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Nontrivial Deformation of $Vec(\mathbb{R})$-Modules of Symbols

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Abstract

In this paper, we introduce a new notion on deformation: *nontrivial* deformation. The action of the Lie algebra of vector fields $\mathrm{Vec}(\mathbb{R})$ on the space of symbols $\mathcal{S}$ is given by the Lie derivative. If we restrict ourselves to the Lie subalgebra $\mathfrak{a}(1)$, we get a family of infinite dimensional $\mathfrak{a}(1)$-modules, we compute the $\mathfrak{a}(1)$-*nonrelative* space $\mathrm{H}^1_{\mathfrak{a}(1)}(\mathrm{Vec}(\mathbb{R}), \mathrm{D}_{\lambda,\nu})$. We study *nontrivial* deformations of this action. We give concrete examples of *nontrivial* deformations. This work is the *nontrivial* case of a results by D. B. Fuchs and Imed Basdouri et al. [D. B. Fuchs, *Coho. of infinite-dim. Lie algebras*, Con. Bu., New York, 1987 and Imed Basdouri et al., *Deformation of* $\mathrm{Vect}_{\mathrm{P}}(\mathbb{R})$*-Modules of Symbols.* J. Geom. Phys. 60 (2010), no. 3, 531-542.]

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1. Introduction

The Lie algebra, $\mathrm{Vec}(\mathbb{R})$, of vector fields on $\mathbb{R}$ naturally acts, by the Lie derivative, on the space

$$\mathcal{F}_\lambda = \left\{ hdx^\lambda \ : \ h \in C^\infty(\mathbb{R}) \right\},$$

of weighted densities of degree λ. The action L^λ_Y on the space $\mathcal{F}_\lambda$ by the vector field $Y\frac{d}{dx}$ is given by

$$L^\lambda_Y = Y\partial + \lambda Y'. \tag{1.1}$$

We define a 2-parameter family of $\mathrm{Vec}(\mathbb{R})$-modules on the space $\mathrm{D}_{\lambda,\nu}$ of linear differential operators : $A \ : \ \mathcal{F}_\lambda \to \mathcal{F}_\nu$. The Lie algebra $\mathrm{Vec}(\mathbb{R})$ acts on $\mathrm{D}_{\lambda,\nu}$ by:

$$L^{\lambda,\nu}_Y(A) = -A \circ L^\lambda_Y + L^\nu_Y \circ A. \tag{1.2}$$

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For arbitrary $\ell \in \mathbb{N}$ and $\gamma \in \mathbb{R}$, we note the space of symbols by:

$$\mathcal{S}_\gamma^\ell = \bigoplus_{j=0}^{\ell} \mathcal{F}_{\gamma-j}$$

and we note the space $\mathcal{D}_\gamma^\ell$ of differential operators on $\mathcal{S}_\gamma^\ell$. That is,

$$\mathcal{D}_\gamma^\ell = \bigoplus_{\gamma-\ell\leq\lambda,\nu\leq\gamma} \mathrm{D}_{\lambda,\nu},$$

where $\gamma-\lambda$ and $\gamma-\nu$ are integers. The Lie derivative by the vector field $Y\frac{d}{dx}$ on $\mathcal{D}_\gamma^\ell$, which will be simply denoted by L_Y, is defined on each component $\mathrm{D}_{\lambda,\nu}$ by $L_Y = L_Y^{\lambda,\nu}$.

In [5] I. Basdouri et al. study the deformations of the structure of $\mathrm{Vect}_\mathrm{P}(\mathbb{R})$-module $\mathcal{S}_\delta^n$ (see also [6, 8, 12, 18, 9, 16] and [1, 13, 14]).

In this work, we compute the $\mathfrak{a}(1)-$*nonrelative* space $\mathrm{H}^1_{\mathfrak{a}(1)}(\mathrm{Vec}(\mathbb{R}), \mathrm{D}_{\lambda,\mu})$. We study the $\mathfrak{a}(1)$-*nontrivial* deformations of the $\mathrm{Vec}(\mathbb{R})$-modules $\mathcal{S}_\gamma$. We show that any $\mathfrak{a}(1)$-*nontrivial* formal deformation is equivalent to its infinitesimal part and we give an examples of $\mathfrak{a}(1)$-*nontrivial* deformation. *Nontrivial* deformation is a very important problemma in Physics and Science and Engineering [**?** 20].

2. Cohomology

Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ and $\mathcal{N}$ be a $\mathfrak{g}$-module. The space of $\mathfrak{h}$-relative q-cochains of $\mathfrak{g}$ with values in $\mathcal{N}$

$$C^q(\mathfrak{g}, \mathfrak{h}; \mathcal{N}) := \mathrm{Hom}_\mathfrak{h}(\wedge^q(\mathfrak{g}); \mathcal{N}).$$

The *coboundary operator* $\delta^n : C^q(\mathfrak{g}, \mathfrak{h}; \mathcal{N}) \to C^{q+1}(\mathfrak{g}, \mathfrak{h}; \mathcal{N})$ is a $\mathfrak{g}$-map verifying $\delta^2 = 0$.

The kernel of δ^q, denoted $Z^q(\mathfrak{g}, \mathfrak{h}; \mathcal{N})$, is the space of $\mathfrak{h}$-relative *q-cocycles*, among them, the elemmaents in the range of δ^{q-1} are called $\mathfrak{h}$-relative *q-coboundaries*. We denote $B^q(\mathfrak{g}, \mathfrak{h}; \mathcal{N})$ the space of q-coboundaries.

The q^{th} $\mathfrak{h}$-relative cohomolgy space is the quotient space

$$H^q(\mathfrak{g}, \mathfrak{h}; \mathcal{N}) = Z^q(\mathfrak{g}, \mathfrak{h}; \mathcal{N})/B^q(\mathfrak{g}, \mathfrak{h}; \mathcal{N}).$$

We will only need the formula for δ^q for degrees 0, 1 and 2: for $\vartheta \in C^0(\mathfrak{g}, \mathfrak{h}; \mathcal{N}) = \mathcal{N}^\mathfrak{h}$, $\delta\vartheta(g) := g\vartheta$, and $\mathfrak{b} \in C^1(\mathfrak{g}, \mathfrak{h}; \mathcal{N})$,

$$\delta\mathfrak{b}(g_1, g_2) := g_1\mathfrak{b}(g_2) - g_2\mathfrak{b}(g_1) - \mathfrak{b}([g_1, g_2])$$

and for $\Omega \in C^2(\mathfrak{g}; \mathfrak{h}, \mathcal{N})$,

$$\delta\Omega(g_1, g_2, g_3) := g_1\Omega(g_2, g_3) - \Omega([g_1, g_2], g_3) + \circlearrowleft (g_1, g_2, g_3), \tag{2.1}$$

where $\circlearrowleft$: the cyclic permutation of the symbols g_1, g_2, g_3. The first $\mathfrak{h}-$*nonrelative* cohomology space $\mathrm{H}^1_\mathfrak{h}(\mathfrak{g}; \mathrm{End}(\mathcal{N})) =: \mathrm{H}^1(\mathfrak{g}; \mathrm{End}(\mathcal{N}))/H^1(\mathfrak{g}, \mathfrak{h}; End(\mathcal{N}))$ classifies infinitesimal $\mathfrak{h}-$*nontrivial* deformations up to equivalence.

In this work, we are interested in the differential $\mathfrak{a}(1)$-*nonrelative* cohomology spaces

$$\mathrm{H}^1_{\mathfrak{a}(1)}(\mathrm{Vect}(\mathbb{R}), \mathrm{D}_{\lambda,\mu}) \qquad \text{and} \qquad \mathrm{H}^2_{\mathfrak{a}(1)}(\mathrm{Vec}(\mathbb{R}), \mathrm{D}_{\lambda,\lambda+k}).$$

and we study the $\mathfrak{a}(1)$-*nontrivial* deformations of the $\mathrm{Vec}(\mathbb{R})$-modules $\mathcal{S}_\gamma$.

2.1. The Space $\mathrm{H}^1_{\mathfrak{a}(1)}(\mathrm{Vec}(\mathbb{R}), \mathrm{D}_{\lambda,\mu})$

The space $\mathrm{H}^1(\mathrm{Vec}(\mathbb{R}), \mathrm{D}_{\lambda,\nu})$ was calculated by Feigin et al. in [17] and cohomology space $\mathrm{H}^1(\mathrm{Vec}(\mathbb{R}), \mathfrak{a}(1); \mathrm{D}_{\lambda,\nu})$ was calculated by the author et al. [11]. So we get $\mathfrak{a}(1)-$*nonrelative* space $\mathrm{H}^1_{\mathfrak{a}(1)}(\mathrm{Vec}(\mathbb{R}), \mathrm{D}_{\lambda,\nu})$, the result is given by

Theorem 2.1. *The space* $\mathrm{H}^1_{\mathfrak{a}(1)}(\mathrm{Vec}(\mathbb{R}), \mathrm{D}_{\lambda,\nu})$ *has the following structure:*

$$\mathrm{H}^1_{\mathfrak{a}(1)}(\mathrm{Vec}(\mathbb{R}), \mathrm{D}_{\lambda,\nu}) \simeq \begin{cases} \mathbb{R} & \text{if} \quad \nu = \lambda, \ \text{for all } \lambda, \\ \mathbb{R} & \text{if} \quad \lambda = 0 \quad \text{and} \quad \nu = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

These spaces are span by the classes of 1-cocycles., $\Lambda_{\lambda,\lambda+k} : \mathrm{Vec}(\mathbb{R}) \to \mathrm{D}_{\lambda,\lambda+k}$, $\forall \ \ Y\frac{d}{dx} \in \mathrm{Vec}(\mathbb{R})$ and $hdx^{\lambda} \in \mathcal{F}_{\lambda}$, we write

$$\Lambda_{\lambda,\lambda+k}(Y\tfrac{d}{dx})(hdx^{\lambda}) = \Lambda_{\lambda,\lambda+k}(Y)(h)dx^{\lambda+k}$$

and we give by $\Lambda_{\lambda,\lambda+k}(Y)(h)$:

$$\Lambda_{\lambda,\lambda}(Y)(h) = X'h,$$
$$\Lambda_{0,1}(Y)(f) = X'h'.$$

We need the following lemma

Lemma 2.2. *Any 1-cocycle* $\Lambda \in Z^1_{\mathfrak{a}(1)}(\mathrm{Vec}(\mathbb{R}), \mathrm{D}_{\lambda,\mu})$ $\mathfrak{a}(1)-$*noninvariant is nonvanishing on* $\mathfrak{a}(1)$.

Proof. The 1-cocycle equation for Λ is written as follows:

$$L^{\lambda,\mu}_{X_F}\Lambda(X_G) - L^{\lambda,\mu}_{X_G}\Lambda(X_F) - \Lambda([X_F, \ X_G]) = 0, \tag{2.3}$$

for X_F, $X_G \in$ Vec($\mathbb{R}$). If Λ is $\mathfrak{a}(1)-$noninvariance for all $X_F \in \mathfrak{a}(1)$, thus, the equation $\mathfrak{a}(1)-$ invariance becomes

$$L^{\lambda,\mu}_{X_F}\Lambda(X_G) - \Lambda([X_F, \ X_G]) \neq 0 \tag{2.4}$$

The difference between the two equations gives:

$$(2.3) - (2.4) = \Lambda(X_F) \neq 0$$

□

Proof. By lemma (2.2), we have that all $\mathfrak{a}(1)-$*nonrelative* cocycles are $\mathfrak{a}(1)-$noninvariant bilinear bidifferential operators. Moreover, Feigin et al. [17] calculated $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vec}(\mathbb{R}); \mathrm{D}_{\lambda,\mu})$:

$$\mathrm{H}^1(\mathrm{Vec}(\mathbb{R}); \mathrm{D}_{\lambda,\nu}) \simeq \begin{cases} \mathbb{R} & \text{if} \quad \nu - \lambda \in \{0, 2, 3, 4\} \text{ for all } \lambda, \\ \mathbb{R}^2 & \text{if} \quad (\lambda, \nu) = (0, 1) \\ \mathbb{R} & \text{if} \quad \lambda \in \{0, -4\} \quad \text{and} \quad \nu = \lambda + 5, \\ \mathbb{R} & \text{if} \quad \lambda = -\frac{5\pm\sqrt{19}}{2} \quad \text{and} \quad \nu = \lambda + 6, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

Moreover, Khaled Basdouri et al. [11] calculated $\mathrm{H}^1(\mathrm{Vec}(\mathbb{R}), \mathfrak{a}(1); \mathrm{D}_{\lambda,\mu})$. The result is as follows

$$\mathrm{H}^1(\mathrm{Vec}(\mathbb{R}), \mathfrak{a}(1); \mathrm{D}_{\lambda,\nu}) \simeq \begin{cases} \mathbb{R} \quad \text{if} & \begin{cases} \nu - \lambda \in \{2, 3, 4\} \text{ for all } \lambda, \\ (\lambda, \nu) = (0, 1), \\ \lambda \in \{0, -4\} \quad \text{and} \quad \nu = \lambda + 5, \\ \lambda = -\frac{5\pm\sqrt{19}}{2} \quad \text{and} \quad \nu = \lambda + 6, \end{cases} \\ 0 \quad \text{otherwise.} \end{cases} \tag{2.6}$$

Using the cohomology (2.6), the cohomology (2.5) and the equation

$$\mathrm{H}^1_{\mathfrak{a}(1)}(\mathrm{Vec}(\mathbb{R});\mathrm{D}_{\lambda,\nu}) =: \ \mathrm{H}^1(\mathrm{Vec}(\mathbb{R});\mathrm{D}_{\lambda,\nu})/\mathrm{H}^1(\mathrm{Vec}(\mathbb{R}),\mathfrak{a}(1);\mathrm{D}_{\lambda,\nu}). \tag{2.7}$$

Hence, the spaces $\mathrm{H}^1_{\mathfrak{a}(1)}(\mathrm{Vec}(\mathbb{R});\mathrm{D}_{\lambda,\nu})$ vanish for $\nu-\lambda \neq 0,1$. By lemma (2.2), so we only have to study $dim(\mathrm{H}^1_{\mathfrak{a}(1)}(\mathrm{Vec}(\mathbb{R});\mathrm{D}_{\lambda,\nu})) = 1$ if $\nu-\lambda = 0,1$. More precisely, these cohomology spaces are spanned by the $\mathfrak{a}(1)-$*nonrelative* cohomology classes of the 1-cocycles, $\Lambda_{\lambda,\lambda}$ and $\Lambda_{0,1}$. A straightforward but long computation leads to the result. □

3. The General Framework: *nontrivial* deformations

In this section, we define *nontrivial* deformations of Lie algebra homomorphisms and introduce the concept of miniversal deformations on commutative algebras.

3.1. The Second $\mathfrak{h}$relative Cohomology Space

Let $\mathfrak{g}$ a Lie algebra, $\mathfrak{h}$ a Lie subalgebra and $\mathcal{N}$ a $\mathfrak{g}$-module, the *cup-product* defined, for two linear maps Λ_1, $\Lambda_2 : \mathfrak{g} \to \mathrm{End}(\mathcal{N})$, is defined by:

$$\begin{aligned} &\Lambda_1 \vee \Lambda_2 : \mathfrak{g}^{\otimes 2} \to \mathrm{End}(\mathcal{N}) \\ &\Lambda_1 \vee \Lambda_2(x_1, x_2) = [\Lambda_1(x_1), \Lambda_2(x_2)] + [\Lambda_2(x_1), \Lambda_1(x_2)]. \end{aligned} \tag{3.1}$$

Therefore, we naturally deduce that the operation (3.1) defines a bilinear map:

$$\mathrm{H}^1_{\mathfrak{h}}(\mathfrak{g},\mathrm{End}(\mathcal{N})) \otimes \mathrm{H}^1_{\mathfrak{h}}(\mathfrak{g},\mathrm{End}(\mathcal{N})) \to \mathrm{H}^2_{\mathfrak{h}}(\mathfrak{g},\mathrm{End}(\mathcal{N})). \tag{3.2}$$

3.2. $\mathfrak{h}-$nontrivial Infinitesimal deformations

Let $\sigma_0 : \mathfrak{g} \to \mathrm{End}(\mathcal{N})$ be an action of a Lie algebra $\mathfrak{g}$ on a vector space $\mathcal{N}$, and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. When studying non-trivial deformations of the action σ_0 of $\mathfrak{g}$, one typically begins by considering *nontrivial* infinitesimal deformations with respect to $\mathfrak{h}$:

$$\sigma = \sigma_0 + t\,\Lambda,$$

verifying

$$[\sigma(x_1), \sigma(x_2)] = \sigma([x_1, x_2]),$$

where t is a formal parameter, $x_1, x_2 \in \mathfrak{g}$, is satisfied in order 1 in t if and only if Λ is a 1-cocycle $\mathfrak{h}-$*nontrivial.*

In the case where this $\mathfrak{h}-$*nonrelative* cohomology space is multidimensional, it is natural to consider $\mathfrak{h}-$*nontrivial* with multiple parameters. If $\dim \mathrm{H}^1_{\mathfrak{h}}(\mathfrak{g};\mathrm{End}(\mathcal{N})) = n$, next, choose the 1-cocycles $\Lambda_1, \ldots, \Lambda_n$ that represent a basis of $\mathrm{H}^1_{\mathfrak{h}}(\mathfrak{g};\mathrm{End}(\mathcal{N}))$ and consider the $\mathfrak{h}-$*nontrivial* infinitesimal deformation

$$\sigma = \sigma_0 + \sum_{\ell=1}^{n} t_\ell\,\Lambda_\ell, \tag{3.3}$$

with independent parameters $t_1, \ldots, t_n$.

The concept of equivalence of deformations in commutative and associative algebras has been studied in publications [3, 4, 10, 14, 18], and the notion of a miniversal deformation is considered fundamental in these works [14, 18].

3.3. Integrability conditions of $\mathfrak{h}-$*nontrivial formal deformations*

Let's consider the problemma of integrability of *nontrivial* infinitesimal deformations. The problemma of nontrival deformation is very important in physics and energy sciences and image processing....[**?** 20]. Starting from the $\mathfrak{h}-$*nontrivial* infinitesimal deformation with respect to $\mathfrak{h}$ (3.3), we seek a formal power series:

$$\sigma = \sigma_0 + \sum_{\ell=1}^{n} t_\ell \, \Lambda_\ell + \sum_{\ell,j} t_\ell t_j \, \sigma^{(2)}_{\ell j} + \cdots , \tag{3.4}$$

whith the higher-order terms, namely $\sigma^{(2)}_{\ell j}, \sigma^{(3)}_{\ell jk}, \dots$, are linear mappings from $\mathfrak{g}$ to $\mathrm{End}(\mathcal{N})$ such that

$$\sigma : \mathfrak{g} \to \mathrm{End}(\mathcal{N}) \otimes \mathbb{C}[[t_1, \dots, t_n]] \tag{3.5}$$

satisfies Λ is a 1-cocycle. The first $\mathfrak{h}-$*nonrelative* cohomology space

$$\mathrm{H}^1_{\mathfrak{h}}(\mathfrak{g}; \mathrm{End}(\mathcal{N})) =: \ \mathrm{H}^1(\mathfrak{g}; \mathrm{End}(\mathcal{N}))/H^1(\mathfrak{g}, \mathfrak{h}; End(\mathcal{N}))$$

classifies $\mathfrak{h}-$*nontrivial* infinitesimal deformations up to equivalence.

However, this problemma often has no solution. Following the approaches in [18] and [14], we will impose additional algebraic relations on the parameters $t_1, \dots, t_n$. Let $\mathcal{I}$ be an ideal of $\mathbb{C}[[t_1, \dots, t_n]]$ spanned by a set of relations; consider the quotient

$$\mathcal{B} = \mathbb{C}[[t_1, \dots, t_n]]/\mathcal{I} \tag{3.6}$$

is a commutative and associative algebra with a neutral elemmaent, and it is possible to study its deformations with respect to the basis $\mathcal{B}$; for more details, see [18]. The map (3.5) sends $\mathfrak{g}$ to $\mathrm{End}(\mathcal{N}) \otimes \mathcal{B}$. The notion of miniversal deformation is fundamental [14, 18].

3.4. Computing the second-order Maurer-Cartan equation

In our study, a *nontrivial* infinitesimal deformation of the action of $\mathrm{Vec}(\mathbb{R})$ on $\mathcal{D}^n_\delta$, related to the Lie algebra $\mathfrak{a}(1)$, takes the following form:

$$L_Y + \mathcal{L}^{(1)}_Y, \tag{3.7}$$

where L_Y denotes the Lie derivative of $\mathcal{D}^n_\delta$ along the vector field $Y\frac{d}{dx}$ given by (1.2), and

$$\mathcal{L}^{(1)}_Y = \begin{cases} \sum_\lambda t_{\lambda,\lambda} \, \Lambda_{\lambda,\lambda}(Y) & \text{if } \gamma \notin (\mathbb{N}+1) \\ t_{0,1}\Lambda_{0,1}(Y) + \sum_\lambda t_{\lambda,\lambda} \, \Lambda_{\lambda,\lambda}(Y) & \text{if } \gamma \in (\mathbb{N}+1), \end{cases}$$

and where $t_{\lambda,\lambda}$ and $t_{0,1}$ are independent parameters, $\gamma - \lambda \in \mathbb{N}$, $\gamma - n \leq \lambda \leq \gamma$ and the 1-cocycles $\Lambda_{\lambda,\lambda}$ and $\Lambda_{0,1}$ are given in theorem (2.1).

Theorem 3.1. *The following conditions are necessary and sufficient for integrability of the* $\mathfrak{a}(1)-$*nontrivial infinitesimal deformation (3.7):*

$$t_{1,1} t_{0,1} = 0 \ . \tag{3.8}$$

Moreover, any $\mathfrak{a}(1)-$*nontrivial formal deformation is equivalent to its infinitesimal part.*

Proof. If $\gamma \notin (\mathbb{N}+1)$, then the parameters t_k can be considered as zero, and in this case, there are no integrability conditions.

Let's assume that the $\mathfrak{a}(1)$-*nontrivial* infinitesimal deformation (3.7) can be integrated to form a formal deformation

$$\mathcal{L}_Y = L_Y + \mathcal{L}^{(1)}_Y + \mathcal{L}^{(2)}_Y + \mathcal{L}^{(3)}_Y + \cdots \tag{3.9}$$

where $\mathcal{L}_Y^{(1)}$ is defined by equation (3.4) and $\mathcal{L}_Y^{(2)}$ is a second-degree polynomial in t whose coefficients belong to $\mathcal{D}_\delta^n$. We determine the conditions related to the second-order terms $\mathcal{L}^{(2)}$. Let us consider the quadratic terms in the homomorphism condition.

$$\mathcal{L}_{[Y,X]} = [\mathcal{L}_Y, \mathcal{L}_X]. \tag{3.10}$$

The homomorphism condition (3.10) gives for the second-order terms the following (Maurer-Cartan) equation

$$-\frac{1}{2}\,(\mathcal{L}^{(1)} \vee \mathcal{L}^{(1)}) = \delta(\mathcal{L}^{(2)}), \tag{3.11}$$

This ensures that the right-hand side of equation (3.11) is automatically a 2-cocycle. In our case, we have:

$$\delta(\mathcal{L}^{(2)})(Y,X) = -\frac{1}{2}(\sum_{\lambda} t_{\lambda,\lambda}\,\Lambda_{\lambda,\lambda} + t_{0,1}\Lambda_{0,1}) \vee (\sum_{\lambda} t_{\lambda,\lambda}\,\Lambda_{\lambda,\lambda} + t_{0,1}\Lambda_{0,1})(Y,X) \tag{3.12}$$

Consider the following 2-cocycles:

$$\Omega_{\lambda,\nu}(Y,X) = -(\sum_{\lambda} t_{\lambda,\lambda}\,\Lambda_{\lambda,\lambda} + t_{0,1}\Lambda_{0,1}) \vee (\sum_{\lambda} t_{\lambda,\lambda}\,\Lambda_{\lambda,\lambda} + t_{0,1}\Lambda_{0,1})(Y,X) \;:\; \mathcal{F}_\lambda \to \mathcal{F}_\nu.$$

(In general, $\Omega_{\lambda,\nu}(Y,X)h = \delta b_{\lambda,\nu}(Y,X)h + Q(t_1,...,t_n)\omega(Y,X)h$, $Q(x_1,...,x_n) \in \mathbb{R}[x]$ must be equal to 0, $\omega \in \mathrm{H}^2_{\mathfrak{a}(1)}(\mathrm{Vec}(\mathbb{R}), \mathrm{D}_{\lambda,\nu})$, Q is the conditions for the integrability and b is the second-order solution.)

The necessary conditions for the $\mathfrak{a}(1)$*nontrivial* infinitesimal deformation of (3.4) to be integrable are that every 2-cocycle $\Omega_{\lambda,\nu}$, with $k \in \{0,1\}$, is a coboundary.

That means

$$\delta b_{\lambda,\nu}(Y,X)f = \Omega_{\lambda,\nu}(Y,X)f, \tag{3.13}$$

where

$$\delta b_{\lambda,\nu}(Y,X)f = b_{\lambda,\nu}([Y,X]f - L_Y^{\lambda,\mu}(b_{\lambda,\nu}(X))f + L_Y^{\lambda,\nu}(b_{\lambda,\nu}(Y))f, \tag{3.14}$$

and, for any $Y\frac{d}{dx} \in \mathrm{Vec}(\mathbb{R})$, the linear map $b_{\lambda,\nu}(Y) : \mathcal{F}_\lambda \to \mathcal{F}_\nu$ is given:

$$b_{\lambda,\nu}(Y)(hdx)^\lambda = \sum_{\ell=0}^{\nu-\lambda+1} a^\ell_{\lambda,\nu} Y^{(\ell)} h^{(k+1-\ell)}(dx)^\nu.$$

We compute successively the 2-cocycles $\Omega_{\lambda,\nu}(Y,X)$:

1) $\Omega_{\lambda,\lambda}(Y,X) = 0$.

$$\Omega_{\lambda,\lambda+1}(Y,X)(h) = \begin{cases} t_{1,1}t_{0,1}\omega_{0,1}(Y,X)h & \text{if } \lambda = 0, \\ 0, & \text{if } \lambda \neq 0. \end{cases}$$

□

We will now prove the following lemma, and then we will conclude based on the Theorem 3.1.

In the sequel, we consider some 2-cocycles $\omega_{\lambda,\nu} : \mathrm{Vec}(\mathbb{R})^{\otimes 2} \to \mathrm{D}_{\lambda,\nu}$. For $Y\frac{d}{dx}$, $X\frac{d}{dx} \in \mathrm{Vec}(\mathbb{R})$ and $hdx^\lambda \in \mathcal{F}_\lambda$, we find

$$\omega_{\lambda,\nu}(Y\tfrac{d}{dx}, X\tfrac{d}{dx})(hdx^\lambda) = \omega_{\lambda,\nu}(Y,X)(h)dx^\nu,$$

but, owing to the simplicity, we give only the expressions of $\omega_{\lambda,\nu}(Y,X)(h)$.

We need the following lemma

Lemma 3.2. *Let $b \in C^1(\mathrm{Vec}(\mathbb{R}), \mathrm{D}_{0,1})$ defined as follows: for $Y\frac{d}{dx} \in \mathrm{Vec}(\mathbb{R})$ and $hdx^\lambda \in \mathcal{F}_\lambda$*

$$b(Y)(h) = \sum_{\ell=0}^{2} a_\ell h^{(2-\ell)} Y^{(\ell)} \tag{3.15}$$

where the coefficients $a_j \in \mathbb{R}$. So the map $\delta b : \mathrm{Vec}(\mathbb{R})^{\otimes 2} \to \mathrm{D}_{0,1}$ is given by

$$\delta b(Y, X)(f) = -a_0(XY'' - X''Y)h' - a_0(XY' - X'Y)h'' \tag{3.16}$$

Proof. A straightforward computation. □ □

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Lemma 3.3. *The $\mathfrak{a}(1)-$nonrelative cohomology spaces $\mathrm{H}^2_{\mathfrak{a}(1)}(\mathrm{Vec}(\mathbb{R}), \mathrm{D}_{0,1})$, is generated by the $\mathfrak{a}(1)-$nonrelative cohomology classes of the nontrivial 2-cocycle $\omega_{0,1}$ defined by*

$$\Omega_{0,1}(Y, X)(h) \quad = (X'Y'' - X''Y')h,$$

Proof. The function $\Omega_{0,1}$ is the cup-product (" $\vee$ ") of the 1-cocycles $\Lambda_{0,1}$ and $\Lambda_{1,1}$. Therefore, $\Omega_{0,1}$ is a 2-cocycle, and it will suffice to show that it is *nontrivial*, since the space $\mathrm{H}^2_{\mathfrak{a}(1)}(\mathrm{Vec}(\mathbb{R}), \mathrm{D}_{0,1})$ is one dimension.

By lemma 3.16, therefore

$$\Omega_{0,1}(Y, X)f \neq \delta b(Y, X)f.$$

Thus $\Omega_{0,1}$ is $\mathrm{H}^2_{\mathfrak{a}(1)}(\mathrm{Vec}(\mathbb{R}), \mathrm{D}_{0,1})$. To complete the proof of lemma 3.3, using the equation 2.7, we must prove that $dim(\mathrm{H}^2_{\mathfrak{a}(1)}(\mathrm{Vec}(\mathbb{R}), \mathrm{D}_{0,1}))$ equal 1. □

The solution $\mathcal{L}^{(2)}$ of equation (3.12) can be chosen to be zero. We will now demonstrate that these conditions are sufficient. By setting the higher-order terms $\mathcal{L}^{(n)}$ (with $n \geq 3$) to zero as well, we obtain a deformation (of first order in t_k). Any $\mathfrak{a}(1)-$*nontrivial* formal deformation of type $\mathfrak{a}(1)$ is equivalent to its infinitesimal part, since different choices of solutions to the Maurer-Cartan equation correspond to equivalent deformations.

4. Examples of $\mathfrak{a}(1)-$*nontrivial* deformations

In this section, we present examples of symbol spaces $\mathcal{D}^n_\delta$: we study the nontrivial formal deformations of these spaces with respect to the Lie algebra $\mathfrak{a}(1)$.

Example 4.1. *Consider $\mathcal{D}^2_2$. The $\mathfrak{a}(1)-$nontrivial infinitesimal deformation of the action of $\mathrm{Vec}(\mathbb{R})$ on $\mathcal{D}^2_2$ is of the form $L_Y + \mathcal{L}^{(1)}_Y$, where $\mathcal{L}^{(1)}_Y$ given by*

$$\mathcal{L}^{(1)}_Y = \sum_{\lambda=1}^{2} t_{\lambda,\lambda+j}\,\Lambda_{\lambda,\lambda+j}(Y) = (t_{0,0}\Lambda_{0,0} + t_{1,1}\Lambda_{1,1} + t_{0,1}\Lambda_{0,1} + t_{2,2}\Lambda_{2,2})(Y). \tag{4.1}$$

We obtain the 1 equation:

$$t_{1,1}t_{0,1} = 0. \tag{4.2}$$

as necessary and sufficient integrability condition of this $\mathfrak{a}(1)-$*nontrivial* infinitesimal deformation.

Proposition 4.2. *There are two $\mathfrak{a}(1)-$nontrivial deformations of the action of $\mathrm{Vec}(\mathbb{R})$ on $\mathcal{D}^2_2$, characterized by three independent parameters:*

$$\mathcal{L}_Y = L_Y + t_0\Lambda_{0,0}(Y) + t_1\Lambda_{1,1}(Y) + t_2\Lambda_{2,2}(Y), \tag{4.3}$$

or

$$\mathcal{L}_Y = L_Y + t_0\Lambda_{0,0}(Y) + t\Lambda_{0,1}(Y) + t_2\Lambda_{2,2}(Y). \tag{4.4}$$

Proof. We have the $\mathfrak{a}(1)-$*nontrivial* deformation (see equation 4.1), and we then study the solutions of equation (4.2).

1- The first case is : $t_{0,1} = 0$, we put $t_1 = t_{1,1}$, $t_2 = t_{2,2}$, $t_0 = t_{0,0}$ and then we have (4.3).

2- The second case is : $t_{1,1} = 0$, we put $t_0 = t_{0,0}$, $t_2 = t_{2,2}$ and $t_{0,1} = t$ and then we have the second $\mathfrak{a}(1)-$*nontrivial* deformation. The solution $\mathcal{L}^{(2)}$ of (3.11) can be chosen to be equal to zero. By choosing the higher-order terms $\mathcal{L}^{(m)}$ with $m \geq 3$ and setting them all equal to zero, we obviously obtain a non-trivial deformation of the algebra $\mathfrak{a}(1)$ (of first order in t). □

Example 4.3. *Consider $\mathcal{D}^3_4$. We obtain in this case*

$$\begin{aligned} \mathcal{L}^{(1)}_Y = \ & t_1\Lambda_{1,1}(Y) + t_2\Lambda_{2,2}(Y) \\ & + t_3\Lambda_{3,3}(Y) + t_4\Lambda_{4,4}(Y). \end{aligned} \tag{4.5}$$

In this case, there is no integrability equation for the $\mathfrak{a}(1)$-nontrivial infinitesimal deformation of $L_Y + \mathcal{L}^{(1)}_X$. So $\mathcal{D}^3_4$ admits a single $\mathfrak{a}(1)-$nontrivial deformation with four independent parameters.

Example 4.4. *Consider $\mathcal{D}^3_3$.*

$$\begin{aligned} \mathcal{L}^{(1)}_Y & = t_{0,0}\Lambda_{0,0}(Y) + t_{0,1}\Lambda_{0,1}(Y) + \\ & + t_{1,1}\Lambda_{1,1}(Y) + t_{2,2}\Lambda_{2,2}(Y) + t_{3,3}\Lambda_{3,3}(Y). \end{aligned} \tag{4.6}$$

We have the unique equation:

$$t_{1,1}t_{0,1} = 0. \tag{4.7}$$

In this case, any *nontrivial* formal deformation of $\mathcal{D}^3_3$ with respect to $\mathfrak{a}(1)$ is equivalent to a $\mathfrak{a}(1)-$*nontrivial* infinitesimal deformation that satisfies this condition (4.7).

We can construct a large number of examples of nontrivial deformations of $\mathcal{D}^3_3$ with respect to the Lie algebra $\mathfrak{a}(1)$, using 2 (or fewer) independent parameters.

However, the nontrivial deformation

$$\mathcal{L}_Y = L_Y + \mathcal{L}^{(1)}_Y$$

is the *nontrivial* deformation miniversal of $\mathcal{D}^3_3$ with respect to $\mathfrak{a}(1)$, where base $\mathcal{B} = \mathbb{C}[t_{0,0}, t_{0,1}, \dots]/\mathcal{I}$ and $\mathcal{I}$ is the ideal generated by $t_{1,1}t_{0,1} = 0$.

5. Problem

The cohomology $\mathrm{H}^1_{\mathfrak{sl}(2)}(\mathrm{Vec}(\mathbb{R}); \mathrm{D}_{\lambda,\mu})$ determines and classifies $\mathfrak{sl}(2)$ *nontrivial* infinitesimal deformations up to equivalence.

Conjecture: Any formal $\mathfrak{sl}(2)$-*nontrivial* deformation is no equivalent to its infinitesimal part.

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