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New Approach to Fractional Milne-type Inequalities

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Abstract

In the literature, there are several Milne-type inequalities by using the convex functions. In this study, we establish the upper and below bounds for fractional Milne-type inequalities by using functions whose second derivatives are bounded instead of convex functions. Moreover, we present new inequalities for Riemann integrals as special cases. We also present an example and a graph to illustrate the main results.

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1. Introduction

The Hermite-Hadamard inequality, recognized as one of the foundational results concerning convex functions, possesses a clear geometric interpretation and has found numerous applications across various fields of mathematics. It has attracted significant attention within elementary mathematical analysis, prompting many researchers to explore generalizations, refinements, extensions, and analogous results for broader classes of functions, particularly through the lens of convexity.

The inequalities originally formulated by C. Hermite and J. Hadamard for convex functions hold a prominent place in the mathematical literature (see, for example, [12, p.137], [8]). These inequalities state that if $\varpi : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$\varpi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \varpi(\varkappa)d\varkappa \leq \frac{\varpi(a)+\varpi(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if ϖ is concave. Some recent papers on Hermite-Hadamrd inequalities, please refer to [1, 2, 14, 16]

In [6] and [7], Dragomir et al. proved the following results connected with the Hermite-Hadamard inequality:

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Theorem 1.1. *Let $\varpi : [a,b] \to \mathbb{R}$ be a twice differentiable mapping such that there exists real constants m and M so that $m \leq \varpi'' \leq M$. Then, the following inequalities hold:*

$$m\frac{(b-a)^2}{24} \leq \frac{1}{b-a}\int_a^b \varpi(\varkappa)d\varkappa - \varpi\left(\frac{a+b}{2}\right) \leq M\frac{(b-a)^2}{24}. \tag{1.2}$$

and

$$m\frac{(b-a)^2}{12} \leq \frac{\varpi(a)+\varpi(b)}{2} - \frac{1}{b-a}\int_a^b \varpi(\varkappa)d\varkappa \leq M\frac{(b-a)^2}{12} \tag{1.3}$$

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

Definition 1.2. Let $\varpi \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}\varpi$ and $J_{b-}^{\alpha}\varpi$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha}\varpi(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_a^{\varkappa} (\varkappa-\varrho)^{\alpha-1}\varpi(\varrho)d\varrho, \quad \varkappa > a$$

and

$$J_{b-}^{\alpha}\varpi(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_{\varkappa}^{b} (\varrho-\varkappa)^{\alpha-1}\varpi(\varrho)d\varrho, \quad \varkappa < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0}\varpi(\varkappa) = J_{b-}^{0}\varpi(\varkappa) = \varpi(\varkappa)$.

For more information about fraction calculus please refer to [9, 10, 11, 13]

In [15], Sarikaya et al. first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.3. *Let $\varpi : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$\varpi\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}\varpi(b) + J_{b-}^{\alpha}\varpi(a)\right] \leq \frac{\varpi(a)+\varpi(b)}{2} \tag{1.4}$$

with $\alpha > 0$.

Moreover, Dragomir give the following another version of Hermite-Hadamard inequality for Riemann-Lioville fractional integrals:

Theorem 1.4. *[5] Let $\varpi : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} \varpi\left(\frac{a+b}{2}\right) &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}\varpi\left(\frac{a+b}{2}\right) + J_{b-}^{\alpha}\varpi\left(\frac{a+b}{2}\right)\right] \\ &\leq \frac{\varpi(a)+\varpi(b)}{2} \end{aligned} \tag{1.5}$$

Theorem 1.5. *[3] Let $\varpi : [a,b] \to \mathbb{R}$ be a twice differentiable function with $a < b$ and $\varpi \in L_1[a,b]$. If ϖ'' is bounded, i.e. $m \leq \varpi''(\varrho) \leq M$, $\varrho \in (a,b)$, $m, M \in \mathbb{R}$, then we have the inequalities*

$$\begin{aligned} &\frac{m(b-a)^2}{8(\alpha+2)} \\ \leq\ &\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}\varpi\left(\frac{a+b}{2}\right) + J_{b-}^{\alpha}\varpi\left(\frac{a+b}{2}\right)\right] - \varpi\left(\frac{a+b}{2}\right) \\ \leq\ &\frac{M(b-a)^2}{8(\alpha+2)} \end{aligned}$$

and

$$\begin{aligned}
&\frac{m(b-a)^2}{4(\alpha+2)} \\
\leq\ & \frac{\varpi(a)+\varpi(b)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}\varpi\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}\varpi\left(\frac{a+b}{2}\right)\right] \\
\leq\ & \frac{M(b-a)^2}{4(\alpha+2)}.
\end{aligned}$$

Now we present some Milne-type inequalities:

Theorem 1.6. *[4]Let $\varpi:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) such that $\varpi\in L_1([a,b])$. If the function $|\varpi'|$ is convex on $[a,b]$, then we have the following Milne-type inequality*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\varpi(a)-\varpi\left(\frac{a+b}{2}\right)+2\varpi(b)\right]-\frac{1}{b-a}\int_a^b\varpi(\varrho)d\varrho\right| \\
\leq\ & \frac{5(b-a)}{24}\left[\left|\varpi'(a)\right|+\left|\varpi'(b)\right|\right].
\end{aligned}$$

Theorem 1.7. *[4] Assume that the assumptions of Theorem 1.6. Then we have*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\varpi(a)-\varpi\left(\frac{a+b}{2}\right)+2\varpi(b)\right]\right. \\
&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}\varpi\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}\varpi\left(\frac{a+b}{2}\right)\right]\right| \\
\leq\ & \frac{\alpha+4}{12(\alpha+1)}(b-a)\left[\left|\varpi'(a)\right|+\left|\varpi'(b)\right|\right],
\end{aligned}$$

for $\alpha>0$.

2. Main Results

In this section, we will prove an upper and below bound for the difference given in Theorem 1.7 by using the functions whose second derivatives are bounded.

Theorem 2.1. *Let $\varpi:[a,b]\to\mathbb{R}$ be a twice differentiable function with $a<b$ and $\varpi\in L_1[a,b]$. If ϖ'' is bounded, i.e. $m\leq\varpi''(\varrho)\leq M$, $\varrho\in(a,b)$, $m,M\in\mathbb{R}$, then we have the inequalities*

$$\begin{aligned}
&\frac{\alpha+8}{8(\alpha+2)}(b-a)^2 m \qquad (2.1) \\
\leq\ & \frac{1}{3}\left[2\varpi(a)-\varpi\left(\frac{a+b}{2}\right)+2\varpi(b)\right] \\
&-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}\varpi\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}\varpi\left(\frac{a+b}{2}\right)\right] \\
\leq\ & \frac{\alpha+8}{8(\alpha+2)}(b-a)^2 M.
\end{aligned}$$

Proof. From the definition of Riemann-Liouville fractional integrals, we get

$$\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}\varpi\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}\varpi\left(\frac{a+b}{2}\right)\right] \qquad (2.2)\\
=&\ \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\left[\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-\varkappa\right)^{\alpha-1}\varpi(\varkappa)d\varkappa+\int_{\frac{a+b}{2}}^{b}\left(\varkappa-\frac{a+b}{2}\right)^{\alpha-1}\varpi(\varkappa)d\varkappa\right]\\
=&\ \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_{a}^{\frac{a+b}{2}}[\varpi(\varkappa)+\varpi(a+b-\varkappa)]\left(\frac{a+b}{2}-\varkappa\right)^{\alpha-1}d\varkappa.
\end{aligned}$$

Using the identity (2.2), we get

$$\begin{aligned}
&\frac{1}{3}\left[2\varpi(a)-\varpi\left(\frac{a+b}{2}\right)+2\varpi(b)\right] \qquad (2.3)\\
&-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}\varpi\left(\frac{a+b}{2}\right)+J_{b-}\varpi\left(\frac{a+b}{2}\right)\right]\\
=&\ \frac{1}{3}\left[2\varpi(a)-\varpi\left(\frac{a+b}{2}\right)+2\varpi(b)\right]\\
&-\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_{a}^{\frac{a+b}{2}}[\varpi(\varkappa)+\varpi(a+b-\varkappa)]\left(\frac{a+b}{2}-\varkappa\right)^{\alpha-1}d\varkappa\\
=&\ \frac{2^{\alpha-1}\alpha}{3(b-a)^{\alpha}}\int_{a}^{\frac{a+b}{2}}\left[\begin{array}{c}4\varpi(a)-2\varpi\left(\frac{a+b}{2}\right)+4\varpi(b)\\ -3\varpi(\varkappa)-3\varpi(a+b-\varkappa)\end{array}\right]\left(\frac{a+b}{2}-\varkappa\right)^{\alpha-1}d\varkappa.
\end{aligned}$$

Using the facts that

$$\varpi(\varkappa)-\varpi(a)=\int_{a}^{\varkappa}\varpi'(\varrho)d\varrho$$

and

$$\varpi(b)-\varpi(a+b-\varkappa)=\int_{a+b-\varkappa}^{b}\varpi'(\varrho)d\varrho,$$

we get

$$\begin{aligned}
&\varpi(a)+\varpi(b)-\varpi(\varkappa)-\varpi(a+b-\varkappa) \qquad (2.4)\\
=&\int_{a+b-\varkappa}^{b}\varpi'(\varrho)d\varrho-\int_{a}^{\varkappa}\varpi'(\varrho)d\varrho=\int_{a}^{\varkappa}\varpi'(a+b-u)du-\int_{a}^{\varkappa}\varpi'(\varrho)d\varrho=\int_{a}^{\varkappa}\left[\varpi'(a+b-\varrho)-\varpi'(\varrho)\right]d\varrho.
\end{aligned}$$

On the other hand, we have

$$\varpi(a)-\varpi\left(\frac{a+b}{2}\right)=-\int_{a}^{\frac{a+b}{2}}\varpi'(\varrho)d\varrho \qquad (2.5)$$

and

$$\varpi(b)-\varpi\left(\frac{a+b}{2}\right)=\int_{\frac{a+b}{2}}^{b}\varpi'(\varrho)d\varrho. \tag{2.6}$$

By (2.5) and (2.6), we get

$$\begin{aligned}
&\varpi(a)+\varpi(b)-2\varpi\left(\frac{a+b}{2}\right) \qquad (2.7)\\
&= \int_{\frac{a+b}{2}}^{b}\varpi'(\varrho)d\varrho-\int_{a}^{\frac{a+b}{2}}\varpi'(\varrho)d\varrho=\int_{a}^{\frac{a+b}{2}}\varpi'(a+b-\varrho)d\varrho-\int_{a}^{\frac{a+b}{2}}\varpi'(\varrho)d\varrho=\int_{a}^{\frac{a+b}{2}}\left[\varpi'(a+b-\varrho)-\varpi'(\varrho)\right]d\varrho.
\end{aligned}$$

We also have

$$\varpi'(a+b-\varrho)-\varpi'\left(\varrho\right)=\int_{\varrho}^{a+b-\varrho}\varpi''(u)du. \tag{2.8}$$

By using the equality (2.8) and $m<\varpi''(u)<M$, $u\in(a,b)$, we obtain,

$$m\int_{\varrho}^{a+b-\varrho}du\le\int_{\varrho}^{a+b-\varrho}\varpi''(u)du\le M\int_{\varrho}^{a+b-\varrho}du$$

i.e.

$$m\left(a+b-2\varrho\right)\le\varpi'(a+b-\varrho)-\varpi'(\varrho)\le M\left(a+b-2\varrho\right). \tag{2.9}$$

By integrating the inequality (2.9) with respect to ϱ on $[a,\varkappa]$, from (2.4), we get

$$\begin{aligned}
&m\left[\left(\frac{b-a}{2}\right)^2-\left(\frac{a+b}{2}-\varkappa\right)^2\right] \qquad (2.10)\\
\le\ & \varpi(a)+\varpi(b)-\varpi(\varkappa)-\varpi(a+b-\varkappa)\\
\le\ & M\left[\left(\frac{b-a}{2}\right)^2-\left(\frac{a+b}{2}-\varkappa\right)^2\right].
\end{aligned}$$

That is,

$$\begin{aligned}
&m\left(\varkappa-a\right)\left(b-\varkappa\right) \qquad (2.11)\\
\le\ & \varpi(a)+\varpi(b)-\varpi(\varkappa)-\varpi(a+b-\varkappa)\\
\le\ & M\left(\varkappa-a\right)\left(b-\varkappa\right).
\end{aligned}$$

Similarly, integrating the inequality (2.9) with respect to ϱ on $\left[a,\frac{a+b}{2}\right]$, from (2.7), we obtain

$$m\left(\frac{b-a}{2}\right)^2\le\varpi(a)+\varpi(b)-2\varpi\left(\frac{a+b}{2}\right)\le M\left(\frac{b-a}{2}\right)^2. \tag{2.12}$$

By using the inequalities (2.11) and (2.12), we get

$$\begin{aligned} & 3m\left(\varkappa-a\right)\left(b-\varkappa\right)+m\left(\frac{b-a}{2}\right)^2 \qquad (2.13)\\ \leq\ & 4\varpi(a)-2\varpi\left(\frac{a+b}{2}\right)+4\varpi(b)-3\varpi(\varkappa)-3\varpi(a+b-\varkappa)\\ \leq\ & 3M\left(\varkappa-a\right)\left(b-\varkappa\right)+M\left(\frac{b-a}{2}\right)^2. \end{aligned}$$

Multiplying the inequality (2.13) by $\frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\left(\frac{a+b}{2}-\varkappa\right)^{\alpha-1}$ and integrating the resultant inequality with respect to $\varkappa$ on $\left[a,\frac{a+b}{2}\right]$, we establish

$$\begin{aligned} & \frac{m2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}}\left[3\left(\varkappa-a\right)\left(b-\varkappa\right)+\left(\frac{b-a}{2}\right)^2\right]\left(\frac{a+b}{2}-\varkappa\right)^{\alpha-1}d\varkappa\\ \leq\ & \frac{2^{\alpha-1}\alpha}{3\left(b-a\right)^{\alpha}}\int_a^{\frac{a+b}{2}}\left[\begin{array}{c}4\varpi(a)-2\varpi\left(\frac{a+b}{2}\right)+4\varpi(b)\\ -3\varpi(\varkappa)-3\varpi(a+b-\varkappa)\end{array}\right]\left(\frac{a+b}{2}-\varkappa\right)^{\alpha-1}d\varkappa\\ \leq\ & \frac{M2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}}\left[3\left(\varkappa-a\right)\left(b-\varkappa\right)+\left(\frac{b-a}{2}\right)^2\right]\left(\frac{a+b}{2}-\varkappa\right)^{\alpha-1}d\varkappa. \end{aligned}$$

By the equality (2.3) and equality

$$\begin{aligned} & \frac{2^{\alpha-1}\alpha}{(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}}\left[3\left(\varkappa-a\right)\left(b-\varkappa\right)+\left(\frac{b-a}{2}\right)^2\right]\left(\frac{a+b}{2}-\varkappa\right)^{\alpha-1}d\varkappa\\ =\ & \frac{\alpha+8}{8(\alpha+2)}\left(b-a\right)^2, \end{aligned}$$

we get the desired inequality (2.1).

This completes the proof. □

Remark 2.2. If we choose $\alpha=1$ in Theorem 2.1, then we have the following inequality

$$\begin{aligned} & \frac{3}{8}\left(b-a\right)^2m\\ \leq\ & \frac{1}{3}\left[2\varpi\left(a\right)-\varpi\left(\frac{a+b}{2}\right)+2\varpi\left(b\right)\right]-\frac{1}{b-a}\int_a^b\varpi\left(\varrho\right)d\varrho\\ \leq\ & \frac{3}{8}\left(b-a\right)^2M. \end{aligned}$$

This is a new contribution for the literature. This Milne inequality holds for the functions whose second derivatives are bounded.

Example 2.3. Let consider the function $\varpi:[-1,2]\to\mathbb{R}$ defined by $\varpi\left(\varrho\right)=\varrho^5$. It is clear that $\varpi''(\varrho)=20\varrho^3$ which is not convex on $[-1,2]$ but bounded such that $-20\leq\varpi''(\varrho)\leq160$. Therefore we have the left hand side and right hand side of the inequality (2.1) as

$$\frac{\alpha+8}{8(\alpha+2)}\left(b-a\right)^2m=-\frac{45\left(\alpha+8\right)}{2(\alpha+2)}$$

and

$$\frac{\alpha+8}{8(\alpha+2)}(b-a)^2 M = \frac{180\,(\alpha+8)}{(\alpha+2)}$$

respectively. We also have

$$\frac{1}{3}\left[2\varpi(a) - \varpi\left(\frac{a+b}{2}\right) + 2\varpi(b)\right] = \frac{1}{3}\left[62 - \frac{1}{32}\right] = \frac{661}{32}.$$

By definition of Riemann-Liouville fractional integrals, we have

$$J^{\alpha}_{a+}\varpi\left(\frac{a+b}{2}\right) = \frac{1}{\Gamma(\alpha)}\int_{-1}^{\frac{1}{2}}\left(\frac{1}{2}-\varrho\right)^{\alpha-1}\varrho^5 d\varrho$$
$$= \frac{1}{2^{\alpha+5}\Gamma(\alpha)}\left[\frac{3^{\alpha}}{\alpha} - \frac{5\cdot 3^{\alpha+1}}{\alpha+1} + \frac{10\cdot 3^{\alpha+2}}{\alpha+2} - \frac{10\cdot 3^{\alpha+3}}{\alpha+3} + \frac{5\cdot 3^{\alpha+4}}{\alpha+4} - \frac{3^{\alpha+5}}{\alpha+5}\right]$$

and

$$J^{\alpha}_{b-}\varpi\left(\frac{a+b}{2}\right) = \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{2}}^{2}\left(\varrho-\frac{1}{2}\right)^{\alpha-1}\varrho^5 d\varrho$$
$$= \frac{1}{2^{\alpha+5}\Gamma(\alpha)}\left[\frac{3^{\alpha}}{\alpha} + \frac{5\cdot 3^{\alpha+1}}{\alpha+1} + \frac{10\cdot 3^{\alpha+2}}{\alpha+2} + \frac{10\cdot 3^{\alpha+3}}{\alpha+3} + \frac{5\cdot 3^{\alpha+4}}{\alpha+4} + \frac{3^{\alpha+5}}{\alpha+5}\right].$$

Thus, the midterm of the inequality (2.1) becomes

$$\frac{1}{3}\left[2\varpi(a) - \varpi\left(\frac{a+b}{2}\right) + 2\varpi(b)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}\varpi\left(\frac{a+b}{2}\right) + J_{b-}\varpi\left(\frac{a+b}{2}\right)\right]$$
$$= \frac{661}{32} - \frac{\alpha}{32}\left[\frac{1}{\alpha} + \frac{90}{\alpha+2} + \frac{405}{\alpha+4}\right].$$

From the inequality (2.1), we have

$$-\frac{45\,(\alpha+8)}{2(\alpha+2)} \le \frac{661}{32} - \frac{\alpha}{32}\left[\frac{1}{\alpha} + \frac{90}{\alpha+2} + \frac{405}{\alpha+4}\right] \le \frac{180\,(\alpha+8)}{(\alpha+2)}. \tag{2.14}$$

One can see the validity of the inequality (2.14) in Figure 1.

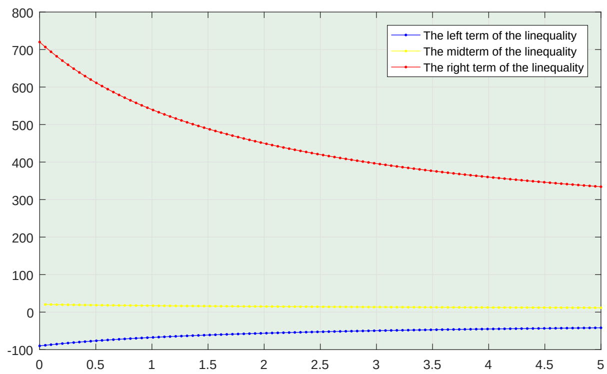

Figure 1: Comparison of the terms of the inequality (2.14)

3. Concluding Remarks

In this study, we obtained fractional Milne-type inequalities by using the functions whose second derivatives are bounded instead of convex functions. The method in this paper may applied for the any other type fractional integrals in the future works

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