



Journal of Prime Research in Mathematics



Fixed Point and Coupled Fixed Point Theorems for a new generalized S -Pata-type contraction in S -Metric Space with Applicability to Boundary Value Problem

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Abstract

This study advances the field of fixed point theory within the framework of S metric space. We first refine the notion of a generalized S -Pata-type contraction by introducing supplementary cross terms into the contractive condition. Within this extended setting, we derive new fixed point theorems that both enhance the earlier results. It addition to widen the applicability of our contractive mappings , we formulate and analyze a coupled fixed point theorem for S -Pata-type contraction, where condition ensuring existence and uniqueness. Finally, the effectiveness of our approach is demonstrated through applications to ordinary differential equations, in particular a boundary value problem for a second-order equation, thereby underlining the practical relevance of the theoretical developments presented.

Keywords: S -metric space, Fixed point, Coupled Fixed point, S -Pata-type contraction.

2010 Mathematics Subject Classification:. 47H10 , 37C25, 54H25, 55M20, 58C30

1. Introduction

The Significance of Metric Space and Fixed Point Theory: A metric space is a set that includes a concept of distance, typically referred to as points, between its components. A metric or distance function is a function that measures distance. Metric space are the most general environment that allows to examine many of the ideas of geometry and mathematical analysis. Rene Maurice Frechet first proposed the concept of an abstract space with metric properties in 1906, while Felix Hausdorff first used the phrase metric space in 1914. The foundation for comprehending convergence, continuity, and other important ideas in non-geometric space was established by Frechet's work. This made it possible for mathematicians to examine functions and sequences in a more comprehensive and adaptable manner. This was significant for the developing functional analysis area. The concept of metric space was subsequently developed and broadened by mathematicians such as Hausdorff and Stefan Banach. Topological spaces were first

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introduced by Hausdorff as an extension of metric space. A major component of Banach's functional analysis work was the metric structure. Metric space eventually emerged as a key component of contemporary mathematics. The primary field of mathematical analysis known as "fixed point theory" studies the conditions under which a function will have a point that is invariant under its action. The Banach contraction principle's key characteristic is that it provides the existence, uniqueness, and sequence of consecutive approximations that converge to a problem solution. The importance of fixed point theory resides in its capacity to prove the existence and uniqueness of solutions to a broad range of issues. For example, in iterative methods for determining equation roots, convergence to a fixed point can be guaranteed under certain conditions, which leads to solutions in real-world applications. In the digital age, The fixed point concepts aid in determining algorithm correctness and give a foundation for comprehending recursive functions.

Overview of S metric Space: A mathematical concept known as a S metric space occurs when a set is fitted with a function known as S metric, which gives a non-negative value for each triplet of elements in the set. This function works similarly to a distance measure but has the special characteristic of requiring three points to define a distance. This generalization of standard metric enables applications in fixed point theory and other fields of mathematics, especially when working with more intricate relationships between elements in a set. In fields like mathematical modeling, dynamic systems, and nonlinear analysis, S -metric space have grown in significance because they allow more flexible solutions for problems where standard metrics might not be enough. Mathematicians can discover novel discoveries and acquire an understanding of intricate relationships across a variety of domains by investigating the special characteristics and ramifications of S -metric space.

Literature Review: In 1912, Brouwer [5] presented the well-known fixed point theorem. This theorem only ensures the existence of a solution and does not provide information on its uniqueness or determination. The Brouwer theorem provides no information regarding the location of fixed points. These tools are useful for computing function zeros. Approximation theory, game theory, and other scientific fields including engineering, economics, and optimization theory may all benefit from the Schauder [19] fixed point theorem. The majority of analytic problems lack a compact environment, therefore the compactness constraint is highly strict. It makes sense to loosen the compactness constraint in order to demonstrate the theorem. Liouville first proposed the sequential approximation approach, then Picard methodically refined it. Then Banach [2] formulated what is now known as the Banach contraction principle. In several scientific fields, the Banach contraction principle serves as a foundation for existence and uniqueness theorems. This theorem demonstrates the use of fixed point theory in analysis as well as the unifying potential of functional analytic approaches. According to Adewale et al. [1], the fixed point theorems were demonstrated in multiplicative S -metric space. Chand, D., & Rohen, Y. [7] invented the concept of $\alpha - \beta - \psi$ -contractive mapping in S -metric space. Several writers including Khomdram, B., & Rohen, Y., [12], Priyobarta, N., Rohen, Y., & Mlaiki, N. [14] and Sedghi, S., Shobe, N., & Aliouche, A. [20] demonstrated a few common coupled fixed point theorems, in S_b -metric space, complex-valued S_b -metric space and proved a fixed point theorem on Complete S -metric space that satisfies a contraction. Two coupled fixed point theorems for multi-valued nonlinear contraction mappings in partially ordered metric space were developed by Samet, B., & Vetro, C. [18]. Contractions are essential to proving the presence of fixed points in different kinds of metric space in the study of fixed point theory. While traditional definitions of contractions focus on metric space with standard distance measurements, Pata [13], have extended these concepts to more general contexts, especially in S -metric space. Common fixed point theorems under Pata's condition in complex-valued metric space were shown by Berrah, K. et al. [4]. Based on Vittorino Patta's findings Chakraborty, M., & Samanta, S.K. [6] established a generalization of Kannan's fixed point theorem. Roy., Saha, P., & Choudhury, B. S. [15] have used w -distance to extend results in metric fixed point theory. Using this idea, coupled fixed point theorems with Pata-type inequalities have been proved and applied to integral equations. Chand, D., et al. [8] expanded an S -metric space on Pata's method. Eshaghi, M. et al. [9] established a theorem of coupled fixed point pertaining to the Contraction of Pata in partially ordered complete metric space that have the mixed monotone characteristic. The presence of a fixed point for Pata-type Zamfirescu mapping in a complete metric space was demonstrated by Jacob, G. K., et al. [10]. The fixed point theorem of Chatterjea was expanded by Kadelburg, Z., & Radenovic, S. [11], based on a Pata's result, and a coupled fixed point result in ordered metric space and common fixed point outcomes of Pata-type for two maps are also established. In the setting of Complete metric space, Saleem, N., et al. [16] found that some α -Pata-proximal admissible contraction of type-I and type-II had optimal coincidence point and best proximity point

outcomes. Three distinct contractive conditions were introduced by Saleem, N., et al. [17]: (1) an α - θ -Geraghty-Pata-Suzuki-type proximal contraction, (2) an α - θ -generalized Geraghty-Pata-Suzuki-type proximal contraction, and (3) an α - θ -modified Geraghty-Pata-Suzuki-type proximal contraction.

Research gap: While generalized S-Pata-type contraction have been studied in S metric space. Existing results are limited to contraction conditions involving only self-comparative terms of the mapping. In particular, the role of additional cross term such as $S(\varsigma, \zeta, \varrho)$ has not yet been explored in the contraction framework. And Coupled fixed point theorems have been extensively studied in metric and partially ordered metric spaces, particularly in connection with Pata-type contractions. However, to the best of our knowledge, there has been no investigation of coupled fixed point results for Pata-type contractions within the broader setting of S metric space. Since S metric generalize classical metric structures and allow more flexibility in analysis, extending coupled fixed point theorems to this framework represents a significant gap in the literature.

Manuscript Organization: The paper is organized as follows. In chapter 2, we present the fundamental definitions and lemmas that are essential for our main results. Chapter 3 introduces a new S-Pata-type contraction and establishes the corresponding fixed point theorem. In chapter 4, we study the monotone property in S metric space and $\bar{S}$ space, and prove a coupled fixed point theorem in this framework. Chapter 5 is devoted to the application of our results to a boundary value problem for a second-order ordinary differential equation. Finally, Chapter 6 contains the conclusion.

2. Fundamentals

Some definitions used in the primary theorems are reviewed in this section. Throughout this entire paper, we utilize the norm $\|\varsigma\| = S(\varsigma, \varsigma, \varsigma_0)$ in a S metric space $(\mathcal{L}, S)$.

Definition 2.1. [20] Let $\mathcal{L}$ be a nonempty set and a function $S: \mathcal{L}^3 \rightarrow [0, \infty)$ is said to be S metric if and only if for all $\varsigma, \zeta, \varrho, \check{t} \in \mathcal{L}$; then the following conditions hold:

- 1.) $S(\varsigma, \zeta, \varrho) \geq 0$,
- 2.) $S(\varsigma, \zeta, \varrho) = 0$ if and only if $\varsigma = \zeta = \varrho$,
- 3.) $S(\varsigma, \zeta, \varrho) \leq S(\varsigma, \varsigma, \check{t}) + S(\zeta, \zeta, \check{t}) + S(\varrho, \varrho, \check{t})$

Thus $(\mathcal{L}, S)$ is said to be a S-metric space.

Definition 2.2. [20] $S(\varsigma, \varsigma, \zeta) = S(\zeta, \zeta, \varsigma)$ is an equality that exists within a S metric space $(\mathcal{L}, S)$.

Definition 2.3. [20] Let $(\mathcal{L}, S)$ be an S-metric space. Suppose $\{\varsigma_i\}$ is a sequence in $\mathcal{L}$ and $\varsigma \in \mathcal{L}$, then

- (i) $\{\varsigma_i\}$ converge to ς if and only if $S(\varsigma_i, \varsigma_i, \varsigma) \rightarrow 0$ hold as $i \rightarrow \infty$.
- (ii) A sequence $\varsigma_i \in \mathcal{L}$ is said to be a Cauchy sequence provided for each $\epsilon > 0$ there is a natural number $\varsigma_0 \in \mathbb{N}$ such that $S(\varsigma_i, \varsigma_i, \varsigma_j) < \epsilon$, $\forall i, j \geq i_0$.
- (iii) If every Cauchy sequence in $\mathcal{L}$ is converges to a limit point in $\mathcal{L}$, then $(\mathcal{L}, S)$ is complete.

Lemma 2.4. [8] Let $(\mathcal{L}, S)$ be an S-metric space and suppose the sequence $\{\varsigma_i\}$ on $\mathcal{L}$ is not Cauchy while satisfying $\lim_{i \rightarrow \infty} S(\varsigma_i, \varsigma_i, \varsigma_{i+1}) = 0$, then for any given $\epsilon > 0$, we can find two subsequences $\{\varsigma_{i_k}\}$ and $\{\varsigma_{j_k}\}$ of $\{\varsigma_i\}$, consider any $\epsilon > 0$ we have

$$\lim_{k \rightarrow \infty} S(\varsigma_{i_k+1}, \varsigma_{i_k+1}, \zeta_{i_k+1}) = \epsilon^+ \\ \lim_{k \rightarrow \infty} S(\varsigma_{i_k}, \varsigma_{i_k}, \varsigma_{i_k}) = \lim_{k \rightarrow \infty} S(\varsigma_{i_k+1}, \varsigma_{i_k+1}, \varsigma_{i_k}) = \lim_{k \rightarrow \infty} S(\varsigma_{i_k}, \varsigma_{i_k}, \varsigma_{i_k+1}) = \epsilon$$

Definition 2.5. [3] An element $(\varsigma, \zeta) \in \mathcal{L} \times \mathcal{L}$ is referred as a coupled fixed point of the mapping $\mathfrak{J}: \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ if it satisfies $\mathfrak{J}(\varsigma, \zeta) = \varsigma$ and $\mathfrak{J}(\zeta, \varsigma) = \zeta$.

3. Fixed point theorems for New Pata-type contraction

This section explores fixed point theorem for New Pata-type contraction, establishes within the structure of an S -metric space.

Definition 3.1. In the S metric space $(\mathcal{L}, S)$, a self mapping $\partial : \mathcal{L} \rightarrow \mathcal{L}$ is referred to as a New S -Pata-type Contraction mapping if it adheres to the following condition

$$S(\partial\varsigma, \partial\zeta, \partial\varrho) \leq \frac{1-\epsilon}{4} \left\{ S(\varsigma, \zeta, \varrho) + S(\varsigma, \varsigma, \partial\varsigma) + S(\zeta, \zeta, \partial\zeta) + S(\varrho, \varrho, \partial\varrho) \right\} \\ + \hbar\epsilon^{\mathfrak{m}}\Xi(\epsilon) \left[1 + \|\varsigma\| + \|\zeta\| + \|\varrho\| + \|\partial\varsigma\| + \|\partial\zeta\| + \|\partial\varrho\| \right]^{\mathfrak{n}} \quad (3.1)$$

holds for any $\epsilon \in [0, 1)$ and fixed constants $\hbar \geq 0$, $\mathfrak{m} \geq 1$ and $\mathfrak{n} \in [0, \mathfrak{m}]$ and $\Xi : [0, 1] \rightarrow [0, \infty]$ to be a non decreasing and continuous vanishing at 0.

Definition 3.2. Let $(\mathcal{L}, S)$ be a S metric space. A self mapping $\partial : \mathcal{L} \rightarrow \mathcal{L}$ is said to have New S -Pata-type Contraction mapping of type-II if it satisfies the following inequality

$$S(\partial\varsigma, \partial\varsigma, \partial\zeta) \leq \frac{1-\epsilon}{4} \left\{ S(\varsigma, \varsigma, \zeta) + 2S(\varsigma, \varsigma, \partial\varsigma) + S(\zeta, \zeta, \partial\zeta) \right\} \\ + \hbar\epsilon^{\mathfrak{m}}\Xi(\epsilon) \left[1 + 2\|\varsigma\| + \|\zeta\| + 2\|\partial\varsigma\| + \|\partial\zeta\| \right]^{\mathfrak{n}} \quad (3.2)$$

holds for any $\epsilon \in [0, 1)$ and fixed constants $\hbar \geq 0$, $\mathfrak{m} \geq 1$ and $\mathfrak{n} \in [0, \mathfrak{m}]$ and $\Xi : [0, 1] \rightarrow [0, \infty]$ to be a non decreasing and continuous vanishing at 0.

Example 3.3. Consider the S -metric space $(\mathcal{L}, S)$, where $\mathcal{L} = [0, 1]$. The S -metric is specified by $S(\varrho, \sigma, \varsigma) = |\varrho - \sigma| + |\sigma - \varsigma| + |\varsigma - \varrho|$. Consider a self-map on $\partial : \mathcal{L} \rightarrow \mathcal{L}$ that is defined as

$$\partial\varrho = \frac{\varrho}{8} \in [0, 1]$$

Clearly $\partial([0, 1]) = [0, \frac{1}{8}] \subseteq [0, 1]$, where $\hbar = \frac{1}{8}$, $\mathfrak{m} = \mathfrak{n} = \frac{1}{4}$, $\Xi(\epsilon) = \epsilon^{\frac{1}{4}}$.

$$S(\partial\varrho, \partial\varrho, \partial\sigma) = S\left(\frac{\varrho}{8}, \frac{\varrho}{8}, \frac{\sigma}{8}\right) \\ \leq \frac{1}{8} \left[S(\varrho, \varrho, \partial\varrho) + S(\varrho, \varrho, \partial\varrho) + S(\sigma, \sigma, \partial\sigma) + S(\varrho, \varrho, \sigma) \right] \\ \leq \frac{1-\epsilon}{4} \left[S(\varrho, \varrho, \partial\varrho) + S(\varrho, \varrho, \partial\varrho) + S(\sigma, \sigma, \partial\sigma) + S(\varrho, \varrho, \sigma) \right] \\ + \left(\frac{1}{8} - \frac{1-\epsilon}{4} \right) \left[S(\varrho, \varrho, \partial\varrho) + S(\varrho, \varrho, \partial\varrho) + S(\sigma, \sigma, \partial\sigma) + S(\varrho, \varrho, \sigma) \right] \\ = \frac{1-\epsilon}{4} \left[S(\varrho, \varrho, \partial\varrho) + S(\varrho, \varrho, \partial\varrho) + S(\sigma, \sigma, \partial\sigma) + S(\varrho, \varrho, \sigma) \right] \\ + \frac{1}{8}\epsilon^{\frac{1}{2}} \left[S(\varrho, \varrho, \partial\varrho) + S(\varrho, \varrho, \partial\varrho) + S(\sigma, \sigma, \partial\sigma) + S(\varrho, \varrho, \sigma) \right] \\ \leq \frac{1-\epsilon}{4} \left[S(\varrho, \varrho, \partial\varrho) + S(\varrho, \varrho, \partial\varrho) + S(\sigma, \sigma, \partial\sigma) + S(\varrho, \varrho, \sigma) \right] \\ + \frac{1}{8}\epsilon^{\frac{1}{4}}\epsilon^{\frac{1}{4}} \left[S(\varrho, \varrho, \partial\varrho) + S(\varrho, \varrho, \partial\varrho) + S(\sigma, \sigma, \partial\sigma) + S(\varrho, \varrho, \sigma) \right]$$

This example satisfies the S -Pata-type contraction.

Theorem 3.4. Let $\partial : \mathcal{L} \rightarrow \mathcal{L}$ be a self-map on the complete S metric space $(\mathcal{L}, S)$ satisfying the following condition:

$$S(\partial\varsigma, \partial\zeta, \partial\varrho) \leq \frac{1-\epsilon}{4} \left\{ S(\varsigma, \zeta, \varrho) + S(\varsigma, \varsigma, \partial\varsigma) + S(\zeta, \zeta, \partial\zeta) + S(\varrho, \varrho, \partial\varrho) \right\} \\ + \hbar\epsilon^{\mathfrak{M}}\Xi(\epsilon) \left\{ 1 + \|\varsigma\| + \|\zeta\| + \|\varrho\| + \|\partial\varsigma\| + \|\partial\zeta\| + \|\partial\varrho\| \right\}^{\mathfrak{M}} \quad (3.3)$$

for every $\epsilon \in [0, 1)$ and $\varsigma, \zeta, \varrho \in \mathcal{L}$. Under this condition, ∂ possesses a unique fixed point ς^* . Furthermore, for any initial point $\varsigma_0 \in \mathcal{L}$ the sequence defined by $\varsigma_i = \partial\varsigma_{i-1}$ converges to ς^* as $\lim_{i \rightarrow \infty}$.

Proof. Select a point ς_0 from $\mathcal{L}$, and set $\varsigma_i = \partial\varsigma_{i-1}$. Define

$$C_i = S(\varsigma_i, \varsigma_i, \varsigma_0) = \|\varsigma_i\|.$$

Monotonicity: To prove monotonicity, in (3.3) we are taking $\epsilon = 0$ and using definition (2.2) we have

$$S(\varsigma_{i+1}, \varsigma_{i+1}, \varsigma_i) = S(\partial\varsigma_i, \partial\varsigma_i, \partial\varsigma_{i-1}) \\ \leq \frac{1}{4} \left\{ S(\varsigma_i, \varsigma_i, \varsigma_{i-1}) + S(\varsigma_i, \varsigma_i, \partial\varsigma_i) + S(\varsigma_i, \varsigma_i, \partial\varsigma_i) + S(\varsigma_{i-1}, \varsigma_{i-1}, \partial\varsigma_{i-1}) \right\} \\ \leq \frac{1}{4} \left\{ S(\varsigma_i, \varsigma_i, \varsigma_{i-1}) + S(\varsigma_i, \varsigma_i, \varsigma_{i+1}) + S(\varsigma_i, \varsigma_i, \varsigma_{i+1}) + S(\varsigma_{i-1}, \varsigma_{i-1}, \varsigma_i) \right\} \\ 4S(\varsigma_{i+1}, \varsigma_{i+1}, \varsigma_i) \leq S(\varsigma_i, \varsigma_i, \varsigma_{i-1}) + S(\varsigma_{i+1}, \varsigma_{i+1}, \varsigma_i) + S(\varsigma_{i+1}, \varsigma_{i+1}, \varsigma_i) + S(\varsigma_i, \varsigma_i, \varsigma_{i-1}) \\ 4S(\varsigma_{i+1}, \varsigma_{i+1}, \varsigma_i) - 2S(\varsigma_{i+1}, \varsigma_{i+1}, \varsigma_i) \leq 2S(\varsigma_i, \varsigma_i, \varsigma_{i-1}) \\ S(\varsigma_{i+1}, \varsigma_{i+1}, \varsigma_i) \leq S(\varsigma_i, \varsigma_i, \varsigma_{i-1}) \\ \Rightarrow S(\varsigma_{i+1}, \varsigma_{i+1}, \varsigma_i) \leq S(\varsigma_i, \varsigma_i, \varsigma_{i-1}) \leq \dots \leq S(\varsigma_1, \varsigma_1, \varsigma_0) = C_1 \quad (3.4)$$

Therefore, $S(\varsigma_{i+1}, \varsigma_{i+1}, \varsigma_i)$ is monotone decreasing and bounded below by 0.

Boundedness: Next, our aim is to show that the sequence C_i is a bounded.

$$C_i = S(\varsigma_i, \varsigma_i, \varsigma_0) \\ \leq [S(\varsigma_i, \varsigma_i, \varsigma_{i+1}) + S(\varsigma_i, \varsigma_i, \varsigma_{i+1}) + S(\varsigma_0, \varsigma_0, \varsigma_{i+1})] \\ \leq [S(\varsigma_{i+1}, \varsigma_{i+1}, \varsigma_i) + S(\varsigma_{i+1}, \varsigma_{i+1}, \varsigma_i) + S(\varsigma_0, \varsigma_0, \varsigma_{i+1})] \\ \leq 2S(\varsigma_{i+1}, \varsigma_{i+1}, \varsigma_i) + [S(\varsigma_0, \varsigma_0, \varsigma_1) + S(\varsigma_0, \varsigma_0, \varsigma_1) + S(\varsigma_{i+1}, \varsigma_{i+1}, \varsigma_1)] \\ S(\varsigma_i, \varsigma_i, \varsigma_0) \leq 2S(\varsigma_{i+1}, \varsigma_{i+1}, \varsigma_i) + [S(\varsigma_1, \varsigma_1, \varsigma_0) + S(\varsigma_1, \varsigma_1, \varsigma_0) + S(\varsigma_{i+1}, \varsigma_{i+1}, \varsigma_1)] \\ \leq 2S(\varsigma_{i+1}, \varsigma_{i+1}, \varsigma_i) + 2S(\varsigma_1, \varsigma_1, \varsigma_0) + S(\varsigma_{i+1}, \varsigma_{i+1}, \varsigma_1)$$

Using the monotonicity established above equation (3.4),

$$S(\varsigma_i, \varsigma_i, \varsigma_0) \leq 2C_1 + 2C_1 + S(\partial\varsigma_i, \partial\varsigma_i, \partial\varsigma_0) \\ C_i \leq 4C_1 + \frac{1-\epsilon}{4} [S(\varsigma_i, \varsigma_i, \varsigma_0) + S(\varsigma_i, \varsigma_i, \varsigma_{i+1}) + S(\varsigma_i, \varsigma_i, \varsigma_{i+1}) + S(\varsigma_0, \varsigma_0, \varsigma_1)] \\ + \hbar\epsilon^{\mathfrak{M}}\Xi(\epsilon) \left\{ 1 + \|\varsigma_i\| + \|\varsigma_i\| + \|\varsigma_0\| + \|\partial\varsigma_i\| + \|\partial\varsigma_i\| + \|\partial\varsigma_0\| \right\}^{\mathfrak{N}} \\ \leq 4C_1 + \frac{1-\epsilon}{4} [C_i + C_1 + C_1 + C_1] + \hbar\epsilon^{\mathfrak{M}}\Xi(\epsilon) \left\{ 1 + 2\|\varsigma_i\| + 2\|\partial\varsigma_i\| + C_1 \right\}^{\mathfrak{N}} \\ \leq 4C_1 + \frac{1-\epsilon}{4} [C_i + 3C_1] + \hbar\epsilon^{\mathfrak{M}}\Xi(\epsilon) \left\{ 1 + 2\|\varsigma_i\| + 2\|\varsigma_{i+1}\| + \|\varsigma_1\| \right\}^{\mathfrak{M}}$$

$$\begin{aligned}
&\leq 4C_1 + \frac{1-\epsilon}{4}[C_i + 3C_1] + \hbar\epsilon^{\mathfrak{m}}\Xi(\epsilon)\{1 + 2C_i + 2C_1 + C_i + C_1\}^{\mathfrak{m}} \\
&\leq 4C_1 + \frac{1-\epsilon}{4}[C_i + 3C_1] + \hbar\epsilon^{\mathfrak{m}}\Xi(\epsilon)\{1 + 3C_i + 3C_1\}^{\mathfrak{m}} \\
C_i &\leq \frac{1-\epsilon}{4}(C_i) + a\epsilon^{\mathfrak{m}}\Xi(\epsilon)C_i^{\mathfrak{m}} + b \text{ for some } a, b > 0.
\end{aligned}$$

Assume C_i is not bounded. Then, there exists a sub sequence $C_{i_k} \rightarrow \infty$. Therefore, we can represent it as follows:

$$\left(\frac{3 + \epsilon_i}{4}\right)C_{i_k} \leq a\epsilon_k^{\mathfrak{m}}\Xi(\epsilon_k)C_{i_k}^{\mathfrak{m}} + b.$$

Let's now assume that taking

$$\epsilon_k = \frac{4 + 4b - 3C_{i_k}}{C_{i_k}}, \Rightarrow 1 + b \leq a\epsilon_k^{\mathfrak{m}}\Xi(\epsilon_k)1 \leq b(1 + a)^{\mathfrak{m}}\Xi(\epsilon_k) \rightarrow 0.$$

This contradiction confirms that C_i is bounded.

To show successive differences vanish:

Moreover, $S(\varsigma_{i+1}, \varsigma_{i+1}, \varsigma_i)$ is a sequence of monotonically decreasing and with lower bound of 0, there exists a $\tau_1 \geq 0$ such that

$$S(\varsigma_{i+1}, \varsigma_{i+1}, \varsigma_i) \rightarrow \tau_1.$$

Then

$$\begin{aligned}
S(\varsigma_{i+1}, \varsigma_{i+1}, \varsigma_i) &\leq S(\partial\varsigma_i, \partial\varsigma_i, \partial\varsigma_{i-1}) \leq \frac{1-\epsilon}{4}\left\{S(\varsigma_i, \varsigma_i, \varsigma_{i-1}) + S(\varsigma_i, \varsigma_i, \partial\varsigma_i) + S(\varsigma_i, \varsigma_i, \partial\varsigma_i) \right. \\
&\quad \left. + S(\varsigma_{i-1}, \varsigma_{i-1}, \partial\varsigma_{i-1})\right\} + K\epsilon^{\mathfrak{m}}\Xi(\epsilon) : K > 0 \\
&\leq \frac{1-\epsilon}{4}\left\{2S(\varsigma_i, \varsigma_i, \varsigma_{i+1}) + S(\varsigma_{i-1}, \varsigma_{i-1}, \varsigma_i) + S(\varsigma_i, \varsigma_i, \varsigma_{i-1})\right\} + K\epsilon^{\mathfrak{m}}\Xi(\epsilon) \\
&\leq \frac{1-\epsilon}{4}\left\{2S(\varsigma_i, \varsigma_i, \varsigma_{i+1}) + 2S(\varsigma_{i-1}, \varsigma_{i-1}, \varsigma_i)\right\}
\end{aligned}$$

where $K > 0$ and as $i \rightarrow \infty$ we get

$$\begin{aligned}
\tau_1 &\leq \frac{1-\epsilon}{4}[2\tau_1 + 2\tau_1] + K\epsilon^{\mathfrak{m}}\Xi(\epsilon) \\
\tau_1 &\leq \tau_1 - \epsilon\tau_1 + K\epsilon^{\mathfrak{m}}\Xi(\epsilon) \\
&\leq K\epsilon^{\mathfrak{m}-1}\Xi(\epsilon) \\
\tau_1 &= 0.
\end{aligned}$$

Proof of the Cauchy sequence:

To demonstrate that $\{\varsigma_i\}$ is a Cauchy sequence, let us instead assume that $\{\varsigma_i\}$ is not. For every $\tau_2 > 0$, Lemma (2.4) indicates the existence of two subsequences of sequence $\{\varsigma_i\}$, namely $\{\varsigma_{i_k}\}$ and $\{\varsigma_{j_k}\}$ such that

$$\lim_{k \rightarrow \infty} S(\varsigma_{i_k}, \varsigma_{i_k}, \varsigma_{j_k}) = \tau_2^+ \text{ and } \lim_{k \rightarrow \infty} S(\varsigma_{i_k-1}, \varsigma_{i_k-1}, \varsigma_{j_k-1}) = \tau_2.$$

Hence,

$$\begin{aligned}
 S(\varsigma_{i_k}, \varsigma_{i_k}, \varsigma_{j_k}) &= S(\partial \varsigma_{i_k-1}, \partial \varsigma_{i_k-1}, \partial \varsigma_{j_k-1}) \\
 &\leq \frac{1-\epsilon}{4} \left\{ S(\varsigma_{i_k-1}, \varsigma_{i_k-1}, \varsigma_{j_k-1}) + S(\varsigma_{i_k-1}, \varsigma_{i_k-1}, \partial \varsigma_{i_k-1}) \right. \\
 &\quad \left. + S(\varsigma_{i_k-1}, \varsigma_{i_k-1}, \partial \varsigma_{i_k-1}) + S(\varsigma_{j_k-1}, \varsigma_{j_k-1}, \partial \varsigma_{j_k-1}) \right\} + K\epsilon^{\mathfrak{M}}\Xi(\epsilon) \\
 &\leq \frac{1-\epsilon}{4} \left\{ S(\varsigma_{i_k-1}, \varsigma_{i_k-1}, \varsigma_{j_k-1}) + 2S(\varsigma_{i_k-1}, \varsigma_{i_k-1}, \varsigma_{i_k}) + S(\varsigma_{j_k-1}, \varsigma_{j_k-1}, \varsigma_{j_k}) \right\} + K\epsilon^{\mathfrak{M}}\Xi(\epsilon)
 \end{aligned}$$

Considering the aforementioned limit and allowing for $k \rightarrow \infty$ we get

$$\begin{aligned}
 \tau_2 &\leq \frac{1-\epsilon}{4} [4\tau_2] + K\epsilon^{\mathfrak{M}}\Xi(\epsilon) \\
 \tau_2 &\leq K\epsilon^{\mathfrak{M}-1}\Xi(\epsilon)
 \end{aligned}$$

This inequality holds for every $\forall \epsilon \in [0, 1)$, and then lets $\epsilon \rightarrow 0^+$ together with $\lim_{\epsilon \rightarrow 0} \epsilon^{\mathfrak{M}-1}\Xi(\epsilon) = 0 \Rightarrow \tau_2 = 0$.

Which contradicts itself. Hence, the sequence $\{\varsigma_i\}$ is Cauchy. We have ς^* in $\mathcal{L}$ such that $\{\varsigma_i\} \rightarrow \varsigma^*$, taking into account the completeness of $\mathcal{L}$. Now We need to demonstrate that, for any $i \in \mathbb{N}$, ς^* is a fixed point for the mapping ∂ .

$$\begin{aligned}
 S(\varsigma^*, \varsigma^*, \partial \varsigma^*) &\leq S(\varsigma^*, \varsigma^*, \varsigma_{i+1}) + S(\varsigma^*, \varsigma^*, \varsigma_{i+1}) + S(\partial \varsigma^*, \partial \varsigma^*, \partial \varsigma_i) \\
 &\leq 2S(\varsigma^*, \varsigma^*, \varsigma_{i+1}) + \frac{1-\epsilon}{4} \left\{ S(\varsigma^*, \varsigma^*, \varsigma_i) + S(\varsigma^*, \varsigma^*, \partial \varsigma^*) \right. \\
 &\quad \left. + S(\varsigma^*, \varsigma^*, \partial \varsigma^*) + S(\varsigma_i, \varsigma_i, \partial \varsigma_i) \right\} + K\epsilon^{\mathfrak{M}}\Xi(\epsilon) \\
 &\leq 2S(\varsigma^*, \varsigma^*, \varsigma_{i+1}) + \frac{1-\epsilon}{4} \left\{ S(\varsigma^*, \varsigma^*, \varsigma_i) + 2S(\varsigma^*, \varsigma^*, \partial \varsigma^*) \right. \\
 &\quad \left. + S(\varsigma_i, \varsigma_i, \varsigma_{i+1}) \right\} + K\epsilon^{\mathfrak{M}}\Xi(\epsilon)
 \end{aligned}$$

letting $i \rightarrow \infty$ and $\epsilon = 0$ we have

$$\begin{aligned}
 S(\varsigma^*, \varsigma^*, \partial \varsigma^*) &\leq \frac{1}{2} [S(\varsigma^*, \varsigma^*, \partial \varsigma^*)] \\
 S(\varsigma^*, \varsigma^*, \partial \varsigma^*) &= 0, \varsigma^* = \partial \varsigma^*.
 \end{aligned}$$

Uniqueness: Suppose ς^* and ζ^* are two fixed points

$$\begin{aligned}
 S(\varsigma^*, \varsigma^*, \zeta^*) &= S(\partial \varsigma^*, \partial \varsigma^*, \partial \zeta^*) \\
 &\leq \frac{1-\epsilon}{4} \left\{ S(\varsigma^*, \varsigma^*, \zeta^*) + S(\varsigma^*, \varsigma^*, \partial \varsigma^*) + S(\varsigma^*, \varsigma^*, \partial \varsigma^*) \right. \\
 &\quad \left. + S(\zeta^*, \zeta^*, \partial \zeta^*) \right\} + K\epsilon^{\mathfrak{M}}\Xi(\epsilon) \\
 &\leq \frac{1-\epsilon}{4} \{ S(\varsigma^*, \varsigma^*, \zeta^*) \} + K\epsilon^{\mathfrak{M}}\Xi(\epsilon) = 0 \\
 \varsigma^* &= \zeta^*
 \end{aligned}$$

Hence ς^* is a Unique fixed point of the function ∂ .

□

4. Coupled fixed point theorems for New Pata-type contraction

This section explores Coupled fixed point theorem for New Pata-type contraction, establishes within the structure of an S -metric space.

Definition 4.1. Let $(\mathcal{L}, S)$ be a S -metric space endowed with a partial order $\leq$ and $\mathfrak{J} : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$. If for any $\varsigma, \zeta \in \mathcal{L}$, $\mathfrak{J}(\varsigma, \zeta)$ is monotone non decreasing in ς and is monotone non-increasing in ζ , then we say that $\mathfrak{J}$ has the mixed monotone property on $\mathcal{L}$

$$\begin{aligned}\varsigma_1, \varsigma_2 \in \mathcal{L}, \varsigma_1 \leq \varsigma_2 &\Rightarrow \mathfrak{J}(\varsigma_1, \zeta) \leq \mathfrak{J}(\varsigma_2, \zeta) \\ \zeta_1, \zeta_2 \in \mathcal{L}, \zeta_1 \leq \zeta_2 &\Rightarrow \mathfrak{J}(\varsigma, \zeta_1) \geq \mathfrak{J}(\varsigma, \zeta_2)\end{aligned}$$

Definition 4.2. In an S -metric space $(\mathcal{L}, S)$, the mapping $\bar{S} : \mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{L}^2 \rightarrow [0, \infty)$ is defined as

$$\bar{S}[(\varsigma, \zeta), (\check{u}, \check{v}), (\check{a}, \check{b})] = S(\varsigma, \check{u}, \check{a}) + S(\zeta, \check{v}, \check{b})$$

for every $((\varsigma, \zeta), (\check{u}, \check{v}), (\check{a}, \check{b})) \in \mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{L}^2$. This defines a metric on $\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{L}^2$ denoted by $\bar{S}$ for simplicity. In an S metric space $(\mathcal{L}, S)$, give an arbitrary element $(\varsigma_0, \zeta_0) \in \mathcal{L} \times \mathcal{L}$, we define

$$\|x, y\| = S[(\varsigma, \zeta), (\varsigma, \zeta), (\varsigma_0, \zeta_0)] \forall (\varsigma, \zeta) \in \mathcal{L} \times \mathcal{L}.$$

Theorem 4.3. Let $(\mathcal{L}, S)$ be a complete S metric space, and let $\mathfrak{J} : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ be a continuous mapping with the mixed monotone property on $\mathcal{L}$. Suppose $\hbar \geq 0$, $\mathfrak{m} \geq 1$ and $\mathfrak{p} \in [0, \mathfrak{m}]$ are constants, and there exist points $\varsigma_0, \zeta_0 \in \mathcal{L}$ such that $\varsigma_0 \leq \mathfrak{J}(\varsigma_0, \zeta_0)$ and $\mathfrak{p}_0 \leq \mathfrak{J}(\zeta_0, \varsigma_0)$. If the inequality holds $\forall 0 \leq \epsilon < 1$ and $(\varsigma, \zeta), (\check{u}, \check{v}), (\check{a}, \check{b}) \in \mathcal{L} \times \mathcal{L} \times \mathcal{L}$ with $\check{u}, \check{a} \leq \varsigma, \zeta \leq \check{v}, \check{b}$,

$$\begin{aligned}S(\mathfrak{J}(\varsigma, \zeta), \mathfrak{J}(\check{u}, \check{v}), \mathfrak{J}(\check{a}, \check{b})) &\leq (1 - \epsilon)S[(\varsigma, \zeta), (\check{u}, \check{v}), (\check{a}, \check{b})] \\ &\quad + \hbar \epsilon^{\mathfrak{m}} \Xi(\epsilon) [1 + \|\varsigma, \zeta\| + \|\check{u}, \check{v}\| + \|\check{a}, \check{b}\|]^{\mathfrak{p}}\end{aligned}\quad (4.1)$$

then $\mathfrak{J}$ possesses a coupled fixed point (ς^*, ζ^*) . Additionally, define $\mathfrak{J}^i = \mathfrak{J} \circ \mathfrak{J} \circ \dots \circ \mathfrak{J}$ (i times). Then the following inequality holds:

$$S[(\varsigma^*, \zeta^*), (\varsigma^*, \zeta^*), (\mathfrak{J}^i(\varsigma_0, \zeta_0), \mathfrak{J}^i(\zeta_0, \varsigma_0))] \leq \frac{2}{3} \|\varsigma^*, \zeta^*\| \quad (4.2)$$

where a positive constant K satisfies

$$K \leq 2\hbar [1 + 4\|\varsigma^*, \zeta^*\| + 4S((\varsigma_0, \zeta_0), (\varsigma_0, \zeta_0), (\zeta_0, \varsigma_0))]^{\mathfrak{p}}$$

Proof. Set $\mathfrak{J}(\varsigma_0, \zeta_0) = \varsigma_1$ and $\mathfrak{J}(\zeta_0, \varsigma_0) = \zeta_1$. So $\varsigma_1 \geq \varsigma_0, \zeta_1 \leq \zeta_0$.

Now from letting $\varsigma_2 = \mathfrak{J}(\varsigma_1, \zeta_1), \zeta_2 = \mathfrak{J}(\zeta_1, \varsigma_1)$ and choosing the notations

$$\begin{aligned}\mathfrak{J}^2(\varsigma_0, \zeta_0) &= \mathfrak{J}[\mathfrak{J}(\varsigma_0, \zeta_0), \mathfrak{J}(\zeta_0, \varsigma_0)] = \mathfrak{J}(\varsigma_1, \zeta_1) = \varsigma_2 \\ \mathfrak{J}^2(\zeta_0, \varsigma_0) &= \mathfrak{J}[\mathfrak{J}(\zeta_0, \varsigma_0), \mathfrak{J}(\varsigma_0, \zeta_0)] = \mathfrak{J}(\zeta_1, \varsigma_1) = \zeta_2\end{aligned}$$

By using Definition 4.1 we have,

$$\begin{aligned}\varsigma_2 &= \mathfrak{J}^2(\varsigma_0, \zeta_0) = \mathfrak{J}(\varsigma_1, \zeta_1) \geq \mathfrak{J}(\varsigma_0, \zeta_0) = \varsigma_1 \Rightarrow \varsigma_2 \geq \varsigma_1 \\ \zeta_2 &= \mathfrak{J}^2(\zeta_0, \varsigma_0) = \mathfrak{J}(\zeta_1, \varsigma_1) \leq \mathfrak{J}(\zeta_0, \varsigma_0) = \zeta_1 \Rightarrow \zeta_2 \leq \zeta_1.\end{aligned}$$

We get the two sequences by continuing the above argument $\{\varsigma_i\}_{i=0}^{\infty}$ and $\{\zeta_i\}_{i=0}^{\infty}$ such that

$\varsigma_0 \leq \varsigma_1 \leq \dots \leq \varsigma_i \leq \dots$ and $\zeta_0 \geq \zeta_1 \geq \dots \geq \zeta_i \geq \dots$.

Furthermore for $i = 1, 2, \dots$ we set,

$$\begin{aligned}\varsigma_i &= \mathfrak{J}(\varsigma_{i-1}, \zeta_{i-1}) = \mathfrak{J}^i(\varsigma_0, \zeta_0) \\ \zeta_i &= \mathfrak{J}(\zeta_{i-1}, \varsigma_{i-1}) = \mathfrak{J}^i(\zeta_0, \varsigma_0) \\ \text{and } C_i &= \left\| \mathfrak{J}^i(\varsigma_0, \zeta_0), \mathfrak{J}^i(\zeta_0, \varsigma_0) \right\| \\ &= S((\varsigma_i, \zeta_i), (\varsigma_i, \zeta_i), (\varsigma_0, \zeta_0))\end{aligned}$$

Using the above setting of ς_i and ζ_i , we can write

$$S(\mathfrak{J}(\varsigma_i, \zeta_i), \mathfrak{J}(\varsigma_i, \zeta_i), \mathfrak{J}(\varsigma_{i-1}, \zeta_{i-1})) = S\left[\left(\mathfrak{J}^{i+1}(\varsigma_0, \zeta_0), \mathfrak{J}^{i+1}(\zeta_0, \varsigma_0)\right), \left(\mathfrak{J}^{i+1}(\varsigma_0, \zeta_0), \mathfrak{J}^{i+1}(\zeta_0, \varsigma_0)\right), \left(\mathfrak{J}^i(\varsigma_0, \zeta_0), \mathfrak{J}^i(\zeta_0, \varsigma_0)\right)\right]$$

Applying the equation (4.1) to the above expression and taking $\epsilon = 0$ we obtain,

$$\begin{aligned}S\left[\left(\mathfrak{J}^{i+1}(\varsigma_0, \zeta_0), \mathfrak{J}^{i+1}(\zeta_0, \varsigma_0)\right), \left(\mathfrak{J}^{i+1}(\varsigma_0, \zeta_0), \mathfrak{J}^{i+1}(\zeta_0, \varsigma_0)\right), \left(\mathfrak{J}^i(\varsigma_0, \zeta_0), \mathfrak{J}^i(\zeta_0, \varsigma_0)\right)\right] &\leq S\left[\left(\mathfrak{J}^i(\varsigma_0, \zeta_0), \mathfrak{J}^i(\zeta_0, \varsigma_0)\right), \left(\mathfrak{J}^i(\varsigma_0, \zeta_0), \mathfrak{J}^i(\zeta_0, \varsigma_0)\right), \right. \\ &\quad \left. \left(\mathfrak{J}^{i-1}(\varsigma_0, \zeta_0), \mathfrak{J}^{i-1}(\zeta_0, \varsigma_0)\right)\right] \\ S\left[\left(\mathfrak{J}^{i+1}(\varsigma_0, \zeta_0), \mathfrak{J}^{i+1}(\zeta_0, \varsigma_0)\right), \left(\mathfrak{J}^{i+1}(\varsigma_0, \zeta_0), \mathfrak{J}^{i+1}(\zeta_0, \varsigma_0)\right), \left(\mathfrak{J}^i(\varsigma_0, \zeta_0), \mathfrak{J}^i(\zeta_0, \varsigma_0)\right)\right] &\leq S\left[\left(\mathfrak{J}^i(\varsigma_0, \zeta_0), \mathfrak{J}^i(\zeta_0, \varsigma_0)\right), \left(\mathfrak{J}^i(\varsigma_0, \zeta_0), \mathfrak{J}^i(\zeta_0, \varsigma_0)\right), \right. \\ &\quad \left. \left(\mathfrak{J}^{i-1}(\varsigma_0, \zeta_0), \mathfrak{J}^{i-1}(\zeta_0, \varsigma_0)\right)\right] \\ &\leq \vdots \\ &\leq S\left[\left(\mathfrak{J}(\varsigma_0, \zeta_0), \mathfrak{J}(\zeta_0, \varsigma_0)\right), \left(\mathfrak{J}(\varsigma_0, \zeta_0), \mathfrak{J}(\zeta_0, \varsigma_0)\right), \left(\varsigma_0, \zeta_0\right)\right] \\ S\left[\left(\mathfrak{J}^{i+1}(\varsigma_0, \zeta_0), \mathfrak{J}^{i+1}(\zeta_0, \varsigma_0)\right), \left(\mathfrak{J}^{i+1}(\varsigma_0, \zeta_0), \mathfrak{J}^{i+1}(\zeta_0, \varsigma_0)\right), \left(\mathfrak{J}^i(\varsigma_0, \zeta_0), \mathfrak{J}^i(\zeta_0, \varsigma_0)\right)\right] &\leq C_1 \tag{4.3} \\ S((\varsigma_i, \zeta_i), (\varsigma_i, \zeta_i), (\varsigma_0, \zeta_0)) &\leq 2S[(\varsigma_i, \zeta_i), (\varsigma_i, \zeta_i), (\varsigma_{i+1}, \zeta_{i+1})] \\ &\quad + S[(\varsigma_0, \zeta_0), (\varsigma_0, \zeta_0), (\varsigma_{i+1}, \zeta_{i+1})] \\ &\leq 2S[(\varsigma_{i+1}, \zeta_{i+1}), (\varsigma_{i+1}, \zeta_{i+1}), (\varsigma_i, \zeta_i)] \\ &\quad + 2S[(\varsigma_0, \zeta_0), (\varsigma_0, \zeta_0), (\varsigma_1, \zeta_1)] \\ &\quad + S[(\varsigma_{i+1}, \zeta_{i+1}), (\varsigma_{i+1}, \zeta_{i+1}), (\varsigma_1, \zeta_1)] \\ &\leq 2C_1 + 2C_1 + S\left[\left(\mathfrak{J}^{i+1}(\varsigma_0, \zeta_0), \mathfrak{J}^{i+1}(\zeta_0, \varsigma_0)\right), \right. \\ &\quad \left. \left(\mathfrak{J}^{i+1}(\varsigma_0, \zeta_0), \mathfrak{J}^{i+1}(\zeta_0, \varsigma_0)\right), \left(\mathfrak{J}(\varsigma_0, \zeta_0), \mathfrak{J}(\zeta_0, \varsigma_0)\right)\right] \\ S((\varsigma_i, \zeta_i), (\varsigma_i, \zeta_i), (\varsigma_0, \zeta_0)) &\leq 4C_1 \\ &\quad + S\left[\left(\mathfrak{J}^{i+1}(\varsigma_0, \zeta_0), \mathfrak{J}^{i+1}(\zeta_0, \varsigma_0), \mathfrak{J}(\varsigma_0, \zeta_0)\right)\right] \\ &\quad + S\left[\left(\mathfrak{J}^{i+1}(\zeta_0, \varsigma_0), \mathfrak{J}^{i+1}(\zeta_0, \varsigma_0), \mathfrak{J}(\zeta_0, \varsigma_0)\right)\right] \\ &\leq 4C_1 \\ &\quad + S\left[\mathfrak{J}(\mathfrak{J}^i(\varsigma_0, \zeta_0), \mathfrak{J}^i(\zeta_0, \varsigma_0)), \mathfrak{J}(\mathfrak{J}^i(\varsigma_0, \zeta_0), \right. \\ &\quad \left. \mathfrak{J}^i(\zeta_0, \varsigma_0)), \mathfrak{J}(\varsigma_0, \zeta_0)\right] + S\left[\mathfrak{J}(\mathfrak{J}^i(\zeta_0, \varsigma_0), \mathfrak{J}^i(\varsigma_0, \zeta_0)) \right. \\ &\quad \left. \mathfrak{J}(\mathfrak{J}^i(\zeta_0, \varsigma_0), \mathfrak{J}^i(\varsigma_0, \zeta_0)), \mathfrak{J}(\zeta_0, \varsigma_0)\right] \\ &\leq 4C_1 \\ &\quad + \frac{1-\epsilon}{3} S\left[\left(\mathfrak{J}^i(\varsigma_0, \zeta_0), \mathfrak{J}^i(\zeta_0, \varsigma_0)\right), \right. \\ &\quad \left. \left(\mathfrak{J}^i(\varsigma_0, \zeta_0), \mathfrak{J}^i(\zeta_0, \varsigma_0)\right), \left(\varsigma_0, \zeta_0\right)\right]\end{aligned}$$

$$\begin{aligned}
& + \hbar \epsilon^{\mathfrak{m}} \Xi(\epsilon) \left[1 + 2 \left\| \mathfrak{J}^{\mathfrak{i}}(s_0, \zeta_0), \mathfrak{J}^{\mathfrak{i}}(\zeta_0, s_0) \right\| + \left\| s_0, \zeta_0 \right\| \right]^{\mathfrak{n}} \\
& + \frac{1-\epsilon}{3} S \left[\left(\mathfrak{J}^{\mathfrak{i}}(\zeta_0, s_0), \mathfrak{J}^{\mathfrak{i}}(s_0, \zeta_0) \right), \left(\mathfrak{J}^{\mathfrak{i}}(\zeta_0, s_0), \right. \right. \\
& \left. \left. \mathfrak{J}^{\mathfrak{i}}(s_0, \zeta_0) \right), (\zeta_0, s_0) \right] \\
& + \hbar \epsilon^{\mathfrak{m}} \Xi(\epsilon) \left[1 + 2 \left\| \mathfrak{J}^{\mathfrak{i}}(\zeta_0, s_0), \mathfrak{J}^{\mathfrak{i}}(s_0, \zeta_0) \right\| + \left\| \zeta_0, s_0 \right\| \right]^{\mathfrak{n}}
\end{aligned}$$

But from the definition of $\|\cdot\|$ we have,

$$\begin{aligned}
\left[1 + 2 \left\| \mathfrak{J}^{\mathfrak{i}}(s_0, \zeta_0), \mathfrak{J}^{\mathfrak{i}}(\zeta_0, s_0) \right\| + \left\| s_0, \zeta_0 \right\| \right]^{\mathfrak{n}} & \leq 1 + 2C_{\mathfrak{i}} \leq 1 + 2C_{\mathfrak{i}} + 4S(s_0, s_0, \zeta_0) \\
\left[1 + 2 \left\| \mathfrak{J}^{\mathfrak{i}}(\zeta_0, s_0), \mathfrak{J}^{\mathfrak{i}}(s_0, \zeta_0) \right\| + \left\| \zeta_0, s_0 \right\| \right]^{\mathfrak{n}} & \leq 1 + 2C_{\mathfrak{i}} \leq 1 + 2C_{\mathfrak{i}} + 4S(s_0, s_0, \zeta_0)
\end{aligned}$$

So for $\mathfrak{m} \geq \mathfrak{n}$ there are some $D, E > 0$ such that,

$$\begin{aligned}
C_{\mathfrak{i}} & \leq 4C_1 \\
& + \frac{1-\epsilon}{3}(C_{\mathfrak{i}}) + \hbar \epsilon^{\mathfrak{m}} \Xi(\epsilon) \left[1 + 2C_{\mathfrak{i}} + 4S(s_0, s_0, \zeta_0) \right]^{\mathfrak{n}} \\
& + \frac{1-\epsilon}{3}(C_{\mathfrak{i}}) + \hbar \epsilon^{\mathfrak{m}} \Xi(\epsilon) \left[1 + 2C_{\mathfrak{i}} + 4S(s_0, s_0, \zeta_0) \right]^{\mathfrak{n}} \\
& \leq \frac{2}{3}(1-\epsilon)C_{\mathfrak{i}} + E\epsilon^{\mathfrak{m}} \Xi(\epsilon) C_{\mathfrak{i}}^{\mathfrak{m}} + D \\
C_{\mathfrak{i}} - \frac{2}{3}(1-\epsilon)C_{\mathfrak{i}} & \leq E\epsilon^{\mathfrak{m}} \psi(\mathfrak{m}) C_{\mathfrak{i}} + D
\end{aligned}$$

Suppose $C_{\mathfrak{i}}$ is not bounded $\exists$ subsequence $C_{i_k} \rightarrow \infty$.

$$C_{i_k} - \frac{(2-2\epsilon_{i_k})}{3} C_{i_k} \leq E\epsilon_{i_k}^{\mathfrak{m}} \psi(\epsilon_{i_k}) C_{i_k}^{\mathfrak{m}} + D \left(\frac{1+2\epsilon_{i_k}}{3} \right) C_{i_k} \leq E\epsilon_{i_k}^{\mathfrak{m}} \psi(\epsilon_{i_k}) C_{i_k}^{\mathfrak{m}} + D$$

This is valid by assumption $\forall 0 \leq \epsilon < 1$ applied to for all $i \in \mathbb{N}$ in the sub sequence C_{i_k} then the choice $\epsilon_{i_k} = \frac{3+3D-C_{i_k}}{2C_{i_k}}$ leads to the following contradiction.

$$\begin{aligned}
\left[\frac{1 + 2 \left(\frac{3+3D-C_{i_k}}{2C_{i_k}} \right)}{3} \right] C_{i_k} & \leq E\epsilon_{i_k}^{\mathfrak{m}} \psi(\epsilon_{i_k}) C_{i_k}^{\mathfrak{m}} + D \\
1 & \leq E\epsilon_{i_k}^{\mathfrak{m}} \psi(\epsilon_{i_k}) C_{i_k}^{\mathfrak{m}} \rightarrow 0 \text{ as } i_k \rightarrow \infty
\end{aligned}$$

Then the sequence $\{C_i\}_{i=1}^{\infty}$ is bounded.

To Prove: $\{\mathfrak{J}^i(s_0, \zeta_0)\}$ and $\{\mathfrak{J}^i(\zeta_0, s_0)\}$ are Cauchy Sequences

$$\begin{aligned}
 & S\left(\mathfrak{J}^{i+j+1}(s_0, \zeta_0), \mathfrak{J}^{i+j+1}(s_0, \zeta_0), \mathfrak{J}^{i+1}(s_0, \zeta_0)\right) \\
 & + S\left(\mathfrak{J}^{i+j+1}(\zeta_0, s_0), \mathfrak{J}^{i+j+1}(\zeta_0, s_0), \mathfrak{J}^{i+1}(\zeta_0, s_0)\right) = S\left[\mathfrak{J}^{i+j}(s_0, \zeta_0), \mathfrak{J}^{i+j}(\zeta_0, s_0), \mathfrak{J}^{i+j}(s_0, \zeta_0)\right. \\
 & \quad \left., \mathfrak{J}^{i+j}(\zeta_0, s_0), \mathfrak{J}^i(s_0, \zeta_0), \mathfrak{J}^i(\zeta_0, s_0)\right] \\
 & \quad + S\left[\mathfrak{J}^{i+j}(\zeta_0, s_0), \mathfrak{J}^{i+j}(s_0, \zeta_0)\right. \\
 & \quad \left., \mathfrak{J}^{i+j}(\zeta_0, s_0), \mathfrak{J}^{i+j}(s_0, \zeta_0), \mathfrak{J}^i(\zeta_0, s_0), \mathfrak{J}^i(s_0, \zeta_0)\right] \\
 & \leq \frac{1-\epsilon}{3} S\left[\left(\mathfrak{J}^{i+j}(s_0, \zeta_0), \mathfrak{J}^{i+j}(\zeta_0, s_0)\right)\right. \\
 & \quad \left., \left(\mathfrak{J}^{i+j}(s_0, \zeta_0), \mathfrak{J}^{i+j}(\zeta_0, s_0)\right), \left(\mathfrak{J}^i(s_0, \zeta_0), \mathfrak{J}^i(\zeta_0, s_0)\right)\right] \\
 & \quad + \hbar \epsilon^{\mathfrak{m}} \Xi(\epsilon) \left[1 + 2\|\mathfrak{J}^{i+j}(s_0, \zeta_0), \mathfrak{J}^{i+j}(\zeta_0, s_0)\| \right. \\
 & \quad \left. + \|\mathfrak{J}^i(s_0, \zeta_0), \mathfrak{J}^i(\zeta_0, s_0)\|\right]^{\mathfrak{p}} \\
 & \quad + \frac{1-\epsilon}{3} S\left[\left(\mathfrak{J}^{i+j}(\zeta_0, s_0), \mathfrak{J}^{i+j}(s_0, \zeta_0)\right), \right. \\
 & \quad \left. \left(\mathfrak{J}^{i+j}(\zeta_0, s_0), \mathfrak{J}^{i+j}(s_0, \zeta_0)\right), \left(\mathfrak{J}^i(\zeta_0, s_0), \mathfrak{J}^i(s_0, \zeta_0)\right)\right] \\
 & \quad + \hbar \epsilon^{\mathfrak{m}} \Xi(\epsilon) \left[1 + 2\|\mathfrak{J}^{i+j}(\zeta_0, s_0), \mathfrak{J}^{i+j}(s_0, \zeta_0)\| \right. \\
 & \quad \left. + \|\mathfrak{J}^i(\zeta_0, s_0), \mathfrak{J}^i(s_0, \zeta_0)\|\right]^{\mathfrak{p}}
 \end{aligned} \tag{4.4}$$

We have

$$\begin{aligned}
 \|\mathfrak{J}^{i+j}(s_0, \zeta_0), \mathfrak{J}^{i+j}(\zeta_0, s_0)\| + \|\mathfrak{J}^i(s_0, \zeta_0), \mathfrak{J}^i(\zeta_0, s_0)\| & \leq C_{i+j} + C_i \\
 & \leq C_{i+j} + C_i + 4S\left((s_0, \zeta_0), (s_0, \zeta_0), (\zeta_0, s_0)\right) \\
 \|\mathfrak{J}^{i+j}(\zeta_0, s_0), \mathfrak{J}^{i+j}(s_0, \zeta_0)\| + \|\mathfrak{J}^i(\zeta_0, s_0), \mathfrak{J}^i(s_0, \zeta_0)\| & \leq C_{i+j} + C_i \\
 & \leq C_{i+j} + C_i + 4S\left((s_0, \zeta_0), (s_0, \zeta_0), (\zeta_0, s_0)\right)
 \end{aligned}$$

From above (4.4) becomes,

$$\begin{aligned}
 & S\left(\mathfrak{J}^{i+j+1}(s_0, \zeta_0), \mathfrak{J}^{i+j+1}(s_0, \zeta_0), \mathfrak{J}^{i+1}(s_0, \zeta_0)\right) \\
 & + S\left(\mathfrak{J}^{i+j+1}(\zeta_0, s_0), \mathfrak{J}^{i+j+1}(\zeta_0, s_0), \mathfrak{J}^{i+1}(\zeta_0, s_0)\right) \leq \frac{2}{3}(1-\epsilon) \\
 & \quad \left[S\left[\mathfrak{J}^{i+j}(s_0, \zeta_0), \mathfrak{J}^{i+j}(s_0, \zeta_0), \mathfrak{J}^i(s_0, \zeta_0)\right] \right. \\
 & \quad \left. + S\left[\mathfrak{J}^{i+j}(\zeta_0, s_0), \mathfrak{J}^{i+j}(\zeta_0, s_0), \mathfrak{J}^i(\zeta_0, s_0)\right]\right] \\
 & \quad + 2\hbar \epsilon^{\mathfrak{m}} \Xi(\epsilon) \left[1 + 2C_{i+j} + C_i \right. \\
 & \quad \left. + 4S\left((s_0, \zeta_0), (s_0, \zeta_0), (\zeta_0, s_0)\right)\right]^{\mathfrak{p}}
 \end{aligned}$$

For a fixed j Set

$$\begin{aligned}
 K & = \sup_{i \in \mathbb{N}} 2\hbar \left[1 + 2C_{i+j} + 2C_i + 4S\left((s_0, \zeta_0), (s_0, \zeta_0), (\zeta_0, s_0)\right)\right]^{\mathfrak{p}} \\
 \text{and } \epsilon & = 1 - \left(\frac{i}{1+i}\right)^{\mathfrak{m}} \leq \frac{\mathfrak{m}}{i+1}. \text{ So}
 \end{aligned}$$

$$\begin{aligned}
& S\left(\mathfrak{J}^{i+j+1}(\varsigma_0, \zeta_0), \mathfrak{J}^{i+j+1}(\varsigma_0, \zeta_0), \mathfrak{J}^{i+1}(\varsigma_0, \zeta_0)\right) \\
& + S\left(\mathfrak{J}^{i+j+1}(\zeta_0, \varsigma_0), \mathfrak{J}^{i+j+1}(\zeta_0, \varsigma_0), \mathfrak{J}^{i+1}(\zeta_0, \varsigma_0)\right) \leq \frac{2}{3} \left(\frac{i}{i+1}\right)^{\mathfrak{M}} \\
& \quad \left\{ S\left[\mathfrak{J}^{i+j}(\varsigma_0, \zeta_0), \mathfrak{J}^{i+j}(\varsigma_0, \zeta_0), \mathfrak{J}^i(\varsigma_0, \zeta_0)\right] \right. \\
& \quad \left. + S\left[\mathfrak{J}^{i+j}(\zeta_0, \varsigma_0), \mathfrak{J}^{i+j}(\zeta_0, \varsigma_0), \mathfrak{J}^i(\zeta_0, \varsigma_0)\right] \right\} \\
& \quad + K \left(\frac{\mathfrak{M}}{i+1}\right)^{\mathfrak{M}} \mathcal{E} \left(\frac{\mathfrak{M}}{i+1}\right) \\
& (i+1)^{\mathfrak{M}} S\left(\mathfrak{J}^{i+j+1}(\varsigma_0, \zeta_0), \mathfrak{J}^{i+j+1}(\varsigma_0, \zeta_0), \mathfrak{J}^{i+1}(\varsigma_0, \zeta_0)\right) \\
& + S\left(\mathfrak{J}^{i+j+1}(\zeta_0, \varsigma_0), \mathfrak{J}^{i+j+1}(\zeta_0, \varsigma_0), \mathfrak{J}^{i+1}(\zeta_0, \varsigma_0)\right) \leq \frac{2}{3} i^{\mathfrak{M}} \left\{ S\left[\mathfrak{J}^{i+j}(\varsigma_0, \zeta_0), \mathfrak{J}^{i+j}(\varsigma_0, \zeta_0), \mathfrak{J}^i(\varsigma_0, \zeta_0)\right] \right. \\
& \quad \left. + S\left[\mathfrak{J}^{i+j}(\zeta_0, \varsigma_0), \mathfrak{J}^{i+j}(\zeta_0, \varsigma_0), \mathfrak{J}^i(\zeta_0, \varsigma_0)\right] \right\} \\
& \quad + K (\mathfrak{M})^{\mathfrak{M}} \mathcal{E} \left(\frac{\mathfrak{M}}{i+1}\right) \\
& \text{Setting } r_i = i^{\mathfrak{M}} \left\{ S\left[\mathfrak{J}^{i+j}(\varsigma_0, \zeta_0), \mathfrak{J}^{i+j}(\varsigma_0, \zeta_0), \mathfrak{J}^i(\varsigma_0, \zeta_0)\right] \right. \\
& \quad \left. + S\left[\mathfrak{J}^{i+j}(\zeta_0, \varsigma_0), \mathfrak{J}^{i+j}(\zeta_0, \varsigma_0), \mathfrak{J}^i(\zeta_0, \varsigma_0)\right] \right\} \\
& \quad + K (\mathfrak{M})^{\mathfrak{M}} \mathcal{E} \left(\frac{\mathfrak{M}}{i+1}\right) \\
& r_{i+1} \leq \frac{2}{3} r_i + K (\mathfrak{M})^{\mathfrak{M}} \mathcal{E} \left(\frac{\mathfrak{M}}{i+1}\right) \\
& \leq \frac{2}{3} r_{i-1} + K (\mathfrak{M})^{\mathfrak{M}} \mathcal{E} \left(\frac{\mathfrak{M}}{i}\right) + K (\mathfrak{M})^{\mathfrak{M}} \mathcal{E} \left(\frac{\mathfrak{M}}{i+1}\right) \\
& \leq \dots \\
& \vdots \\
& \leq \frac{2}{3} r_0 + K (\mathfrak{M})^{\mathfrak{M}} \sum_{t=1}^{i+1} \mathcal{E} \left(\frac{\mathfrak{M}}{t}\right) \\
& = K (\mathfrak{M})^{\mathfrak{M}} \sum_{t=1}^{i+1} \mathcal{E} \left(\frac{\mathfrak{M}}{t}\right)
\end{aligned} \tag{4.5}$$

Therefore

$$\begin{aligned}
& S\left[\mathfrak{J}^{i+j}(\varsigma_0, \zeta_0), \mathfrak{J}^{i+j}(\varsigma_0, \zeta_0), \mathfrak{J}^i(\varsigma_0, \zeta_0)\right] + S\left[\mathfrak{J}^{i+j}(\zeta_0, \varsigma_0), \mathfrak{J}^{i+j}(\zeta_0, \varsigma_0), \mathfrak{J}^i(\zeta_0, \varsigma_0)\right] \leq K \left(\frac{\mathfrak{M}}{i}\right)^{\mathfrak{M}} \sum_{t=1}^{i+1} \mathcal{E} \left(\frac{\mathfrak{M}}{t}\right) \\
& S\left[\mathfrak{J}^{i+j}(\varsigma_0, \zeta_0), \mathfrak{J}^{i+j}(\varsigma_0, \zeta_0), \mathfrak{J}^i(\varsigma_0, \zeta_0)\right] + S\left[\mathfrak{J}^{i+j}(\zeta_0, \varsigma_0), \mathfrak{J}^{i+j}(\zeta_0, \varsigma_0), \mathfrak{J}^i(\zeta_0, \varsigma_0)\right] = K w_i(\mathfrak{M})
\end{aligned} \tag{4.6}$$

Taking the limit as $i \rightarrow \infty$ we obtain the following

$$S\left[\mathfrak{J}^{i+j}(\varsigma_0, \zeta_0), \mathfrak{J}^{i+j}(\varsigma_0, \zeta_0), \mathfrak{J}^i(\varsigma_0, \zeta_0)\right] + S\left[\mathfrak{J}^{i+j}(\zeta_0, \varsigma_0), \mathfrak{J}^{i+j}(\zeta_0, \varsigma_0), \mathfrak{J}^i(\zeta_0, \varsigma_0)\right] \rightarrow 0.$$

Thus $\{\mathfrak{J}^i(\varsigma_0, \zeta_0)\}$ and $\{\mathfrak{J}^i(\zeta_0, \varsigma_0)\}$ form Cauchy sequence in $\mathcal{L}$. Since $\mathcal{L}$ is a complete metric space, there exists $\varsigma^*, \zeta^* \in \mathcal{L}$ such that

$$\lim_{i \rightarrow \infty} \mathfrak{J}^i(\varsigma_0, \zeta_0) = \lim_{i \rightarrow \infty} \varsigma_i = \varsigma^*$$

and

$$\lim_{j \rightarrow \infty} \mathfrak{J}^j(\zeta_0, s_0) = \lim_{j \rightarrow \infty} \zeta_j = \zeta^*.$$

$$\begin{aligned} \lim_{i \rightarrow \infty} S \left[(\varsigma^*, \zeta^*), (\varsigma^*, \zeta^*), (\mathfrak{J}(\varsigma_{i-1}, \zeta_{i-1}), \mathfrak{J}(\zeta_{i-1}, \varsigma_{i-1})) \right] &= \lim_{i \rightarrow \infty} S \left[(\varsigma^*, \zeta^*), (\varsigma^*, \zeta^*), (\mathfrak{J}^i(s_0, \zeta_0), \mathfrak{J}^i(\zeta_0, s_0)) \right] \\ &\leq \lim_{i, j \rightarrow \infty} S \left[(\mathfrak{J}^{i+j}(s_0, \zeta_0), \mathfrak{J}^{i+j}(\zeta_0, s_0)), (\mathfrak{J}^{i+j}(s_0, \zeta_0), \mathfrak{J}^{i+j}(\zeta_0, s_0)) \right. \\ &\quad \left. , (\mathfrak{J}^i(s_0, \zeta_0), \mathfrak{J}^i(\zeta_0, s_0)) \right] \leq Kw_i(\mathfrak{M}) \end{aligned}$$

And from the continuity of $\mathfrak{J}$ we have $\mathfrak{J}(\varsigma^*, \zeta^*) = \varsigma^*$ and $\mathfrak{J}(\zeta^*, \varsigma^*) = \zeta^*$.

$$\begin{aligned} S \left[(\varsigma^*, \zeta^*), (\varsigma^*, \zeta^*), [\mathfrak{J}^i(s_0, \zeta_0), \mathfrak{J}^i(\zeta_0, s_0)] \right] &= S \left[(\mathfrak{J}(\varsigma^*, \zeta^*), \mathfrak{J}(\zeta^*, \varsigma^*)), (\mathfrak{J}(\varsigma^*, \zeta^*), \mathfrak{J}(\zeta^*, \varsigma^*)), (\mathfrak{J}^i(s_0, \zeta_0), \mathfrak{J}^i(\zeta_0, s_0)) \right] \\ &= S \left[\mathfrak{J}(\varsigma^*, \zeta^*), \mathfrak{J}(\varsigma^*, \zeta^*), \mathfrak{J}^i(s_0, \zeta_0) \right] + S \left[\mathfrak{J}(\zeta^*, \varsigma^*), \mathfrak{J}(\zeta^*, \varsigma^*), \mathfrak{J}^i(\zeta_0, s_0) \right] \\ &= S \left[\mathfrak{J}(\varsigma^*, \zeta^*), \mathfrak{J}(\varsigma^*, \zeta^*), \mathfrak{J}(\mathfrak{J}^{i-1}(s_0, \zeta_0), \mathfrak{J}^{i-1}(\zeta_0, s_0)) \right] \\ &\quad + S \left[\mathfrak{J}(\zeta^*, \varsigma^*), \mathfrak{J}(\zeta^*, \varsigma^*), \mathfrak{J}(\mathfrak{J}^{i-1}(\zeta_0, s_0), \mathfrak{J}^{i-1}(s_0, \zeta_0)) \right] \\ &= \frac{1}{3} S \left[(\varsigma^*, \zeta^*), (\varsigma^*, \zeta^*), (\mathfrak{J}^{i-1}(s_0, \zeta_0), \mathfrak{J}^{i-1}(\zeta_0, s_0)) \right] \\ &\quad + \frac{1}{3} S \left[(\zeta^*, \varsigma^*), (\zeta^*, \varsigma^*), (\mathfrak{J}^{i-1}(\zeta_0, s_0), \mathfrak{J}^{i-1}(s_0, \zeta_0)) \right] \\ &= \frac{2}{3} S \left[(\varsigma^*, \zeta^*), (\varsigma^*, \zeta^*), (\mathfrak{J}^{i-1}(s_0, \zeta_0), \mathfrak{J}^{i-1}(\zeta_0, s_0)) \right] \\ &\vdots \\ &\leq \frac{2}{3} S \left[(\varsigma^*, \zeta^*), (\varsigma^*, \zeta^*), (s_0, \zeta_0) \right] = \frac{2}{3} \|\varsigma^*, \zeta^*\| \\ \Rightarrow C_i &= S \left[(\mathfrak{J}^i(s_0, \zeta_0), \mathfrak{J}^i(\zeta_0, s_0)), (\mathfrak{J}^i(s_0, \zeta_0), \mathfrak{J}^i(\zeta_0, s_0)), (s_0, \zeta_0) \right] \\ &\leq 2S \left[(\mathfrak{J}^i(s_0, \zeta_0), \mathfrak{J}^i(\zeta_0, s_0)), (\mathfrak{J}^i(s_0, \zeta_0), \mathfrak{J}^i(\zeta_0, s_0)), (\varsigma^*, \zeta^*) \right] \\ &\quad + S \left[(s_0, \zeta_0), (s_0, \zeta_0), (\varsigma^*, \zeta^*) \right] \\ &= 2S \left[(\varsigma^*, \zeta^*), (\varsigma^*, \zeta^*), (\mathfrak{J}^i(s_0, \zeta_0), \mathfrak{J}^i(\zeta_0, s_0)) \right] \\ &\quad + S \left[(\varsigma^*, \zeta^*), (\varsigma^*, \zeta^*), (s_0, \zeta_0) \right] \\ &= 2 \left(\frac{2}{3} \|\varsigma^*, \zeta^*\| \right) + \frac{2}{3} \|\varsigma^*, \zeta^*\| \\ &= 2 \|\varsigma^*, \zeta^*\| \end{aligned}$$

Hence (4.5) becomes,

$$\begin{aligned} K &= \sup_{i \in \mathbb{N}} 2\hbar [1 + 2C_i + 4S((s_0, \zeta_0), (s_0, \zeta_0), (\zeta_0, s_0))]^p \\ K &\leq 2\hbar [1 + 4\|\varsigma^*, \zeta^*\| + 4S((s_0, \zeta_0), (s_0, \zeta_0), (\zeta_0, s_0))]^p \end{aligned}$$

□

5. Application to Ordinary Differential Equations

Building on the results from the preceding section, the primary seeking of this section is to ascertain whether the boundary value problems possess a solution. Specifically, we consider the following boundary value problem, which

involves a second-order differential equation.

$$\begin{cases} \frac{-d^2\chi}{d\eta^2} &= \varphi(\eta, \chi(\eta)), \eta \in [0, 1] \\ \chi(0) &= \chi(1) = 0 \end{cases} \quad (5.1)$$

where $\varphi : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ denotes a continuous function. The associated Green function for equation (5.1) is

$$\Gamma(\eta, \xi) = \begin{cases} \eta(1 - \xi) & , 0 \leq \eta \leq \xi \leq 1 \\ \xi(1 - \eta) & , 0 \leq \xi \leq \eta \leq 1 \end{cases}$$

For the boundary value problem (5.1), the integral equation can be written as,

$$\chi(\eta) = \int_0^1 \Gamma(\eta, \xi) \varphi(\xi, \chi(\xi)) d\xi \quad \forall \eta \in [0, 1] \quad (5.2)$$

Consider a set $C([0, 1])$ that includes all continuous functions over $[0, 1]$. This collection equipped with the metric described below is an established fact. $S(\chi, \iota, \kappa) = \|\chi - \iota\|_\infty + \|\iota - \kappa\|_\infty + \|\kappa - \chi\|_\infty = \max_{\eta \in [0, 1]} (|\chi - \iota| + |\iota - \kappa| + |\kappa - \chi|)$ is a Complete S -metric space. We define a self-map $\partial : C([0, 1]) \rightarrow C([0, 1])$ as follows.

$$\partial\chi(\eta) = \int_0^1 \Gamma(\eta, \xi) \varphi(\xi, \chi(\xi)) d\xi \quad \eta \in [0, 1] \quad (5.3)$$

Theorem 5.1. *If the inequality $|\varphi(\eta, a) - \varphi(\eta, b)| \leq L|a - b|$, is holds $\forall \eta \in [0, 1]$, $L \in [0, 2)$ and $a, b \in \mathbb{R}$ then the Boundary Value Problem (5.1) admits at least one solution $\chi^* \in [0, 1]$, where the mapping $\partial : C([0, 1]) \rightarrow C([0, 1])$ is given by equation (5.3).*

Proof. It is well-established that solving the integral equation (5.2) is equivalent to solving equation (5.1). In other words, addressing problem (5.1) involves identifying a function $\chi^* \in C([0, 1])$ that fulfills the condition $\partial\chi^* = \chi^*$, where ∂ is outlined in the equation (5.3). Therefore, problem (5.1) reduces to determining a fixed point of the operator ∂ in the function space $C([0, 1])$. Let us now consider $\chi, \iota \in C([0, 1])$ such that $S(\chi(\eta), \chi(\eta), \iota(\eta)) \geq 0 \quad \forall \eta \in [0, 1]$.

$$\begin{aligned} |\partial\chi(\eta) - \partial\iota(\eta)| &= \left| \int_0^1 \Gamma(\eta, \xi) [\varphi(\xi, \chi(\xi)) - \varphi(\xi, \iota(\xi))] d\xi \right| \\ &\leq \int_0^1 \Gamma(\eta, \xi) |\varphi(\xi, \chi(\xi)) - \varphi(\xi, \iota(\xi))| d\xi \\ &\leq \left(\sup_{\eta \in [0, 1]} \int_0^1 \Gamma(\eta, \xi) d\xi \right) [L|\chi(\eta) - \iota(\eta)|] \\ &= \frac{1}{8} [L|\chi(\eta) - \iota(\eta)|] \end{aligned}$$

Note that for all $\eta \in [0, 1]$, $\int_0^1 \Gamma(\eta, \xi) d\xi = \frac{\eta}{2} - \frac{\eta^2}{2} \Rightarrow \sup_{\eta \in [0, 1]} \int_0^1 \Gamma(\eta, \xi) d\xi = \frac{1}{8}$

$$S(\partial\chi(\eta), \partial\chi(\eta), \partial\iota(\eta)) \leq \frac{L}{8} \left[S(\chi(\eta), \chi(\eta), \iota(\eta)) + 2S(\chi(\eta), \chi(\eta), \partial\chi(\eta)) + S(\iota(\eta), \iota(\eta), \partial\iota(\eta)) \right]$$

Hence

$$\left(1 - \frac{L}{8}\right) S(\partial\chi(\eta), \partial\chi(\eta), \partial\iota(\eta)) \leq \frac{L}{8} \left[S(\chi(\eta), \chi(\eta), \iota(\eta)) + 2S(\chi(\eta), \chi(\eta), \partial\chi(\eta)) + S(\iota(\eta), \iota(\eta), \partial\iota(\eta)) \right]$$

and therefore

$$S(\partial\chi(\eta), \partial\chi(\eta), \partial\iota(\eta)) \leq \frac{L}{8-L} \left[S(\chi(\eta), \chi(\eta), \iota(\eta)) + 2S(\chi(\eta), \chi(\eta), \partial\chi(\eta)) + S(\iota(\eta), \iota(\eta), \partial\iota(\eta)) \right]$$

Since $L < 2$ we have $\frac{L}{8-L} \leq \frac{1}{3}$. Consequently, there exists $\epsilon \in [0, 1)$ such that $\frac{L}{8-L} \leq \frac{1-\epsilon}{4}$

$$\begin{aligned} S(\partial\chi(\eta), \partial\chi(\eta), \partial\iota(\eta)) &= \frac{1-\epsilon}{4} \left[S(\chi(\eta), \chi(\eta), \iota(\eta)) + 2S(\chi(\eta), \chi(\eta), \partial\chi(\eta)) + S(\iota(\eta), \iota(\eta), \partial\iota(\eta)) \right] \\ &\quad + \left(\frac{1}{8} - \frac{1-\epsilon}{4} \right) \left[S(\chi(\eta), \chi(\eta), \iota(\eta)) + 2S(\chi(\eta), \chi(\eta), \partial\chi(\eta)) \right. \\ &\quad \left. + S(\iota(\eta), \iota(\eta), \partial\iota(\eta)) \right] \\ &\leq \frac{1-\epsilon}{4} \left[S(\chi(\eta), \chi(\eta), \iota(\eta)) + 2S(\chi(\eta), \chi(\eta), \partial\chi(\eta)) + S(\iota(\eta), \iota(\eta), \partial\iota(\eta)) \right] \\ &\quad + \frac{1}{4} \epsilon^{\frac{8}{4}} [2\|\chi\| + \|\iota\| + 2\|\partial\chi\| + \|\partial\iota\|] \\ &= \frac{1-\epsilon}{4} \left[S(\chi(\eta), \chi(\eta), \iota(\eta)) + 2S(\chi(\eta), \chi(\eta), \partial\chi(\eta)) + S(\iota(\eta), \iota(\eta), \partial\iota(\eta)) \right] \\ &\quad + \frac{1}{4} \epsilon^2 \epsilon^0 [1 + 2\|\chi\| + \|\iota\| + 2\|\partial\chi\| + \|\partial\iota\|] \end{aligned}$$

Therefore, the mapping ∂ meets the criteria in Theorem (3.4), where $\alpha = 2, \beta = 1, \hbar = \frac{1}{4}$, and $\Xi(\epsilon) = \epsilon^0 = 1$. As a result, ∂ has a single distinct fixed point $\chi^* \in C([0, 1])$, such that $\partial\chi^* = \chi^*$. Consequently, χ^* is the sole solution to the Boundary Value Problem (5.1). $\square$

6. Conclusion

In this paper, we have developed new results in the setting of S metric space by introducing and studying New S -Pata-type contraction and establishing a Fixed Point and Coupled Fixed Point theorems under this framework. Moreover, the applicability of the obtained results was demonstrated by solving a boundary value problem for a second-order ordinary differential equation, highlighting the practical utility of our work.

Declarations

Funding:

This article has not received any funding.

Conflict of interest:

The authors affirm that they have no conflicts of interest.

Authors' contributions:

The article was written with equal contributions from each author. The completed work was read and approved by each author.

Data Availability:

There is no related data in manuscript.

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