then $c_{1,2,0} = c_{1,1,1} = c_{1,0,2} = 0$.
   Thus, the 1-cocycle is trivial.

(iv) The case when $\mu - \lambda_1 - \lambda_2 = 3$

Any 1-cocycle should retains the following general form:

$$c(X, f_1, f_2) = \sum_{i+j+k=4, i\geq 1} c_{i,j,k} X^{(i)} f_1^{(j)} f_2^{(k)} dx^{\mu}$$

The 1-cocycle condition is equivalent to the following system:

$$\begin{array}{ll} c_{1,0,3}(3+3\lambda_2) + c_{1,1,2}\lambda_1 = 0, & c_{1,0,3}(1+3\lambda_2) + c_{1,2,1}\lambda_1 = 0, \\ c_{1,0,3}\lambda_2 + c_{1,3,0}\lambda_1 = 0, & c_{1,1,2}(1+2\lambda_2) + c_{1,2,1}(1+2\lambda_1) = 0, \\ c_{1,1,2}\lambda_2 + c_{1,3,0}(1+3\lambda_1) = 0, & c_{1,2,1}\lambda_2 + c_{1,3,0}(3+3\lambda_1) = 0, \\ c_{2,0,2}\lambda_2 + c_{2,2,0}\lambda_1 = 0, & c_{3,0,1}\lambda_2 + c_{3,1,0}\lambda_1 + 2c_{4,0,0} = 0. \end{array}$$

The system is equivalent and consequently the space $\mathrm{Z}^1(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2), \mathcal{D}_{\lambda,\lambda_2,\mu})$ is 6-dimensional.
Let us study the triviality of this 1-cocycle. A straightforward computation shows that

$$\begin{aligned} L_X T_3^{\bar{\lambda}} = & - (\varepsilon_{0,3}(3+3\lambda_2) + \varepsilon_{1,2}\lambda_1) X'' f_1 f_2'' - (\varepsilon_{1,2}(1+2\lambda_2) + \varepsilon_{2,1}(1+2\lambda_1)) X'' f_1' f_2' \\ & - (\varepsilon_{2,1}\lambda_2 + a_{3,0}(3+3\lambda_1)) X'' f_1'' f_2 - (\varepsilon_{1,2}\lambda_2 + \varepsilon_{3,0}(1+3\lambda_1)) X''' f_1' f_2 \\ & - (\varepsilon_{0,3}(1+3\lambda_2) + \varepsilon_{2,1}\lambda_1) X''' f_1 f_2' - (\varepsilon_{0,3}\lambda_2 + \varepsilon_{3,0}\lambda_1) X^{(4)} f_1 f_2 \end{aligned}$$

1. Now we will characterize all values of λ and λ_2 for which $\mathrm{B}^1(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2), \mathcal{D}_{\bar{\lambda},\mu})$ is three-dimensional. A easy computation proves that these values are (0,0), $(\frac{-2}{3}, \frac{-2}{3})$, (0,-2) and (-2,0). Hence, the cohomology group is three-dimensional.

2. If λ and λ_2 are not like above, then the space $\mathrm{B}^1(\mathrm{V}ec(\mathbb{R}), \mathfrak{so}(2), \mathcal{D}_{\bar{\lambda}\mu})$ is 5-dimensional Hence, the cohomology group is one-dimensional.

(v) The case when $\mu-\lambda-\lambda_2=4$

Any 1-cocycle should retains the following general form:

$$c(X,f_1,f_2)=\sum_{i+j+k=5,i\geq 1} c_{i,j,k}X^{(i)}f_1^{(j)}f_2^{(k)}dx^{\mu}$$

The 1-cocycle condition is equivalent to the following system

$$\begin{cases}\lambda_1\ c_{2,3,0}-\lambda_1\ c_{4,1,0}+\lambda_2\ c_{2,0,3}-\lambda_2\ c_{4,0,1}=5\ c_{5,0,0}\\ \lambda_1\ c_{2,2,1}-\lambda_1\ c_{3,1,1}+(1+3\ \lambda_2)\ c_{2,0,3}-(1+2\ \lambda_2)\ c_{3,0,2}\\ (1+3\ \lambda_1)\ c_{2,3,0}-(1+2\ \lambda_1)\ c_{3,2,0}+\lambda_2\ c_{2,1,2}-\lambda_2\ c_{3,1,1}=2\ c_{4,1,0}\end{cases}$$

The space of solutions of the system above is 7-dimensional.
Let us study the triviality of this 1-cocycle. A straightforward computation shows that

$$\begin{array}{rcl} L_XT_4^{\bar\lambda}= & - & (\varepsilon_{4,0}(6+4\lambda_1)+\varepsilon_{3,1}\lambda_2)X''f_1'''f_2-(\varepsilon_{3,1}(3+3\lambda_1)+\varepsilon_{2,2}(1+2\lambda_2))X''f_1''f_2'\\ & - & (\varepsilon_{2,2}(1+2\lambda_1)+\varepsilon_{1,3}(3+3\lambda_2))X''f_1'f_2''-(\varepsilon_{1,3}\lambda_1+\varepsilon_{0,4}(6+4\lambda_2))X''f_1f_2'''\\ & - & (\varepsilon_{4,0}(4+6\lambda_1)+\varepsilon_{2,2}\lambda_2)X'''f_1''f_2-(\varepsilon_{3,1}(1+3\lambda_1)+\varepsilon_{1,3}(1+3\lambda_2))X'''f_1'f_2'\\ & - & (\varepsilon_{2,2}\lambda_1+\varepsilon_{0,4}(4+6\lambda_2))X'''f_1f_2''-(\varepsilon_{4,0}(1+4\lambda_1)+\varepsilon_{1,3}\lambda_2)X^{(4)}f_1'f_2\\ & - & (\varepsilon_{3,1}\lambda_1+\varepsilon_{0,4}(1+4\lambda_2))X^{(4)}f_1f_2'-(\varepsilon_{4,0}\lambda_1+\varepsilon_{0,4}\lambda_2)X^{(4)}f_1f_2 \end{array}$$

A direct computation proves that, for all values of $\bar\lambda$ the space $\mathrm{B}^1(\mathrm{V}ec(\mathbb{R}),\mathfrak{so}(2),\mathcal{D}_{\bar\lambda,\mu})$ is 5-dimensional. Hence, the cohomology group is two-dimensional.

(vi) The case when $\mu-\lambda_2-\lambda_2=5$

Any 1-cocycle should retains the following general form:

$$c(X,f_1,f_2)=\sum_{i+j+k=6,i\geq 1} c_{i,j,k}X^{(i)}f_1^{(j)}f_2^{(k)}dx^{\mu}$$

A straightforward computation shows that the above 1-cocycle condition reads:

The 1-cocycle condition is equivalent to the following system:

$$\begin{array}{ll}
c_{1,5,0}(10+5\lambda_1)+c_{1,4,1}\lambda_2=0, & c_{1,5,0}(10+10\lambda_1)+c_{1,3,2}\lambda_2=0,\\
c_{1,5,0}(5+10\lambda_1)+c_{1,2,3}\lambda_2=0, & c_{1,5,0}(1+5\lambda_1)+c_{1,1,4}\lambda_2=0,\\
c_{1,5,0}\lambda_1+c_{1,0,5}\lambda_2=0, & c_{1,4,1}(6+4\lambda_1)+c_{1,3,2}(1+2\lambda_2)=0,\\
c_{1,4,1}(4+6\lambda_1)+c_{1,2,3}(1+3\lambda_2)=0, & c_{1,4,1}(1+4\lambda_1)+c_{1,1,4}(1+4\lambda_2)=0,\\
c_{1,4,1}\lambda_1+c_{1,0,5}(1+5\lambda_2)=0, & c_{1,3,2}(3+3\lambda_1)+c_{1,2,3}(3+3\lambda_2)=0,\\
c_{1,3,2}(1+3\lambda_1)+c_{1,1,4}(4+6\lambda_2)=0, & c_{1,3,2}\lambda_1+c_{1,0,5}(5+10\lambda_2)=0,\\
c_{1,2,3}(1+2\lambda_1)+c_{1,1,4}(6+4\lambda_2)=0, & c_{1,2,3}\lambda_1+c_{1,0,5}(10+10\lambda_2)=0,\\
c_{1,1,4}\lambda_1+c_{1,0,5}(10+5\lambda_2)=0, & \\
c_{2,4,0}(4+6\lambda_1)+c_{2,2,2}\lambda_2=0, & c_{2,4,0}(1+4\lambda_)+c_{2,1,3}\lambda_2=0,\\
c_{2,4,0}\lambda_1+c_{2,0,4}\lambda_2=0, & c_{2,3,1}(1+3\lambda_1)+c_{2,1,3}(1+3\lambda_2)=0,\\
c_{2,3,1}\lambda_1+c_{2,0,4}(1+4\lambda_2)=0, & c_{2,3,1}\lambda_1+c_{2,0,4}(4+6\lambda_2)=0,\\
(c_{4,2,0}-c_{3,3,0})\lambda_1+(c_{4,0,2}-c_{3,0,3})\lambda_2+5c_{6,0,0}=0, & c_{3,3,0}(3+3\lambda_1)+c_{3,2,1}\lambda_2+2c_{4,2,0}=0,\\
c_{3,2,1}(1+2\lambda_1)+c_{3,1,2}(1+2\lambda_2)+2c_{4,1,1}=0, & c_{3,1,2}\lambda_1+c_{3,0,3}(3+3\lambda_2)+2c_{4,0,2}=0,\\
c_{4,2,0}(1+2\lambda_1)+c_{4,1,1}\lambda_2+5c_{5,1,0}=0, & \\
c_{4,1,1}\lambda_1+c_{4,0,2}(1+2\lambda_2)+5c_{5,0,1}=0, & c_{5,1,0}\lambda_1+c_{5,0,1}\lambda_2+9c_{6,0,0}=0.
\end{array}$$

The space of solutions of the system above is 9-dimensional for $\bar\lambda\in\{(0,-4),(0,0),(-4,0)\}$, and 8-dimensional otherwise. According to these values, let us study the space $\mathrm{B}^1(\mathrm{V}ec(\mathbb{R}),\mathfrak{so}(2),\mathcal{D}_{\lambda_1,\lambda_2,\mu})$.
A direct computation proves that Let us study the triviality of this 1-cocycle. A straightforward computation shows that

$$\begin{aligned}
L_X T_5^{\bar\lambda} = \;&-\; (\varepsilon_{5,0}(10+5\lambda_1)+\varepsilon_{4,1}\lambda_2)Y''f_1^{(4)}f_2 - (\varepsilon_{4,1}(6+4\lambda_1)+\varepsilon_{3,2}(1+2\lambda_2))X''f_1'''f_2' \\
&-\; (\varepsilon_{3,2}(3+3\lambda_1)+\varepsilon_{2,3}(3+3\lambda_2))X''f_1''f_2'' - (\varepsilon_{2,3}(1+2\lambda_1)+\varepsilon_{1,4}(6+4\lambda_2))X''f_1'f_2''' \\
&-\; (\varepsilon_{1,4}\lambda_1+\varepsilon_{0,5}(10+5\lambda_2))X''f_1f_2^{(4)} - (\varepsilon_{5,0}(10+10\lambda_1)+\varepsilon_{3,2}\lambda_2)X'''f_1'''f_2 \\
&-\; (\varepsilon_{4,1}(4+6\lambda_1)+\varepsilon_{2,3}(1+3\lambda_2))Y'''f_1''f_2' - (\varepsilon_{3,2}(1+3\lambda_1)+\varepsilon_{1,4}(4+6\lambda_2))X'''f_1'f_2'' \\
&-\; (\varepsilon_{2,3}\lambda_1+\varepsilon_{0,5}(10+10\lambda_2))X'''f_1f_2''' - (\varepsilon_{5,0}(5+10\lambda_1)+\varepsilon_{2,3}\lambda_2)X^{(4)}f''g \\
&-\; (\varepsilon_{4,1}(1+4\lambda_1)+\varepsilon_{1,4}(1+4\lambda_2))X^{(4)}f_1'f_2' - (\varepsilon_{3,2}\lambda_1+\varepsilon_{0,5}(5+10\lambda_2))X^{(4)}f_1f_2'' \\
&-\; (\varepsilon_{5,0}(1+5\lambda_1)+\varepsilon_{1,4}\lambda_2)Y^{(5)}f_1'f_2 - (\varepsilon_{4,1}\lambda_1+\varepsilon_{0,5}(1+5\lambda_2))X^{(5)}f_1f_2' \\
&-\; (\varepsilon_{5,0}\lambda_1+\varepsilon_{0,5}\lambda_2)X^{(6)}f_1f_2
\end{aligned}$$

For all values of $\bar\lambda$ the space $\mathrm{B}^1(\mathrm{V}ec(\mathbb{R}),\mathfrak{so}(2),\mathcal{D}_{\bar\lambda,\mu})$ is 6-dimensional; generated by

$$\begin{aligned}
\mathcal{B}_1(X,f_1,f_2) = \;& \lambda_2\; X^{(6)}f_1f_2 + (1+5\;\lambda_2)\;X^{(5)}f_1f_2' + (5+10\;\lambda_2)\;X^{(4)}f_1f_2^{(2)} \\
& +(10+10\;\lambda_2)\;X^{(3)}f_1f_2^{(3)} + (10+5\;\lambda_2)\;X^{(2)}\phi\psi^{(4)}. \\
\mathcal{B}_2(X,f_1,f_2) = \;& \lambda_2\; X^{(5)}f_1'f_2 + (1+4\;\lambda_2)\;X^{(4)}f_1'f_2i' + (4+6\;\lambda_2)\;X^{(3)}f_1'f_2^{(2)} \\
& +(6+4\;\lambda_2)\;X^{(2)}f_1'f_2^{(3)} + \lambda_1\;X^{(2)}f_1f_2^{(4)}. \\
\mathcal{B}_3(X,f_1,f_2) = \;& \lambda_1\; X^{(6)}f_1f_2 + (1+5\;\lambda_1\;X^{(5)}f_1'f_2 + (5+10\;\lambda_1)\;X^{(4)}f_1^{(2)}f_2 \\
& +(10+10\;\lambda_1)\;X^{(3)}f_1^{(3)}f_2 + (10+5\;\lambda_1)\;X^{(2)}f_1^{(4)}\psi. \\
\mathcal{B}_4(X,f_1,\psi) = \;& \lambda\; X^{(4)}f_1f_2^{(2)} + (1+3\;\lambda_1)\;X^{(3)}f_1'f_2^{(2)} + (1+2\;\lambda_2)\;X^{(2)}f_1^{(3)}f_2' \\
& +(3+3\;\lambda_1)\;X^{(2)}f_1^{(2)}f_2^{(2)} + \lambda_2\;X^{(3)}f_1^{(3)}f_2. \\
\mathcal{B}_5(X,f_1,f_2) = \;& \lambda_1\; X^{(5)}f_1f_2' + (1+4\;\lambda_1)\;X^{(4)}f_1'f_2' + (4+6\;\lambda_1)\;X^{(3)}f_1^{(2)}f_2' \\
& +(6+4\;\lambda_1)\;X^{(2)}f_1^{(3)}f_2' + \lambda_2\;X^{(2)}\phi^{(4)}\psi. \\
\mathcal{B}_6(X,\phi,f_2) = \;& \lambda_2\; X^{(4)}f_1^{(2)}f_2 + (1+3\;\lambda_2)\;X^{(3)}f_1^{(2)}f_2' + (3+3\;\lambda_2)\;X^{(2)}f_1^{(2)}f_2^{(2)} \\
& +(1+2\;\lambda_1)\;X^{(2)}f_1'f_2^{(3)} + \lambda_1\;X^{(3)}f_1f_2^{(3)}.
\end{aligned}$$

Hence, the cohomology group is three-dimensional for $(\lambda_1,\lambda_2) = (0,-4)$, (0,0) and (-4,0), and two-dimensional otherwise.

(vii) The case when $\mu-\lambda-\lambda_2 = 6$

Any 1-cocycle should retains the following general form:

$$c(X,f_1,f_2) = \sum_{i+j+k=7,i\geq 1} c_{i,j,k}X^{(i)}f_1^{(j)}f_2^{(k)}dx^\mu$$

A straightforward computation shows that the above 1-cocycle condition reads:

$$\begin{array}{ll}
c_{1,6,0}(15+6\lambda_1)+c_{1,5,1}\lambda_2=0, & c_{1,6,0}(20+15\lambda_1)+c_{1,4,2}\lambda_2=0,\\
c_{1,6,0}(15+20\lambda_1)+c_{1,3,3}\lambda_2=0, & c_{1,6,0}(6+15\lambda_1)+c_{1,2,4}\lambda_2=0,\\
c_{1,6,0}(1+6\lambda_1)+c_{1,1,5}\lambda_2=0, & c_{1,6,0}\lambda+c_{1,0,6}\lambda_2=0,\\
c_{1,5,1}(10+5\lambda_1)+c_{1,4,2}(1+2\lambda_2)=0, & c_{1,5,1}(10+10\lambda_1)+c_{1,3,3}(1+3\lambda_2)=0,\\
c_{1,5,1}(5+10\lambda_1)+c_{1,2,4}(1+4\lambda_2)=0, & c_{1,5,1}(1+5\lambda_1)+c_{1,1,5}(1+5\lambda_2)=0,\\
c_{1,5,1}\lambda_1+c_{1,0,6}(1+6\lambda_2)=0, & c_{1,4,2}(6+4\lambda_1)+c_{1,3,3}(3+3\lambda_2)=0,\\
c_{1,4,2}(4+6\lambda_1)+c_{1,2,4}(4+6\lambda_2)=0, & c_{1,4,2}(1+4\lambda_1)+c_{1,1,5}(5+10\lambda_2)=0,\\
c_{1,4,2}\lambda_1+c_{1,0,6}(6+15\lambda_2)=0, & c_{1,3,3}(3+3\lambda_1)+c_{1,2,4}(6+4\lambda_2)=0,\\
c_{1,3,3}(1+3\lambda_1)+c_{1,1,5}(10+10\lambda_2)=0, & c_{1,3,3}\lambda_1+c_{1,0,6}(15+20\lambda_2)=0,\\
c_{1,2,4}(1+2\lambda_1)+c_{1,1,5}(10+5\lambda_2)=0, & c_{1,2,4}\lambda_1+c_{1,0,6}(20+15\lambda_2)=0,\\
c_{1,1,5}\lambda_1+c_{1,0,6}(15+6\lambda_2)=0, & c_{2,5,0}(10+10\lambda_1)+c_{2,3,2}\lambda_2=0,\\
c_{2,5,0}(5+10\lambda_1)+c_{2,2,3}\lambda_2=0, & c_{2,5,0}(1+5\lambda_1)+c_{2,1,4}\lambda_2=0,\\
c_{2,5,0}\lambda_1+c_{2,0,5}\lambda_2=0, & c_{2,4,1}(4+6\lambda_1)+c_{2,2,3}(1+3\lambda_2)=0,\\
c_{2,4,1}(1+4\lambda_1)+c_{2,1,4}(1+4\lambda_2)=0, & c_{2,4,1}\lambda_1+c_{2,0,5}(1+5\lambda_2)=0,\\
c_{2,3,2}(1+3\lambda_1)+c_{2,1,4}(4+6\lambda_2)=0, & c_{2,3,2}\lambda_1+c_{2,0,5}(5+10\lambda_2)=0,\\
c_{2,2,3}\lambda_1+c_{2,0,5}(10+10\lambda_2)=0, & c_{3,4,0}(6+4\lambda_1)+c_{3,3,1}\lambda_2+2c_{4,3,0}=0,\\
c_{3,3,1}(3+3\lambda_1)+c_{3,2,2}(1+2\lambda_2)+2c_{4,2,1}=0, & c_{6,1,0}\lambda_1+c_{6,0,1}\lambda_2+14c_{7,0,0}=0,\\
c_{3,2,2}(1+2\lambda_1)+c_{3,1,3}(3+3\lambda_2)+2c_{4,1,2}=0, & c_{3,1,3}\lambda_1+c_{3,0,4}(6+4\lambda_2)+2c_{4,0,3}=0,\\
c_{4,3,0}(1+3\lambda_1)-c_{3,4,0}(1+4\lambda_1)+(c_{4,1,2}-c_{3,1,3})\lambda_2 & \\
+5c_{6,1,0}=0, & c_{4,3,0}(3+3\lambda_1)+c_{4,2,1}\lambda_2+5c_{4,0,3}=0,\\
(c_{3,3,1}-c_{4,2,1})\lambda_1-c_{3,0,4}(1+4\lambda_2)+c_{4,0,3}(1+3\lambda_2) & \\
+5c_{6,0,1}=0, & c_{4,1,2}\lambda+c_{4,0,3}(3+3\lambda_2)+5c_{5,0,2}=0,\\
c_{4,2,1}(1+2\lambda_1)+c_{4,1,2}(1+2\lambda_2)+5c_{5,1,1}=0, & c_{5,2,0}(1+2\lambda_1)+c_{5,1,1}\lambda_2+9c_{6,1,0}=0,\\
c_{5,2,0}\lambda_1+c_{5,0,2}\lambda_2-c_{3,4,0}\lambda_1-c_{3,0,4}\lambda_2+14c_{7,0,0}=0, & c_{5,1,1}\lambda_1+c_{5,0,2}(1+2\lambda_2)+9c_{6,0,1}=0.
\end{array}$$

The space of solutions of the system above is 10-dimensional for $(\lambda_1,\lambda_2)=(\frac{-5\pm\sqrt{19}}{2},0),(0,\frac{-5\pm\sqrt{19}}{2}),(\frac{-5-\sqrt{19}}{2},\frac{-5+\sqrt{19}}{2}),(\frac{-5+\sqrt{19}}{2},\frac{-5-\sqrt{19}}{2})$ and 9-dimensional otherwise. Let us study the space $\mathrm{B}^1(\mathrm{V}ec(\mathbb{R}),\mathfrak{so}(2),\mathcal{D}_{\bar{\lambda},\mu})$.

Let us study the triviality of this 1-cocycle. A straightforward computation shows that

$$\begin{array}{rcl}
L_XT_6^{\bar{\lambda}}= & - & (\varepsilon_{6,0}(15+6\lambda_1)+\varepsilon_{5,1}\lambda_2)X''f_1^{(5)}f_2-(\varepsilon_{5,1}(10+5\lambda_1)+\varepsilon_{4,2}(1+2\lambda_2))X''f_1^{(4)}f_2'\\
& - & (\varepsilon_{4,2}(6+4\lambda_1)+\varepsilon_{3,3}(3+3\lambda_2))X''f_1'''f_2''-(\varepsilon_{3,3}(3+3\lambda_1)+\varepsilon_{2,4}(6+4\lambda_2))X''f_1''f_2'''\\
& - & (\varepsilon_{2,4}(6+4\lambda_1)+\varepsilon_{1,5}(10+5\lambda_2))X''f_1'f_2^{(4)}-(\varepsilon_{1,5}\lambda_1+\varepsilon_{0,6}(15+6\lambda_2))X''f_1f_2^{(5)}\\
& - & (\varepsilon_{6,0}(20+15\lambda_1)+\varepsilon_{4,2}\lambda_2)X'''f_1^{(4)}f_2-(\varepsilon_{5,1}(10+10\lambda_1)+\varepsilon_{3,3}(1+3\lambda_2))X'''f_1'''f_2'\\
& - & (\varepsilon_{4,2}(4+6\lambda_1)+\varepsilon_{2,4}(4+6\lambda_2))X'''f_1''f_2''-(\varepsilon_{3,3}(1+3\lambda_1)+\varepsilon_{1,5}(10+10\lambda_2))X'''f_1'f_2'''\\
& - & (\varepsilon_{2,4}\lambda_1+\varepsilon_{0,6}(20+15\lambda_2))X'''f_1f_2^{(4)}-(\varepsilon_{3,3}(15+20\lambda_1)+\varepsilon_{6,0}\lambda_2)X^{(4)}f_1'''f_2\\
& - & (\varepsilon_{5,1}(5+10\lambda_1)+\varepsilon_{2,4}(1+4\lambda_2))X^{(4)}f_1''f_2'-(\varepsilon_{1,5}(1+4\lambda_1)+\varepsilon_{4,2}(5+10\lambda_2))X^{(4)}f_1'f_2''\\
& - & (\varepsilon_{3,3}\lambda_1+\varepsilon_{0,6}(15+20\lambda_2))X^{(4)}f_1f_2'''-(\varepsilon_{2,4}(6+15\lambda_1)+\varepsilon_{6,0}\lambda_2)X^{(5)}f_1''f_2\\
& - & (\varepsilon_{1,5}(1+5\lambda_1)+\varepsilon_{5,1}(1+5\lambda_2))X^{(5)}f_1'f_2'-(\varepsilon_{4,2}\lambda_1+\varepsilon_{0,6}(6+15\lambda_2))X^{(5)}f_1f_2''\\
& - & (\varepsilon_{1,5}(1+6\lambda_1)+\varepsilon_{6,0}\lambda_2)X^{(6)}f_1'f_2-(\varepsilon_{5,1}\lambda_1+\varepsilon_{0,6}(1+6\lambda_2))Y^{(6)}f_1f_2'\\
& - & (\varepsilon_{6,0}\lambda_1+\varepsilon_{0,6}\lambda_2)X^{(7)}f_1f_2
\end{array}$$

A straightforward computation shows that, for all values of λ_1 and λ_2 the space $\mathrm{B}^1(\mathrm{V}ec(\mathbb{R}),\mathfrak{so}(2),\mathcal{D}_{\bar{\lambda},\mu})$ is 7-dimensional. Hence, the cohomology group is three-dimensional for $(\lambda_1,\lambda_2)=(\frac{-5\pm\sqrt{19}}{2},0),(0,\frac{-5\pm\sqrt{19}}{2})$, $(\frac{-5-\sqrt{19}}{2},\frac{-5+\sqrt{19}}{2}),(\frac{-5+\sqrt{19}}{2},\frac{-5-\sqrt{19}}{2})$ and two-dimensional otherwise.

(viii)The case when $\mu-\lambda-\lambda_2=7$

Any 1-cocycle should retains the following general form:

$$c(X,f_1,f_2)=\sum_{i+j+k=8,i\geq 1}c_{i,j,k}X^{(i)}f_1^{(j)}f_2^{(k)}dx^{\mu}$$

A straightforward computation shows that the above 1-cocycle condition reads:

$$
\begin{array}{ll}
c_{1,7,0}(21+7\lambda_1)+c_{1,6,1}\lambda_2=0, & c_{1,7,0}(35+21\lambda_1)+c_{1,5,2}\lambda_2=0,\\
c_{1,7,0}(35+355\lambda_1)+c_{1,4,3}\lambda_2=0, & c_{1,7,0}(21+35\lambda_1)+c_{1,3,4}\lambda_2=0,\\
c_{1,7,0}(7+21\lambda_1)+c_{1,2,5}\lambda_2=0, & c_{1,7,0}(1+7\lambda_1)+c_{1,1,6}\lambda_2=0,\\
c_{1,7,0}\lambda_1+c_{1,0,7}\lambda_2=0, & c_{1,6,1}(15+6\lambda_1)+c_{1,5,2}(1+2\lambda_2)=0,\\
c_{1,6,1}(20+15\lambda_1)+c_{1,4,3}(1+3\lambda_2)=0, & c_{1,6,1}(15+20\lambda_1)+c_{1,3,4}(1+4\lambda_2)=0,\\
c_{1,6,1}(6+15\lambda_1)+c_{1,2,5}(1+5\lambda_2)=0, & c_{1,6,1}(1+6\lambda_1)+c_{1,1,6}(1+6\lambda_2)=0,\\
c_{1,6,1}\lambda_1+c_{1,0,7}(1+7\lambda_2)=0, & c_{1,5,2}(10+5\lambda_1)+c_{1,4,3}(3+3\lambda_2)=0,\\
c_{1,5,2}(10+10\lambda_1)+c_{1,3,4}(4+6\lambda_2)=0, & c_{1,5,2}(5+10\lambda_1)+c_{1,2,5}(5+10\lambda_2)=0,\\
c_{1,5,2}(1+5\lambda_1)+c_{1,1,6}(6+15\lambda_2)=0, & c_{1,5,2}\lambda_1+c_{1,0,7}(7+21\lambda_2)=0,\\
c_{1,4,3}(6+4\lambda_1)+c_{1,3,4}(6+4\lambda_2)=0, & c_{1,4,3}(4+6\lambda_1)+c_{1,2,5}(10+10\lambda_2)=0,\\
c_{1,4,3}(1+4\lambda_1)+c_{1,1,6}(15+20\lambda_2)=0, & c_{1,4,3}\lambda_1+c_{1,0,7}(21+35\lambda_2)=0,\\
c_{1,3,4}(3+3\lambda_1)+c_{1,2,5}(10+5\lambda_2)=0, & c_{1,3,4}(1+3\lambda_1)+c_{1,1,6}(20+15\lambda_2)=0,\\
c_{1,3,4}\lambda_1+c_{1,0,7}(35+35\lambda_2)=0, & c_{1,2,5}(1+2\lambda_1)+c_{1,1,6}(15+6\lambda_2)=0,\\
c_{1,2,5}\lambda_1+c_{1,0,7}(35+21\lambda_2)=0, & c_{1,1,6}\lambda_1+c_{1,0,7}(21+7\lambda_2)=0,
\end{array}
$$

The space of solutions of the 1-cocycle condition is 10-dimensional for $(\lambda_1,\lambda_2)\in\{(0,\lambda_2),(\lambda,0),(-\lambda_2-6,\lambda_2),(\frac{-5-\sqrt{19}}{2},-1+\sqrt{19}),(\frac{-5+\sqrt{19}}{2},-1-\sqrt{19}),(\frac{-5+\sqrt{19}}{2},\frac{-5+\sqrt{19}}{2}),(\frac{-5-\sqrt{19}}{2},\frac{-5-\sqrt{19}}{2}),(-1-\sqrt{19},\frac{-5+\sqrt{19}}{2}),(-1+\sqrt{19},\frac{-5-\sqrt{19}}{2})\}$ and 9-dimensional otherwise.

Let us study the triviality of this 1-cocycle. A straightforward computation shows that

$$
\begin{array}{rcl}
L_XT_7^{\bar\lambda}= & - & (\varepsilon_{7,0}(21+7\lambda)+a_{6,1}\lambda_2)X''f^{(6)}g-(\varepsilon_{6,1}(15+6\lambda)+\varepsilon_{5,2}(1+2\lambda_2))_1f_2X''f_1^{(5)}f_2'\\
& - & (\varepsilon_{5,2}(10+5\lambda)+\varepsilon_{4,3}(3+3\lambda_2))X''f_1^{(4)}f_2''-(\varepsilon_{4,3}(6+4\lambda)+\varepsilon_{3,4}(6+4\lambda_2))_1f_2X''f_1'''f_2'''\\
& - & (\varepsilon_{3,4}(3+3\lambda)+\varepsilon_{2,5}(10+5\lambda_2))X''f_1''f_2^{(4)}-(\varepsilon_{2,5}(1+2\lambda)+\varepsilon_{1,6}(15+6\lambda_2))_1f_2X''f_1'f_2^{(5)}\\
& - & (\varepsilon_{1,6}\lambda+\varepsilon_{0,7}(21+7\lambda_2))X''f_1f_2^{(6)}-(\varepsilon_{7,0}(35+21\lambda)+\varepsilon_{5,2}\lambda_2)Y'''f_1^{(5)}f_2\\
& - & (\varepsilon_{6,1}(20+15\lambda)+\varepsilon_{4,3}(1+3\lambda_2))X'''f_1^{(4)}f_2'-(\varepsilon_{5,2}(10+10\lambda)+\varepsilon_{3,4}(4+6\lambda_2))X'''f_1'''f_2''\\
& - & (\varepsilon_{4,3}(4+6\lambda)+\varepsilon_{2,5}(10+10\lambda_2))X'''f_1''f_2'''-(\varepsilon_{3,4}(1+3\lambda)+\varepsilon_{1,6}(20+15\lambda_2))X'''f_1'f_2^{(4)}\\
& - & (\varepsilon_{2,5}\lambda+\varepsilon_{0,7}(35+21\lambda_2))X'''f_1f_2^{(5)}-(\varepsilon_{7,0}(35+35\lambda)+\varepsilon_{4,3}\lambda_2)X^{(4)}f(4)_1f_2\\
& - & (\varepsilon_{6,1}(15+20\lambda)+\varepsilon_{3,4}(1+4\lambda_2))X^{(4)}f_1'''f_2'-(\varepsilon_{5,2}(5+10\lambda)+\varepsilon_{2,5}(5+10\lambda_2))X^{(4)}f_1''f_2''\\
& - & (\varepsilon_{4,3}(1+4\lambda)+\varepsilon_{1,6}(15+20\lambda_2))X^{(4)}f_1'f_2'''-(\varepsilon_{3,4}\lambda+\varepsilon_{0,7}(35+35\lambda_2))X^{(4)}f_1f_2^{(4)}\\
& - & (\varepsilon_{7,0}(21+35\lambda)+\varepsilon_{3,4}\lambda_2)X^{(5)}f_1'''f_2-(\varepsilon_{6,1}(6+15\lambda)+\varepsilon_{2,5}(1+5\lambda_2))X^{(5)}f_1''f_2'\\
& - & (\varepsilon_{5,2}(1+5\lambda)+\varepsilon_{1,6}(6+15\lambda_2))X^{(5)}f_1'f_2''-(\varepsilon_{4,3}\lambda+\varepsilon_{0,7}(21+35\lambda_2))X^{(5)}f_1f_2'''\\
& - & (\varepsilon_{7,0}(7+21\lambda)+\varepsilon_{2,5}\lambda_2)X^{(6)}f_1''f_2-(\varepsilon_{6,1}(1+6\lambda)+\varepsilon_{1,6}(1+6\lambda_2))X^{(6)}f_1'f_2'\\
& - & (\varepsilon_{5,2}\lambda+\varepsilon_{0,7}(7+21\lambda_2))X^{(6)}f_1f_2''-(\varepsilon_{7,0}(1+7\lambda)+\varepsilon_{1,6}\lambda_2)X^{(7)}f_1'f_2\\
& - & (\varepsilon_{6,1}(1+7\lambda)+\varepsilon_{0,7}\lambda_2)X^{(7)}f_1f_2'-(\varepsilon_{7,0}\lambda+\varepsilon_{0,7}\lambda_2)X^{(8)}f_1f_2
\end{array}
$$

This equation is equivalent, for all values of λ_1 and λ_2 the space $\mathrm{B}^1(\mathrm{V}ec(\mathbb{R}),\mathfrak{so}(2),\mathcal{D}_{\lambda_1,\lambda_2,\mu})$ is 8-dimensional.

(ix)The case when $k\geq 8$

For $k=8$, the number of variables generating any 1-cocycle is much smaller than the number of equations coming out from the 1-cocycle condition, for generic $\bar\lambda$. This cohomology class is indeed trivial because the expression $X.T_k^{\bar\lambda}$ is also a 1-cocycle.

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