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Numerical Computations of Fractional Differential Equations in Engineering using the Polynomial Least Squares Method

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Abstract

Fractional differential equations appear in the modeling of science and engineering phenomena including classical mechanics, quantum mechanics, thermodynamics, fluid mechanics, relativity theory and chemical engineering. In this paper, the least squares method is used to find the numerical solution of fractional differential equations appearing in mechanics and engineering. The increasing interest in applications of fractional calculus has motivated the development and study of numerical methods specifically created to solve fractional differential equations. A unique feature is that engineers, physicists and scientists come across processes which lead to involve fractional differential equations. When dealing with more complicated systems with no precise solutions other than approximations, numerical methods are relied on to obtain solutions. The suggested method is used to solve various linear and non-linear problems of constant-order. The proposed scheme is found to be computationally effective with fast convergence.

Keywords: Caputo Fractional Derivative, Fractional Calculus, Numerical Analysis, Fractional Differential Equations, Polynomial Least Squares Method.
2010 MSC: 26A33, 34A08, 65L05.

1. Introduction

The theory of arbitrary complex or real order derivatives and integrals is known as fractional calculus. The definition of derivative and integrals of any real or complex order can be found in fractional calculus, which is a generalization of ordinary calculus. Particularly when the dynamics is impacted by system-specific limitations, these fractional operators may more effectively simulate some real-world occurrences [1]. The classical derivative and integral have geometrical and physical interpretations, which is one of the

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primary advantages of using classical calculus and its applications. It started in 1695 on September 30, when G.F.A. L' Hospital asked Gottfried Wilhelm Leibniz to explain the significance of of $\frac{d^n y}{dx^n}$, where $n = \frac{1}{2}$. Subsequently, in 1832, J. Liouville proposed the first Liouville definition and was based on the equation for differentiating the exponential function [2]. In recent years, a considerable number of academics have been interested in fractional differential equations (FDEs). Many mathematical representations of actual issues have been created with the use of fractional calculus in a variety of engineering and scientific domains, including electromagnetic waves, electrode-electrolyte polarization, viscoelasticity, and dielectric polarization. As solving fractional differential equations analytically presents so many challenges, numerical methods are often a more practical way to get solutions [3, 29].

Due to their capacity to simulate complicated processes, fractional differential equations have garnered a great deal of attention. The numerous applications of FDEs in engineering and science have significantly increased the use of fractional derivatives in modeling processes [4, 22, 23]. Fractional differential equations are studied in a variety of domains, including biomathematics, plasma physics, control systems, mathematical biology, elasticity, quantum mechanics, fluid mechanics, optics, bioengineering, complex systems, and others. Contrary to understand Newtonian derivatives, fractional derivatives can be defined in a variety of ways, and even for smooth functions, various definitions typically do not all lead to the same conclusion. There are numerous different fractional derivative operators in fractional calculus, such as Grunwald-Letnikov fractional derivative, the Caputo fractional derivative, the Caputo-Fabrizio fractional derivative, the Riemann-Liouville fractional derivative, Atangana-Baleanu fractional derivative, Hilfer fractional derivative, and many others [5, 28]. The corresponding derivatives are found by applying Langrange method for fractional operators. The order derivative is calculated by computing the n^{th} order derivative over the order integral $(n - \alpha)$.

Fractional derivatives are important for modeling phenomena in various fields of science and engineering, due to their non-locality, which is an inherent characteristic of many complex systems. FDEs are essential for accurately modeling systems with memory and hereditary characteristics, which are prevalent in many physical and engineering applications. However, their complex nature often makes analytical solutions intractable [26, 27]. The fractional differential equations do not, in general, have accurate solutions. Instead, the importance of analytical and numerical approaches grows as they are used to solve fractional differential equations [6]. There have been a lot of effective ways to solve FDEs created recently. These techniques include fixed point theorems, topological degree theory, and monotone iterative methodology. Additionally, numerical solutions are achieved using Adomian decomposition approach, the variational iteration method, the homotopy perturbation method, Haar wavelet an operational method, neural networks, and other techniques. A basic statistical procedure known as the least-squares method is used to determine a regression line, or the line that best fits the given pattern. In other words, the least square method reduces the difference between the actual data and the anticipated values to identify the line or curve that best matches the data. The parameters of the model, that explain the association between two variables, are typically estimated using this technique in linear regression analysis. The least square approach is popular across many academic disciplines, including economics, finance, engineering, and the social sciences, to mention a few [7]. The least squares method offers a reliable and efficient numerical alternative, capable of producing high-accuracy approximations while maintaining computational simplicity. By applying this method to FDEs, our study contributes a practical and versatile approach that enhances the toolbox of available techniques for addressing complex real-world problems [24, 25].

It can be used for both basic and multiple regression analysis. In assessment and regression, the method of least squares is frequently utilized. This technique is reportedly a conventional method in regression analysis for approximating sets of equations where the number of equations exceeds the number of unknowns. The minimizing of the sum of squares of deviations, or the errors in solutions of each equation, is really defined with the least squares approach. Overall, the least square approach offers an effective way for numerically resolving fractional differential equations, even in the absence of closed-form solutions [8].

For the purpose of handling fractional derivatives, it modifies the conventional least squares method [9, 10, 11]. The main goal of the process is to reduce the sum of squares representing the variance between the observed data and the values that were predicted. When dealing with fractional equations, fractional derivatives rather than regular derivatives are used to derive the expected values.
The rest of the paper is arranged as follows: In Section 2, we present essential definitions and preliminaries related to fractional calculus. Section 3, introduces the proposed least squares method for solving fractional differential equations. In Section 4, numerical results are provided to demonstrate the accuracy and efficiency of the method. Finally, Section 5 concludes the paper with a summary of the findings and suggestions for future research.

2. Basic Definitions of Fractional Calculus

Definition 2.1. The Riemann-Liouville derivative [12] of order α is defined as:

$$ {}_{a}^{RL}D_t^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^t (t-v)^{n-\alpha-1} g(v) dv. $$

Definition 2.2. Let g be a continuous function from a set of positive real numbers to real line, i.e. $g : [0, +\alpha] \to \mathbb{R}$, then the fractional operator of Riemann-Liouville is given as:

$$ {}_{0}^{RL}D_t^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^t (t-v)^{n-\alpha-1} g(v) dv, \quad \alpha \in (0,1), t > 0. $$

Definition 2.3. The fractional Caputo derivative [12] of order α is defined as:

$$ {}_{a}^{C}D_t^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-v)^{n-\alpha-1} \left(\frac{d}{dx}\right)^n g(v) dv. $$

Definition 2.4. For a continuous function g from set of positive real numbers to real line, i.e. $g : [0, +\alpha] \to \mathbb{R}$, the Caputo fractional derivative is given as:

$$ {}_{0}^{C}D_t^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-v)^{n-\alpha-1} \left(\frac{d}{dx}\right)^n g(v) dv, \quad \alpha \in (0,1), t > 0. $$

3. Numerical Method Description

The Polynomial Least Squares Method (PLSM) enables us to find polynomial based approximate analytical solutions to the fractional differential equations. The research and development of numerical methods particularly designed to solve fractional differential equations (FDEs) have been inspired by the growing interest in applications of fractional calculus. Least Square Method is effective for obtaining numerical solutions and can be used to simulate models involving fractional differential equations (FDEs) numerically. Additionally, this approach can be used to a wide range of issues, saving time and money when dealing with challenging equations.
For the case of fractional differential equations, we have here extended the least squares method. We have solved some problems by using the least square method. The Caputo derivative is preferred in this work due to its compatibility with classical initial conditions. This makes it more useful for real-world applications in fractional differential equations.
The proposed scheme is highly effective but it has few limitations. It is not effective for solutions with singularities or discontinuities. Additionally, it is not suitable for irregular or multi-dimensional geometries and fails to construct appropriate approximations.

Consider a fractional differential equation (FDE) of the form:

$$D^{\alpha}u(t) + u(t) + f(t) = 0, \tag{3.1}$$

with the conditions:

$$u(t) = 0, \qquad u(1) = 0, \qquad 0 < \alpha \leq 1,$$

using the approximation:

$$u_{app} = \sum_{i=1}^{n} C_i \phi_i. \tag{3.2}$$

Here C_i represents the constants, and $\phi_i = \phi_i(t)$, $i = 1, 2, ..., n$ is test function. We will introduce an operator L for calculations:

$$L[u_{app}] = D^{\alpha}u_{app} + u_{app}.$$

Now, we will define the functional I:

$$I[C_1, C_2, \ldots, C_n] = \frac{1}{\Gamma(\alpha+1)} \int_0^1 \left[L[u_{app}] + f(t)\right]^2 (dt)^{\alpha} \to \min, \tag{3.3}$$

By minimization, which gives a system of equations in $C_1, ..., C_n$:

$$\frac{\partial I[C_1, C_2, \ldots, C_n]}{\partial C_i} = 0, \quad i = 1, 2, \ldots n.$$

from which we obtain the constants $C_1, \ldots, C_n$.
These constants are used in (3.2) to complete the process of getting approximate solution.

4. Applications

In this section, we explore few applications using the proposed methodology.
Example 4.1: As the first example, Consider a multi-order fractional differential equation (FDE) [13, 14]:

$$D^{\alpha_1}u(t) - D^{\alpha_2}u(t) + e^{t-1} + 1 = 0,$$

with boundary conditions:

$$u(0) = 0, \quad u(1) = 0.$$

Solution: The exact solution of this multi-order fractional differential equation is:

$$u(t) = t(1 - e^{t-1}).$$

To solve this problem, we suppose the value of $\alpha_1 = 2$ and $\alpha_2 = 1/2$,

$$D^2u(t) - D^{\frac{1}{2}}u(t) + e^{t-1} + 1 = 0.$$

By using the polynomial least square method, let us consider the approximate solution:

$$u_{app}(t) = C_0 + C_1 t + C_2 t^2.$$

By calculations, we obtain the approximate solution:

$$\int_0^1 \left(-2C_1 - C_1 D^{\frac{1}{2}} t + C_1 D^(1/2) t^2 + e^{t-1} + 1\right) \left(-2 - \frac{2t}{\sqrt{\pi}} + \frac{8t^2}{3\sqrt{\pi}}\right) (dt)^{\frac{1}{2}} = 0,$$

$$u(t) = 0.375t - 0.375t^2.$$

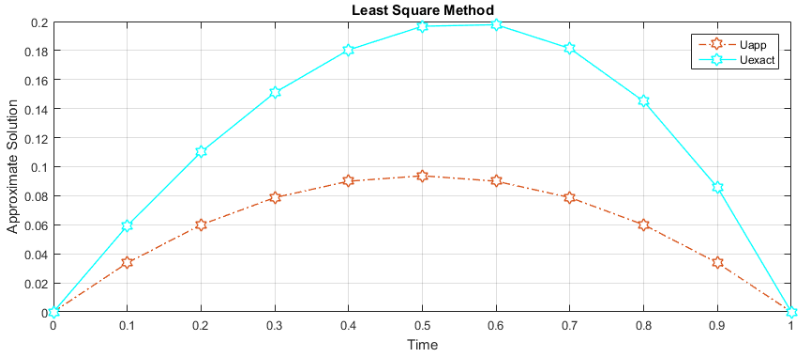

Figure 1: Numerical comparison of exact and approximate solution with LSM for Example 4.1, taking $\alpha_1 = 2$, $\alpha_2 = 0.5$

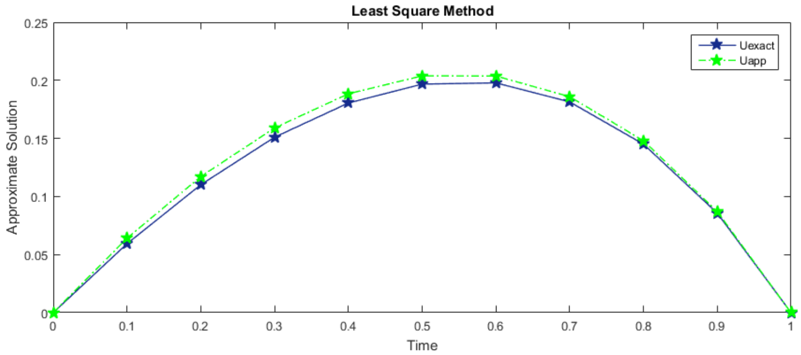

Figure 2: Numerical comparison of exact and approximate solution with LSM for Example 4.1, taking $\alpha_1 = 1.9$, $\alpha_2 = 0.9$

The calculations are performed using MATLAB.
In example 4.1, we initially considered $\alpha_1 = 2$, $\alpha_2 = 0.5$ and produced approximations of the solutions that is provided below, along with a MATLAB figure visible in table 1 and figure 1.

Table 1 Numerical results of Example 4.1 for $\alpha_1 = 2$, $\alpha_2 = 0.5$

T	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
u_{exact}	0.0593	0.1121	0.1532	0.1821	0.1973	0.2000	0.1812	0.1443	0.0852	0.000
u_{app}	0.0346	0.0605	0.0756	0.0895	0.0931	0.0889	0.0756	0.0593	0.0328	0.000
$Absol.Error$	0.0247	0.0516	0.0776	0.0936	0.1042	0.1111	0.1056	0.0850	0.0534	0.000

Now we choose $\alpha_1 = 1.9$, $\alpha_2 = 0.9$ for example 4.1 and produce approximations to the solutions shown in table 2 and figure 2.

Table 2 Numerical results of Example 4.1 for $\alpha_1 = 1.9$, $\alpha_2 = 0.9$

T	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
u_{exact}	0.0679	0.1092	0.1421	0.1793	0.1960	0.1960	0.1783	0.1391	0.0842	0.000
u_{app}	0.0634	0.1181	0.1591	0.1852	0.2030	0.2030	0.1864	0.1472	0.0851	0.0019
$Absol.Error$	0.0045	0.0089	0.0170	0.0059	0.0070	0.0070	0.0081	0.0081	0.0009	0.0019

We vary fractional order further for example 4.1 and take $\alpha_1 = 2$, $\alpha_2 = 1$. The approximation results are given in table 3 and figure 3. Here we note that by taking $\alpha_1 = 2$ and $\alpha_2 = 1$, the solution curves

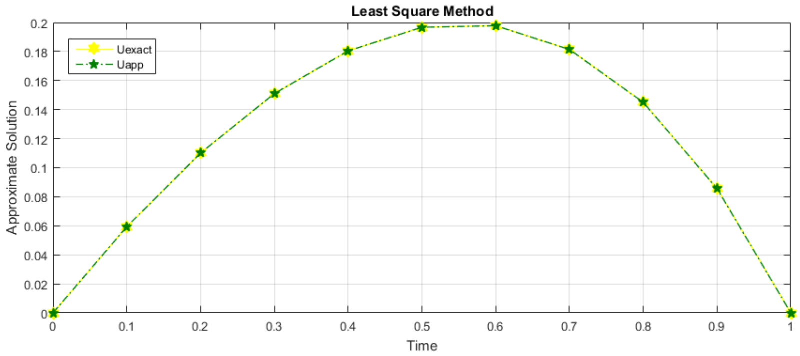

Figure 3: Numerical comparison of exact and approximate solution with LSM for Example 4.3

coincide, indicating that there is no error in approximate solution. Further, table 4 shows comparison of the proposed method approximations with other methods, i.e. Homotopy Perturbation Method(HPM) [19], Haar wavelet Operational Matrix Method(HWOM) [20].

Table 3 Numerical results of Example 4.1 for $\alpha_1 = 2$, $\alpha_2 = 1$

T	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
u_{exact}	0.0593	0.1101	0.1510	0.1804	0.1967	0.1978	0.1814	0.1450	0.0856	0.0000
u_{app}	0.0593	0.1101	0.1510	0.1804	0.1967	0.1978	0.1814	0.1450	0.0856	0.0000
$Absol.Error$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 4 Numerical Comparison of solutions for Example 4.1, taking $\alpha_1 = 2$, $\alpha_2 = 1$

T	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900
u_{exact}	0.0593	0.1101	0.1510	0.1804	0.1967	0.1978	0.1814	0.1450	0.0856
u_{app}	0.0593	0.1101	0.1510	0.1804	0.1967	0.1978	0.1814	0.1450	0.0856
$HWOM$	0.05934	0.11013	0.15102	0.18047	0.19673	0.19673	0.18142	0.14501	0.08564
HPM	0.05934	0.11014	0.15105	0.18048	0.19673	0.19780	0.18142	0.14501	0.08564

Example 4.2: Consider the following fractional differential equation (FDE) [15]:

$$D^{\frac{1}{2}}u(t) + u(t) + u^2(t) = f(t), 0 \leq t \leq 1,$$

with boundary conditions:

$$u(0) = 0,$$

where

$$f(t) = \frac{8t^{3/2}}{3\sqrt{\pi}} + t^2 + t^4.$$

Solution: The exact solution of this problem is

$$u(t) = t^2.$$

$$D^{\frac{1}{2}}u(t) + u(t) + u^2(t) = \frac{8t^{3/2}}{3\sqrt{\pi}} + t^2 + t^4.$$

By using the polynomial least square method, let us consider the approximate solution:

$$u_{app}(t) = C_0 + C_1 t + C_2 t^2,$$

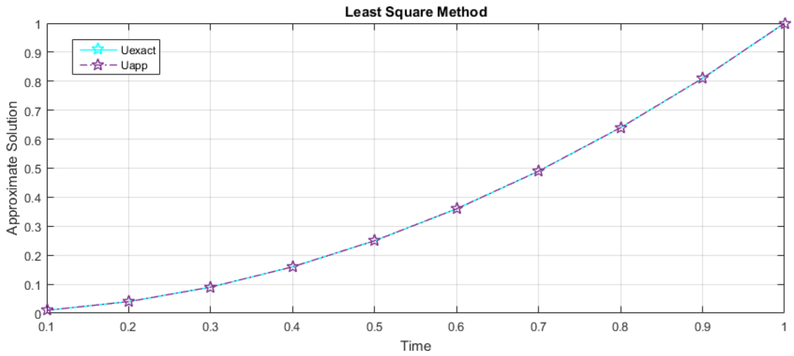

Figure 4: Numerical comparison of exact and approximate solution with LSM for Example 4.2

By using initial condition, we obtain $C_0 = 0$. So, the approximate solution will be:

$$u(t) = C_1 t + C_2 t^2.$$

In this case the corresponding remainder (3.3) is:

$$I[C_1, C_2] = \frac{1}{\Gamma\left(\frac{1}{2}+1\right)} \int_0^1 \left(\frac{2C_1}{\sqrt{\pi}} t^{1/2} + \frac{8C_2}{3\sqrt{\pi}} t^{3/2} + C_1 t + C_2 t^2 + C_1^2 t^2 + C_2^2 t^4 + 2C_1C_2 t^3 - \frac{8t^{3/2}}{3\sqrt{\pi}} - t^2 - t^4 \right)^2 (dt)^{\frac{1}{2}}$$

By solving the above equation, we get the values of constants: $C_1 = 0$ and $C_2 = 1$.
The calculations are performed using MATLAB.

In example 4.2, we selected $\alpha = 0.5$ and computed an exact solution that is provided below, along with a MATLAB figure. By using MATLAB, we compute the associated values and impose the necessary requirements for minimum. These computations can be done precisely (not numerically). We again get the values $d_1 = 0$ and $d_2 = 1$ for the minimum, giving us the exact solution of the problem. The results are presented in table 5 and figure 4.

Table 5 Numerical results of Example 4.2 for $\alpha = 0.5$

T	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
u_{exact}	0.0100	0.0400	0.0900	0.1600	0.2500	0.3600	0.4900	0.6400	0.8100	1.0000
u_{app}	0.0100	0.0400	0.0900	0.1600	0.2500	0.3600	0.4900	0.6400	0.8100	1.0000
$Absol.Error$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Example 4.3: Let us consider another fractional differential equation (FDE) [16]:

$$D^\alpha u(t) + 1 - (1+\alpha)t = 0, 0 < \alpha \le 1,$$

with boundary conditions:

$$u(0) = 0, u(1) = 0.$$

Solution: The exact solution of this problem is,

$$u(t) = -\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+1)}.$$

Consider an approximate solution:

$$u_{app}(t) = +C_1\phi_1 + C_2\phi_2,$$

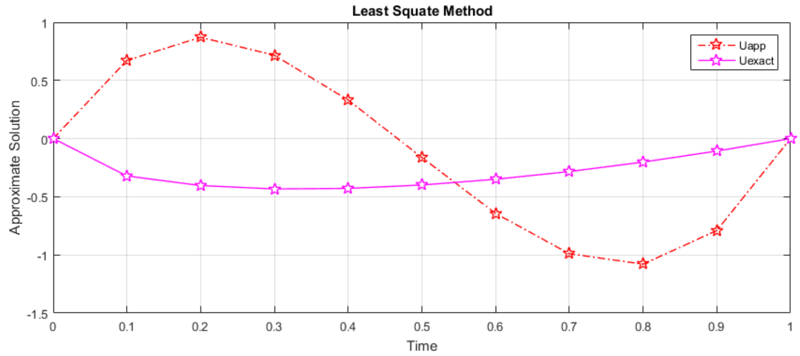

Figure 5: Numerical comparison of exact and approximate solution with LSM for Example 4.3

We obtain:

$$I[C_1, C_2] = \frac{1}{\Gamma(\alpha+1)} \int_0^1 \left[D^\alpha u_{app} + 1 - (1+\alpha)t\right]^2 (dt)^\alpha \to \min,$$

$$\frac{\partial I[C_1, C_2]}{\partial C_1} = 0, \frac{\partial I[C_1, C_2]}{\partial C_2} = 0.$$

$$\int_0^1 \left[C_1 D^\alpha \phi_1 + C_2 D^\alpha \phi_2 + 1 - (1+\alpha)t\right] D^\alpha \phi_1 (dt)^\alpha = 0,$$

$$\int_0^1 \left[C_1 D^\alpha \phi_1 + C_2 D^\alpha \phi_2 + 1 - (1+\alpha)t\right] D^\alpha \phi_2 (dt)^\alpha = 0,$$

and using the notation,

$$A_{ij} = \int_0^1 \left[(D^\alpha \phi_i)(D^\alpha \phi_j)\right] (dt)^\alpha, B_j = -\int_0^1 \left[(1 - (1+\alpha)t)(D^\alpha \phi_i)\right] (dt)^\alpha,$$

here $i, j = 1, 2$. In matrix form it is written as:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}.$$

Finally, the system will be:

$$AC^T = B^T, C^T = A^{-1}B^T.$$

The calculations for values of C_1, C_2 and other calculations are performed using MATLAB.

Table 6 Numerical results of Example 4.3 for $\alpha = 0.5$

T	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
u_{exact}	-0.339	-0.404	-0.434	-0.419	-0.397	-0.359	-0.284	-0.201	-0.109	0.0000
u_{app}	0.659	0.868	0.718	0.334	-0.163	-0.645	-0.992	-1.080	-0.796	0.0000
$Absol.Error$	0.988	1.272	1.152	0.753	0.234	0.286	0.708	0.879	0.687	0.0000

In example 4.3, the fractional order is chosen as $\alpha = 0.5$ and the computed approximations of the solutions are provided in table 6 and figure 5.

Example 4.4: Consider the fractional differential equation (FDE) [17]:

$$D^2 u(t) + D^{\frac{3}{2}} u(t) + u(t) - t - 1 = 0,$$

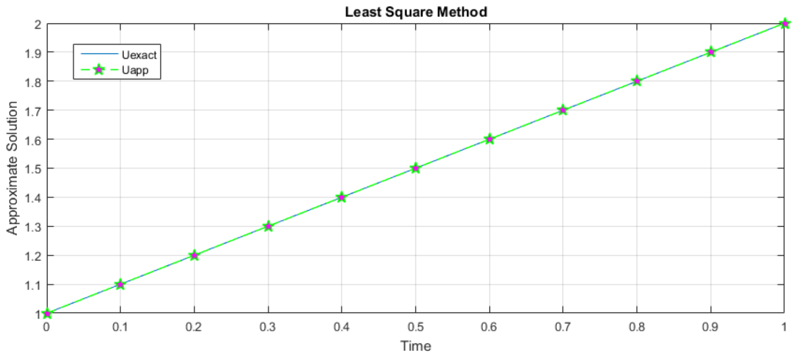

Figure 6: Numerical comparison of exact and approximate solution with LSM for Example 4.4

with boundary conditions:

$$u(0) = 1, u^{'}(0) = 1.$$

Solution: The exact solution of this problem is

$$u(t) = t + 1.$$

$$D^2 u(t) + D^{\frac{3}{2}} u(t) + u(t) - t - 1 = 0,$$

By using the polynomial least square method, let us consider the approximate solution:

$$u_{app}(t) = C_0 + C_1 t + C_2 t^2,$$

By applying initial conditions, we obtain $C_0 = 1$ and $C_1 = 1$. So, the approximate solution will be:

$$u(t) = 1 + t + C_2 t^2.$$

Now, we will define the functional I which satisfy the approximate solution of the problem:

$$I[C_2] = \frac{1}{\Gamma(\alpha + 1)} \int_0^1 [Du(t)]^2 \, (dt)^{\alpha},$$

$$I[C_2] = \frac{1}{\Gamma\left(\frac{3}{2} + 1\right)} \int_0^1 \left[\left(2 + \frac{4}{\sqrt{\pi}} t^{1/2} + t^2 \right) C_2 + \frac{1}{\sqrt{\pi}} t^{-1/2} - \frac{1}{2\sqrt{\pi}} t^{-3/2} \right]^2 (dt)^{\frac{3}{2}}.$$

We solve this above equation and obtain the value of $C_2 = 0$ and $u(t) = 1 + t$.
The calculations are performed using MATLAB.
In example 4.4, we initially selected $\alpha = 1$, and produced approximations of the solutions provided in table 7 and figure 6.
We note here that even if we use a larger degree polynomial in the absence of knowing the exact solution, the computation will still produce the exact solution.

Table 7 Numerical results of Example 4.4

T	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
u_{exact}	1.000	1.100	1.200	1.300	1.400	1.500	1.600	1.700	1.800	1.900	2.000
$LSMapp$	1.000	1.100	1.200	1.300	1.400	1.500	1.600	1.700	1.800	1.900	2.000

Example 4.5: Consider the fractional differential equation (FDE) [18]:

$$D^{\frac{5}{2}} u(t) - 3D^{\frac{3}{2}} u(t) = g(t), \quad t \in [0, 1],$$

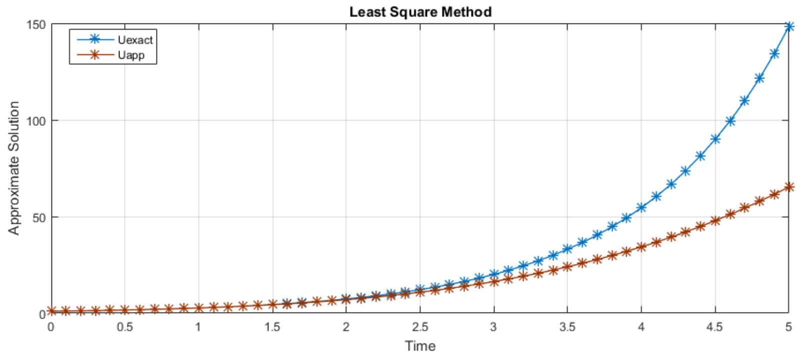

Figure 7: Numerical comparison of exact and approximate solution with LSM for Example 4.5, taking $\beta = 1$

$$u(0) = 1, \quad u'(0) = \beta, \quad u''(0) = \beta^2.$$

Solution: The exact solution of this problem is

$$u(t) = e^{\beta t}.$$

By using the polynomial least square method, let us consider the approximate solution:

$$u_{app} = \sum_{i=1}^{n} C_i \phi_i, i = 1, 2, 3.$$

By applying initial conditions, we obtain $C_0 = 1$ and $C_1 = \beta - 1$ and $C_2 = \frac{\beta^2}{2}$. So, the approximate solution will be:

$$u(t) = 1 + (\beta - 1)t + \frac{\beta^2}{2}t^2 + C_3 t^3.$$

Now, we will define the functional I which satisfy the approximate solution of the problem:

$$I[C_2] = \frac{1}{\Gamma(\alpha + 1)} \int_0^1 [Du(t)]^2 \, (dt)^\alpha.$$

The calculations are performed using MATLAB. We obtain the approximate solution for $\beta = 1$. The results are shown in table 8 and figure 7.

$$u_{app}(t) = 1 + t + 0.5t^2 + 0.16667t^3 + 0.04167t^4.$$

Now, we selected $\beta = 5$, and the approximations produced are shown in table 8 and figure 8.

$$u_{app}(t) = 1 + t + 0.5t^2 + 34.0010t^3 + 51.9962t^4.$$

Table 8 Comparison of approximate solutions for Example 4.5 with $\beta = 1$, $\beta = 5$

β	1	5
STM [18]	1.6×10^{-5}	5.6×10^{-5}
UWCM [21]	5.2×10^{-10}	2.14×10^{-8}
PLSM	1.5×10^{-10}	7.5×10^{-9}

5. Discussions and Conclusion

The major objective of this paper is to design the numerical method for solving fractional differential equations in order to obtain approximate solutions. The Polynomial Least Squares Method gives a simple

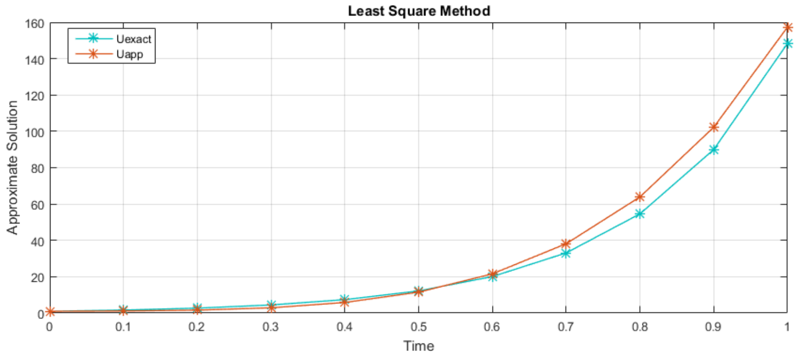

Figure 8: Numerical comparison of exact and approximate solution with LSM for Example 4.5, taking $\beta = 5$

and effective way to calculate analytical approximate solution for fractional differential equations. It is computationally effective and easy to implement for problems involving fractional derivatives. This method is an original innovation that can be used to address problems in applied sciences. The applications that are presented show the accuracy of the method. The presented scheme is preferred with Caputo derivative due to suitable operational matrices. But it is applicable with other fractional derivatives and can be utilized. In that case, matching operational matrix will be required and which will further be integrated to the least squares framework.

This approach provides a faster convergence of numerical solution for fractional differential equations that are more realistic. PLSM offers incredibility precise approximations, particularly for issues requiring smooth solutions. Since fractional derivatives do not need to be evaluated, it is computationally efficient. A large class of fractional differential equations, including ones with non-integer orders, can be solved using PLSM. This approach is reliable and capable of handling erratic or noisy data. The collected results demonstrated the method's effectiveness in comparison to exact solutions and also demonstrated the resemblance between the exact and approximate solutions. Calculations revealed that PLSM is a strong and effective method for locating an excellent solution for linear or non-linear equations as well as for many other physical issues in sciences and engineering. Additionally, the results are shown graphically to further illustrate the process. The most recent developments in the study of fractional differential equations in applied mathematics has given rise to many complex problems which can be approximated with this proposed scheme.

As future research direction, it is recommended to extend the proposed scheme for fractional partial differential equations. Another promising for future research is the application of the proposed scheme to stochastic fractional differential equations. The proposed scheme can also be utilized to develop hybrid computational frameworks for a particular domain. This will enhance the applicability of the proposed scheme to multiple domains.

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