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# Expanded rectangular $b$-metric spaces and fixed point results for nonlinear $F$-contractions

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**Abstract**

The generalization of metric space is one of research directions that focus on fixed point theory. Recently, many generalizations of metric spaces have been introduced and the related fixed point results in these abstract spaces have been established. In this paper, we introduce a new concept called expanded rectangular $b$-metric space by combining the ideas of expanded $b$-metric space and rectangular metric space. We claim that expanded rectangular $b$-metric space is another generalization of metric space, $b$-metric space, expanded $b$-metric space, rectangular metric space, and rectangular $b$-metric space, and we give examples to illustrate this. We also prove fixed point result of nonlinear $F$-contractions under weaker assumptions in such metric spaces. Our fixed point results generalize and improve many of the previous well-known fixed point results. Finally, we apply obtained result to consider the existence of solutions for nonlinear integral equation.

*Keywords:* fixed point; rectangular $b$-metric space; expanded rectangular $b$-metric space; $F$-contraction, nonlinear $F$-contraction.

## 1. Introduction

Due to the widespread application of fixed point theory in mathematics and other sciences, this theory has been developing rapidly in recent years. In particular, several works have been reported on the application of fixed point theory to fractal sets. See [1, 6, 13]. The development of new extensions of metric spaces (for short, *MS*) in the field of fixed point theory has attracted attention.

Here, the research on rectangular metric space (for short, *RMS*) proposed by Branciari [5] in 2000 is of great importance. Rectangular metric cannot be reduced to normal metric, especially its topological structure cannot be compatible with the topological structure of normal metric. Because of this, it is not easy to deal with rectangular metric and is attracting many researchers. Indeed, in the last few years, inspired by Branciari's idea, researchers have generalized many spaces such as $b$-*MS* [4], extended $b$-*MS* [8], $p$-*MS* [14], controlled *MS* [11], and double controlled *MS* [2]. As a result, new spaces have been developed,

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such as rectangular *b-MS* (for short, *Rb-MS*) [7], extended rectangular *b-MS* (for short, *ERbMS*) [3, 12], controlled $b$-Branciari metric type space [17], and triple controlled *MS* [16]. In 2022, Karami et al. [9] have proposed a new generalization of *b-MS* called expanded $b$-metric space (for short, *Eb-MS*), in a way using the composition of function to each term in the right-hand side of the triangle inequality (see also [10]).

In this paper, strongly motivated by the recent study for generalization of metric spaces, we introduce a new metric space, namely expanded rectangular $b$-metric space (for short, *ERb-MS*), which generalizes rectangular metric space and rectangular $b$-metric space. Our new space is a different concept from the extended rectangular $b$-metric spaces introduced earlier. While the extended rectangular b-metric space proposed in [3] is a concept that generalizes a rectangular metric space by multiplying a function on the right-hand side of rectangular inequality, our concept is another generalization of rectangular metric space by using the composition of a function on the right-hand side of the rectangular inequality. And our expanded rectangular b-metric space, unlike the extended rectangular b-metric space proposed in [12], is a new concept using the composition of a function for each term rather than the whole of the right-hand side of rectangular inequality. And we prove the fixed point result for nonlinear $F$-contractions in expanded rectangular $b$-metric space. Moreover, we consider the existence of solution for nonlinear integral equations as an application of presented results.

## 2. Preliminaries

We first recall the concept of rectangular metric space.

**Definition 2.1.** [5] *Let $\aleph \neq \emptyset$. A mapping $\vartheta_R : \aleph \times \aleph \to [0, \infty)$ is called a rectangular metric if*

$(R1)$ $w = z \Leftrightarrow \vartheta_R(w, z) = 0$,

$(R2)$ $\vartheta_R(w, z) = \vartheta_R(z, w)$,

$(R3)$ $\vartheta_R(w, z) \leq \vartheta_R(w, p) + \vartheta_R(p, q) + \vartheta_R(q, z)$,

*for all $w, z \in \aleph$ and all distinct $p, q \in \aleph \backslash \{w, z\}$. Then $(\aleph, \vartheta_R)$ is called a rectangular metric space (for short, RMS).*

In 2015, George et al. [7] defined a rectangular $b$-metric space by combining $b$-metric space with *RMS*.

**Definition 2.2.** [7] *Let $\aleph \neq \emptyset$ and $b \geq 1$ be a given constant. A mapping $\vartheta_{Rb} : \aleph \times \aleph \to [0, \infty)$ is called a rectangular b-metric if*

$(Rb1)$ $w = z \Leftrightarrow \vartheta_{Rb}(w, z) = 0$,

$(Rb2)$ $\vartheta_{Rb}(w, z) = \vartheta_{Rb}(z, w)$,

$(Rb3)$ $\vartheta_{Rb}(w, z) \leq b[\vartheta_{Rb}(w, p) + \vartheta_{Rb}(p, q) + \vartheta_{Rb}(q, z)]$,

*for all $w, z \in \aleph$ and all distinct $p, q \in \aleph \backslash \{w, z\}$. Then $(\aleph, \vartheta_{Rb})$ is called a rectangular b-metric space (for short, Rb-MS).*

Recently, researchers have generalized extended $b$-metric space to various types of extended rectangular $b$-metric spaces by using Branciari's idea.

**Definition 2.3.** [3] *Let $\aleph \neq \emptyset$ and $\theta : \aleph \times \aleph \to [1, \infty)$. A mapping $\vartheta_{ERbA} : \aleph \times \aleph \to [0, \infty)$ is called an extended rectangular b-metric of type (A) if*

$(ERb1\text{-A})$ $w = z \Leftrightarrow \vartheta_{ERbA}(w, z) = 0$,

$(ERb2\text{-A})$ $\vartheta_{ERbA}(w, z) = \vartheta_{ERbA}(z, w)$,

$(ERb3\text{-A})$ $\vartheta_{ERbA}(w, z) \leq \theta(w, z)[\vartheta_{ERbA}(w, p) + \vartheta_{ERbA}(p, q) + \vartheta_{ERbA}(q, z)]$,

*for all $w, z \in \aleph$ and all distinct $p, q \in \aleph \backslash \{w, z\}$. Then $(\aleph, \vartheta_{ERbA})$ is called an extended rectangular b-metric space of type $(A)$ (for short, ERbMS of type $(A)$).*

**Definition 2.4.** [12] *Let $\aleph \neq \emptyset$ and $f : [0, \infty) \to [0, \infty)$ be a strictly increasing continuous function with $f(x) \geq x$ for all $x > 0$ and$f(0) = 0$. A mapping $\vartheta_{ERbB} : \aleph \times \aleph \to [0, \infty)$ is called an extended rectangular b-metric of type (B) if*

($ERb1$-B) $w = z \Leftrightarrow \vartheta_{ERbB}(w, z) = 0$,
($ERb2$-B) $\vartheta_{ERbB}(w, z) = \vartheta_{ERbB}(z, w)$,
($ERb3$-B) $\vartheta_{ERbB}(w, z) \leq f(\vartheta_{ERbB}(w, p) + \vartheta_{ERbB}(p, q) + \vartheta_{ERbB}(q, z))$,
*for all* $w, z \in \aleph$ *and all distinct* $p, q \in \aleph\backslash\{w, z\}$. *Then* $(\aleph, \vartheta_{ERbB})$ *is called an extended rectangular b-metric space of type* $(B)$ *(for short, ERbMS of type* $(B)$*).*

Very recently, the notion of an expanded $b$-metric space as a new generalization of $b$-metric space was first introduced by Karami et al. [9].

Let $\Psi$ denote a set of all functions $\Psi : [0, \infty) \to [0, \infty)$ such that $\psi(x) \geq x$ and $d\psi/dx = \psi'$ is increasing for all $x \geq 0$.

**Definition 2.5.** [9] *Let* $\aleph \neq \emptyset$. *A mapping* $\vartheta_{Eb} : \aleph \times \aleph \to [0, \infty)$ *is called an expanded b-metric if there exists a* $\psi \in \Psi$ *such that*
($Eb1$) $w = z \Leftrightarrow \vartheta_{Eb}(w, z) = 0$,
($Eb2$) $\vartheta_{Eb}(w, z) = \vartheta_{Eb}(z, w)$,
($Eb3$) $\vartheta_{Eb}(w, z) \leq (\psi_{Eb}(w, p)) + (\psi_{Eb}(p, z))$, *for all* $w, z, p \in \aleph$.
*Then* $(\aleph, \vartheta_{Eb})$ *is called an* expanded $b$-metric space *(for short, Eb-MS).*

Obviously, each *MS* is an *Eb-MS* with $\psi(x) = x$, and each *b-MS* is an *Eb-MS* with $\psi(x) = bx$ for $b \geq 1$. That is, *Eb-MS* is a larger class covering both *MS* and *b-MS*.

**Definition 2.6.** [18] *Let* $\mathcal{F}$ *be a set of all functions* $F : (0, \infty) \to (-\infty, +\infty)$ *satisfying:*
($F1$) *$F$ is strictly increasing;*
($F2$) *For all sequence* $\{\beta_t\}$ *of positive real numbers,* $lim_{t\to+\infty}\beta_t = 0 \Leftrightarrow lim_{t\to+\infty}F(\beta_t) = -\infty$;
($F3$) *there exists* $\lambda \in (0, 1)$ *such that* $lim_{\beta\to0^+}\beta^{\lambda} F(\beta) = 0$.

**Definition 2.7.** [18] *A mapping* $\Gamma : \aleph \times \aleph$ *on a MS* $(\aleph, \vartheta)$ *is called an F-contraction if there exist* $F \in \mathcal{F}$ *and* $\tau > 0$ *such that*

$$\tau + F(\vartheta(\Gamma w, \Gamma z)) \leq F(\vartheta(w, z)),$$

*for all* $w, z \in \aleph$ *with* $\Gamma w \neq \Gamma z$.

In 2013, Secelean [15] replaced the condition ($F2$) of Definition 2.7 with the following condition, equivalent and simpler:
($F2'$) $infF = -\infty$ or
($F2''$) There exists a positive real sequence $\{\beta_t\}$ such that $lim_{t\to+\infty}F(\beta_t) = -\infty$.

In 2018, the author in [19] introduced the notion of nonlinear $F$-contraction by replacing $\tau > 0$ with function $\eta : (0, \infty) \to (0, \infty)$.

**Definition 2.8.** [19] *A mapping* $\Gamma : \aleph \times \aleph$ *on a MS* $(\aleph, \vartheta)$ *is called an* $(\eta, F)$ *-contraction if there exist* $F : (0, \infty) \to (-\infty, +\infty)$ *and* $\eta : (0, \infty) \to (0, \infty)$ *such that*
(1) *$F$ satisfies* ($F1$) *and* ($F2'$);
(2) $liminf_{\alpha\to x^+}\eta(\alpha) > 0$ *for all* $x \geq 0$;
(3) $\eta(\vartheta(w, z)) + F(\vartheta(\Gamma w, \Gamma z)) \leq F(\vartheta(w, z))$, *for all* $w, z \in \aleph$ *with* $\Gamma w \neq \Gamma z$.

## 3. Main results

We first propose the concept of expanded rectangular $b$-metric space that generalizes *Eb-MS* using Branciari's idea.

**Definition 3.1.** *Let* $\aleph \neq |\emptyset$. *A mapping* $\vartheta_{ERb} : \aleph \times \aleph \to [0, \infty)$ *is called an expanded rectangular b-metric if there exists a strictly increasing continuous function* $f : [0, \infty) \to [0, \infty)$ *with* $f(0)=0$ *such that*
($ERb1$) $w = z \Leftrightarrow \vartheta_{ERb}(w, z) = 0$,
($ERb2$) $\vartheta_{ERb}(w, z) = \vartheta_{ERb}(z, w)$,

($ERb3$) $\vartheta_{ERb}(w,z) \leq f(\vartheta_{ERb}(w,p)) + f(\vartheta_{ERb}(p,q)) + f(\vartheta_{ERb}(q,z))$,
*for all* $w, z \in \aleph$ *and all distinct* $p, q \in \aleph\backslash\{w,z\}$. *The pair* $(\aleph, \vartheta_{ERb})$ *is called an expanded rectangular b-metric space (for short, ERb-MS).*

*Remark* 3.2.

1. *Definition 3.1 is different from Definition 2.3 and Definition 2.4. ERbMS of type (A) contains the multiplication of function, but ERb-MS contains the composition of function. And ERbMS of type (B) contains the composition of function over the whole right-hand side of rectangular inequality, while ERb-MS contains the composition of function for each term in the right-hand side.*
2. *Every Rb-MS is an ERb-MS with* $f(x) = bx$ *for* $b \geq 1$, *and every RMS is an ERb-MS with* $f(x) = x$. *However, the converse is not generally true.*

The following example shows that *ERb-MS* is a real generalization of *RMS* and *Rb-MS*.

**Example 3.3.** *Let* $\aleph$=$\mathbb{R}$. *Define* $\vartheta_{ERb} : \aleph \times \aleph \to [0,\infty)$ *by* $\vartheta_{ERb}(w,z) = sinh\,|w-z|$ , *for all* $w, z \in \aleph$. *Then* $(\aleph, \vartheta_{ERb})$ *is an ERb-MS with* $f(x) = sinh(\ 3x)$ *for all* $x \geq 0$, *neither RMS nor Rb-MS.*

*Proof.* ($ERb1$) and ($ERb2$) of Definition 3.1 are obvious, and we will see only the condition ($ERb3$). Since $\sinh(x)$ is strictly increasing, for all $\alpha, \beta, \gamma \geq 0$, we get

$$sinh(\ \alpha+\beta+\gamma) \leq sinh(\ 3max\{\ \alpha,\beta,\gamma\}) \leq sinh(\ 3\alpha) + sinh(\ 3\beta) + sinh(\ 3\gamma).$$

Thus, by the above inequality and $sinh(\ x) \geq x$ for all $x \geq 0$, we get

$$\begin{aligned}\vartheta_{ERb}(w,z) = sinh\,|w-z| \ &\leq sinh\,(|w-p| + |p-q| + |q-z|)\\ &\leq sinh\,(sinh\,|w-p| \ + sinh\,|p-q| \ + sinh\,|q-z| \ )\\ &\leq sinh\,(3sinh\,|w-p| \ ) \ + sinh\,(3sinh\,|p-q| \ ) \ + sinh\,(3sinh\,|q-z| \ )\\ &= f(\vartheta_{ERb}(w,p)) + f(\vartheta_{ERb}(p,q)) + f(\vartheta_{ERb}(q,z)),\end{aligned}$$

So, $(\aleph, \vartheta_{ERb})$ is an *ERb-MS* with $f(x) = sinh(\ 3x)$.

However, $sinh\,|w-z|$ is not a rectangular metric on $\mathbb{R}$. In fact, for $w$=5, $p$=3, $q$=1 and $z$=0, we have that $sinh5 \ \approx 74.203 > 8.428 \approx sinh2 \ + sinh1 \ + sinh2$ .

Also, there exists no $b \neq 1$ such that $sinh\,|w-z|$ is a rectangular $b$-metric with constant $b$. In fact, taking$p$=1, $q$=1/2 and $z$=0, we cannot find fixed $b$ such that it holds that $sinhw \leq b[sinh\,|w-1| + 2sinh(1/2)]$ for all $w$∈$\mathbb{R}$. □

**Example 3.4.** *Let* $\aleph = \{0,1,2,3\}$. *Take* $f : [0,\infty) \to [0,\infty)$ *by*$f(x) = 3\sqrt{x}$. *A mapping* $\vartheta_{ERb} : \aleph \times \aleph \to [0,\infty)$ *is symmetric and defined as follows:*
$\vartheta_{ERb}(w,w) = 0$, *for all* $w \in \aleph$;
$\vartheta_{ERb}(0,w) = 1$, *for all* $w \in \{1,2,3\}$;
$\vartheta_{ERb}(1,2) = \widetilde{\vartheta}(1,3) = 2$ *and* $\widetilde{\vartheta}(2,3) = 9$.
*Then,* $(\aleph, \vartheta_{ERb})$ *is an ERb-MS with the function*$f(x) = 3\sqrt{x}$. *While, we have that*

$$\vartheta_{ERb}(2,3) = 9 > 6 = f\left(\vartheta_{ERb}(2,0) + \vartheta_{ERb}(0,1) + \vartheta_{ERb}(1,3)\right),$$

$$\vartheta_{ERb}(2,3) = 9 > 6 = 3+3 = f(\vartheta_{ERb}(2,0)) + f(\vartheta_{ERb}(0,3)).$$

*Hence,* $(\aleph, \vartheta_{ERb})$ *is neither an ERbMS of type (B), nor an Eb-MS.*

From Example 3.4, we can see that *ERb-MS* is a different concept from *ERbMS* of type (B).

The following example shows that for different functions the given mapping $\vartheta_{ERb} : \aleph \times \aleph \to [0,\infty)$ can be an expanded rectangular $b$-metric.

**Example 3.5.** *Let* $\aleph$=$\mathbb{R}$. *We define* $\vartheta_{ERb} : \aleph \times \aleph \to [0,\infty)$ *as*

$$\vartheta_{ERb}(w,z) = e^{|w-z|} - 1, \ \textit{for all } w,z \in \aleph.$$

*Take two functions* $f_1, f_2 : [0,\infty) \to [0,\infty)$ *by*

$$f_1(x) = e^{3x} - 1 \ \textit{and} \ f_2(x) = x^3 + 3x^2 + 3x.$$

*Then* $(\aleph, \vartheta_{ERb})$ *is an ERb-MS with*$f_1$, *but also an ERb-MS with*$f_2$.

*Proof.* We will show only the condition $(ERb3)$ of Definition 3.1. Since $e^x - 1$ is an increasing function, it holds that for all $\alpha, \beta, \gamma \geq 0$,

$$e^{\alpha+\beta+\gamma} - 1 \leq e^{3max\{\ \alpha,\beta,\gamma\}} - 1 \leq (e^{3\alpha} - 1) + (e^{3\beta} - 1) + (e^{3\gamma} - 1).$$

By the above inequality and $|w-z| \leq |w-p| + |p-q| + |q-z|$ for all $w,\ p,\ q,\ z \in\mathbb{R}$, we obtain

$$\vartheta_{ERb}(w,z) = e^{|w-z|} - 1 \leq e^{|w-p|+|p-q|+|q-z|} - 1$$

$$\leq \left(e^{3|w-p|} - 1\right) + \left(e^{3|p-q|} - 1\right) + \left(e^{3|q-z|} - 1\right).$$

From the fact that $e^x - 1 \geq x$ for all $x \geq 0$, it follows that

$$\vartheta_{ERb}(w,z) \leq \left(e^{3\left(e^{|w-p|}-1\right)} - 1\right) + \left(e^{3\left(e^{|p-q|}-1\right)} - 1\right) + \left(e^{3\left(e^{|q-z|}-1\right)} - 1\right)$$

$$= f_1\left(e^{|w-p|} - 1\right) + f_1\left(e^{|p-q|} - 1\right) + f_1\left(e^{|q-z|} - 1\right)$$

$$= f_1(\tilde{\vartheta}(w,p)) + f_1(\tilde{\vartheta}(p,q)) + f_1(\tilde{\vartheta}(q,z)).$$

Thus $(\aleph, \vartheta_{ERb})$ is an *ERb-MS* with $f_1$.

On the other hand, since $a^3 - 1 = (a-1)^3 + 3(a-1)^2 + 3(a-1) = f_2(a-1)$, we get

$$\vartheta_{ERb}(w,z) \leq \left(e^{3|w-p|} - 1\right) + \left(e^{3|p-q|} - 1\right) + \left(e^{3|q-z|} - 1\right)$$

$$= f_2\left(e^{|w-p|} - 1\right) + f_2\left(e^{|p-q|} - 1\right) + f_2\left(e^{|q-z|} - 1\right)$$

$$= f_2(\vartheta_{ERb}(w,p)) + f_2(\vartheta_{ERb}(p,q)) + f_2(\vartheta_{ERb}(q,z)).$$

So, $(\aleph, \vartheta_{ERb})$ is also an *ERb-MS* with $f_2$. □

We can simply define the concepts such as convergence, Cauchyness and completeness in *ERb-MS*.

**Example 3.6.** *Let* $(\aleph, \vartheta_{ERb})$ *be an ERb-MS and* $\{w_t\}_{t\geq 0} \subset \aleph$. *Then*

1. $\{w_t\}$ *is called convergent, if* $lim_{t\to\infty}\vartheta_{ERb}\ (w_t, \hat{w}) = 0$ *for some* $\hat{w} \in \aleph$;
2. $\{w_t\}$ *is called Cauchy, if* $lim_{t,s\to\infty}\vartheta_{ERb}\ (w_t, w_s) = 0$;
3. $(\aleph, \vartheta_{ERb})$ *is called complete if every Cauchy sequence converges.*

In *ERb-MS*, under certain conditions, every sequence has a unique limit.

**Lemma 3.7.** *Let* $\{w_t\}$ *be a Cauchy sequence such that* $w_t \neq w_s$ *whenever* $t \neq s$ *in ERb-MS*$(\aleph, \vartheta_{ERb})$ *, Then,* $\{w_t\}$ *converges at most one point.*

*Proof.* Let $\{w_t\}$ be a Cauchy sequence such that $lim_{t\to\infty}\vartheta_{ERb}\ (w_t, \hat{w}) = 0$ and $lim_{t\to\infty}\vartheta_{ERb}\ (w_t, \hat{z}) = 0$ for some $\hat{w}, \hat{z} \in \aleph$. By $(ERb3)$, we obtain

$$\vartheta_{ERb}(\hat{w}, \hat{z}) \leq f(\vartheta_{ERb}(\hat{w}, w_t)) + f(\vartheta_{ERb}(w_t, w_s)) + f(\vartheta_{ERb}(w_s, \hat{z})).$$

Taking the limit as $t, s \to \infty$, by the continuity of *f*, we get

$$\vartheta_{ERb}(\hat{w}, \hat{z}) \leq f\left(lim_{t\to\infty}\vartheta_{ERb}\ (\hat{w}, w_t)\right) + f\left(lim_{t,s\to\infty} \leq_{ERb}\ (w_t, w_s)\right) + f\left(lim_{s\to\infty}\vartheta_{ERb}\ (w_s, \hat{z})\right)$$

$$\leq 3f(0) = 0.$$

□

**Lemma 3.8.** *Let $(\aleph, \vartheta_{ERb})$ be an ERb-MS with the function f. Suppose that two sequences $\{w_t\}$ and $\{z_t\}$ in $\aleph$ are such that $w_t \to \hat{w}$ and $z_t \to \hat{z}$, with $w_t \neq \hat{w}$, $z_t \neq \hat{z}$ and $\hat{w}?\hat{z}$ for $t \in\mathbb{N}$. Then, it holds that*

$$f^{-1}(\vartheta_{ERb}(\hat{w}, \hat{z})) \leq liminf_{t\to\infty}\vartheta_{ERb}\ \ (w_t, z_t) \leq limsup_{t\to\infty}\vartheta_{ERb}\ \ (w_t, z_t) \leq f(\vartheta_{ERb}(\hat{w}, \hat{z})). \tag{3.1}$$

*And, if $\{w_t\}$ is a Cauchy sequence such that $lim_{t\to\infty}\vartheta_{ERb}\ (w_t, \hat{w}) = 0$ and $w_t \neq w_s$ for infinitely many t, s $\in\mathbb{N}$, $t \neq s$, then*

$$f^{-1}(\vartheta_{ERb}(\hat{w}, \hat{z})) \leq liminf_{t\to\infty}\vartheta_{ERb}\ \ (w_t, \hat{z}) \leq limsup_{t\to\infty}\vartheta_{ERb}\ \ (w_t, \hat{z}) \leq f(\vartheta_{ERb}(\hat{w}, \hat{z})), \tag{3.2}$$

*for all $\hat{z} \in \aleph$ with $\hat{w} \neq \hat{z}$.*

*Proof.* From the rectangular inequality $(ERb3)$, we have

$$\vartheta_{ERb}(\hat{w}, \hat{z}) \leq f(\vartheta_{ERb}(\hat{w}, w_t)) + f(\vartheta_{ERb}(w_t, z_t)) + f(\vartheta_{ERb}(z_t, \hat{z})),$$

$$\vartheta_{ERb}(w_t, z_t) \leq f(\vartheta_{ERb}(w_t, \hat{w})) + f(\vartheta_{ERb}(\hat{w}, \hat{z})) + f(\vartheta_{ERb}(\hat{z}, z_t)).$$

Taking the lower limit and the upper limit in two above inequalities as $t \to \infty$respectively, by the continuity and monotonicity of $f$ and $f(0)$=0 we have

$$\begin{aligned}\vartheta_{ERb}(\hat{w}, \hat{z}) &\leq f(liminf_{t\to\infty}\vartheta_{ERb}\ \ (\hat{w}, w_t)) + f(liminf_{t\to\infty}\vartheta_{ERb}\ \ (w_t, z_t)) + f(liminf_{t\to\infty}\vartheta_{ERb}\ \ (z_t, \hat{z}))\\ &= f(0) + f(liminf_{t\to\infty}\vartheta_{ERb}\ \ (w_t, z_t)) + f(0)\\ &= f(liminf_{t\to\infty}\vartheta_{ERb}\ \ (w_t, z_t)),\end{aligned}$$

and

$$\begin{aligned}limsup_{t\to\infty}\vartheta_{ERb}\ \ (w_t, z_t) &\leq f(limsup_{t\to\infty}\vartheta_{ERb}\ \ (w_t, \hat{w})) + f(\vartheta_{ERb}(\hat{w}, \hat{z})) + f(limsup_{t\to\infty}\vartheta_{ERb}\ \ (\hat{z}, z_t))\\ &= f(0) + f(\vartheta_{ERb}(\hat{w}, \hat{z})) + f(0) = f(\vartheta_{ERb}(\hat{w}, \hat{z})).\end{aligned}$$

From the above two inequalities and the monotonicity of $f^{-1}$, we get

$$f^{-1}(\vartheta_{ERb}(\hat{w}, \hat{z})) \leq liminf_{t\to\infty}\vartheta_{ERb}\ \ (w_t, z_t) \leq limsup_{t\to\infty}\vartheta_{ERb}\ \ (w_t, z_t) \leq f(\vartheta_{ERb}(\hat{w}, \hat{z})),$$

that is, the inequality (3.1) is satisfied.
Also, we have

$$\vartheta_{ERb}(\hat{w}, \hat{z}) \leq f(\vartheta_{ERb}(\hat{w}, w_t)) + f(\vartheta_{ERb}(w_t, w_s)) + f(\vartheta_{ERb}(w_s, \hat{z})),$$

$$\vartheta_{ERb}(w_t, \hat{z}) \leq f(\vartheta_{ERb}(w_t, w_s)) + f(\vartheta_{ERb}(w_s, \hat{w})) + f(\vartheta_{ERb}(\hat{w}, \hat{z})).$$

Taking the lower limit and the upper limit in two above inequalities as $t, s \to \infty$respectively, by the continuity and monotonicity of $f$ and $f(0)$=0 we have

$$\begin{aligned}\vartheta_{ERb}(\hat{w}, \hat{z}) &\leq f(liminf_{t\to\infty}\vartheta_{ERb}\ \ (\hat{w}, w_t)) + f(liminf_{t,s\to\infty}\vartheta_{ERb}\ \ (w_t, w_s)) + f(liminf_{s\to\infty}\vartheta_{ERb}\ \ (w_s, \hat{z}))\\ &= f(0) + f(0) + f(liminf_{t\to\infty}\vartheta_{ERb}\ \ (w_t, \hat{z}))\\ &= f(liminf_{t\to\infty}\vartheta_{ERb}\ \ (w_t, \hat{z})),\end{aligned}$$

and

$$\begin{aligned}limsup_{t\to\infty}\vartheta_{ERb}\ \ (w_t, \hat{z}) &\leq f(limsup_{t,s\to\infty}\vartheta_{ERb}\ \ (w_t, w_s)) + f(limsup_{s\to\infty}\vartheta_{ERb}\ \ (w_s, \hat{w})) + f(\vartheta_{ERb}(\hat{w}, \hat{z}))\\ &= f(0) + f(0) + f(\vartheta_{ERb}(\hat{w}, \hat{z})) = f(\vartheta_{ERb}(\hat{w}, \hat{z})).\end{aligned}$$

From the above two inequalities and the monotonicity of $f^{-1}$, we get

$$f^{-1}(\vartheta_{ERb}(\hat{w}, \hat{z})) \leq liminf_{t\to\infty}\vartheta_{ERb}\ \ (w_t, \hat{z}) \leq limsup_{t\to\infty}\vartheta_{ERb}\ \ (w_t, \hat{z}) \leq f(\vartheta_{ERb}(\hat{w}, \hat{z})).$$

That is, the inequality (3.2) is satisfied. □

**Definition 3.9.** *Let* $(\aleph, \vartheta_{ERb})$ *be an ERb-MS with the function f. A mapping* $\mathit{\Gamma} : \aleph \to \aleph$ *is called an* $(\eta, \tilde{F})$ *-contraction (or nonlinear* $\tilde{F}$*-contraction) if there exist* $\tilde{F} : (0, \infty) \to (-\infty, +\infty)$ *and* $\eta : (0, \infty) \to (0, \infty)$ *such that*

$(NF1)$ $\tilde{F}$ *satisfies* $(F1)$;
$(NF2)$ $liminf_{\alpha \to u^+}\eta\ \ (\alpha) > 0$ *for all* $u \geq 0$;
$(NF3)$ $\eta(\vartheta_{ERb}(w, z)) + \tilde{F}(f(\vartheta_{ERb}(\mathit{\Gamma} w, \mathit{\Gamma} z))) \leq \tilde{F}(\vartheta_{ERb}(w, z))$ *for all* $w, z \in \aleph$ *with* $\mathit{\Gamma} w \neq \mathit{\Gamma} z$.

Now, we give fixed point result for $(\eta, \tilde{F})$ -contraction in *ERb-MS* using only $(F1)$.

**Theorem 3.10.** *Let* $(\aleph, \vartheta_{ERb})$ *be a complete ERb-MS with a strictly increasing continuous function* $f : [0, \infty) \to [0, \infty)$ *such that* $f(x) \geq x$, $\forall x > 0$ *and* $f(0) = 0$, *and let* $\mathit{\Gamma} : \aleph \to \aleph$ *be an* $(\eta, \tilde{F})$ *-contraction. Then* $\mathit{\Gamma}$*has a unique fixed point.*

*Proof.* Take any $w_0 \in \aleph$. Define the sequence $\{w_t\}$ by $w_t = \mathit{\Gamma} w_{t-1} = \mathit{\Gamma}^t w_0$ for all t $\in \mathbb{N}$. If $w_{t_0} = w_{t_0+1}$ for some $t_0 \geq 0$, then $w_{t_0}$ is a fixed point of $\mathit{\Gamma}$. Assume that $w_t \neq w_{t+1}, \forall t \geq 0$. Then, we get $\mathit{\Gamma} w_t \neq \mathit{\Gamma} w_{t+1}$ for all$t \geq 0$. Now we will prove the theorem in four steps.
**Step 1:** The first step is to prove that $lim_{t \to \infty}\vartheta_{ERb}\ (w_{t+1}, w_t) = 0$. Setting $w = w_t$ and $z = w_{t-1}$ in $(NF3)$ and using $(F1)$ and $f(x) \geq x$ for all $x \geq 0$, we get

$$\begin{aligned}\eta(\vartheta_{ERb}(w_t, w_{t-1})) + \tilde{F}(\vartheta_{ERb}(w_{t+1}, w_t)) &\leq (\vartheta_{ERb}(w_t, w_{t-1})) + \tilde{F}(f(\vartheta_{ERb}(w_{t+1}, w_t))) \\ &\leq \tilde{F}(\vartheta_{ERb}(w_t, w_{t-1})), \forall t \geq 0. \end{aligned} \tag{3.3}$$

That is,

$$\tilde{F}(\vartheta_{ERb}(w_{t+1}, w_t)) \leq \tilde{F}(\vartheta_{ERb}(w_t, w_{t-1})) - \eta(\vartheta_{ERb}(w_t, w_{t-1})) \leq \tilde{F}(\vartheta_{ERb}(w_t, w_{t-1})),$$

which implies that $0 < \vartheta_{ERb}(w_{t+1}, w_t) \leq \vartheta_{ERb}(w_t, w_{t-1})$,$\forall t \geq 0$. So the sequence $\{\vartheta_{ERb}(w_{t+1}, w_t)\}$ has a limit $\theta \geq 0$ as$t \to \infty$. Suppose that$\theta > 0$. From $(F1)$, it follows that $lim_{\alpha \to \theta^+}\tilde{F}(\alpha) = \tilde{F}(\theta^+)$ for all $\theta \in (0, \infty)$. Rewriting the inequality (3.3), we get

$$\eta(\vartheta_{ERb}(w_t, w_{t-1})) \leq \tilde{F}(\vartheta_{ERb}(w_t, w_{t-1})) - \tilde{F}(\vartheta_{ERb}(w_{t+1}, w_t)).$$

Taking the limit as $t \to \infty$, we obtain

$$\underset{\alpha \to \theta^+}{liminf}\, \eta(\alpha) = \underset{t \to \infty}{liminf}\, \eta(\vartheta_{ERb}(w_t, w_{t-1}))\eta\tilde{F}(\theta^+) - \tilde{F}(\theta^+) = 0,$$

which is a contradiction with $(NF2)$. Consequently, we have that $\theta = 0$, i.e.,

$$lim_{t \to \infty}\vartheta_{ERb}\ (w_{t+1}, w_t) = 0. \tag{3.4}$$

**Step 2:** Next, we claim that $w_t \neq w_s$ whenever$t \neq s$. In fact, assume that $w_t = w_s$ for some $t > s$, then $w_{t+1} = \mathit{\Gamma} w_t = \mathit{\Gamma} w_s = w_{s+1}$. By $(NF3)$, we obtain

$$\begin{aligned}\tilde{F}(\vartheta_{ERb}(w_{s+1}, w_s)) = \tilde{F}(\vartheta_{ERb}(w_{t+1}, w_t)) &\leq \tilde{F}(f(\vartheta_{ERb}(w_{t+1}, w_t))) \\ &\leq \tilde{F}(\vartheta_{ERb}(w_t, w_{t-1})) - \eta(\vartheta_{ERb}(w_t, w_{t-1})) \\ &< \tilde{F}(\vartheta_{ERb}(w_t, w_{t-1})) \\ &\leq \tilde{F}(f(\vartheta_{ERb}(w_t, w_{t-1}))) \\ &\leq \tilde{F}(\vartheta_{ERb}(w_{t-1}, w_{t-2})) - \eta(\vartheta_{ERb}(w_{t-1}, w_{t-2}))\end{aligned}$$

$$\vdots$$
$$< \tilde{F}(\vartheta_{ERb}(w_{s+1}, w_s)),$$

which is a contradiction.

**Step 3:** We will show that $\{w_t\}$ is Cauchy. From $(F1)$, it follows that discontinuous points of $\tilde{F}$ form at most countable set $\Omega$. Assume that $\{w_t\}$ is not Cauchy. Then, there is $\varepsilon > 0$, $f(\varepsilon) \notin \Omega$that can find two subsequences $\{w_{s_r}\}$ and $\{w_{t_r}\}$ of $\{w_t\}$ for every $r \geq 0$ such that $t_r$ is the smallest index satisfying

$$t_r > s_r > r and \vartheta_{ERb}(w_{s_r}, w_{t_r}) \geq \varepsilon. \tag{3.5}$$

This implies that

$$\vartheta_{ERb}(w_{s_r}, w_{t_r-1}) < \varepsilon and \vartheta_{ERb}(w_{s_r}, w_{t_r-2}) < \varepsilon. \tag{3.6}$$

From $(NF3)$, we have

$$\tilde{F}(f(\vartheta_{ERb}(w_{s_r}, w_{t_r}))) \leq \tilde{F}(\vartheta_{ERb}(w_{s_r-1}, w_{t_r-1})) - \eta(\vartheta_{ERb}(w_{s_r-1}, w_{t_r-1}))$$
$$\leq \tilde{F}(\vartheta_{ERb}(w_{s_r-1}, w_{t_r-1})).$$

By $(F1)$, we have

$$f(\vartheta_{ERb}(w_{s_r}, w_{t_r})) \leq \vartheta_{ERb}(w_{s_r-1}, w_{t_r-1}). \tag{3.7}$$

By using (3.5), (3.6), (3.7), $(ERb3)$ and the fact that $f$ is strictly increasing

$$f(\varepsilon) \leq f(\vartheta_{ERb}(w_{s_r}, w_{t_r})) \leq \vartheta_{ERb}(w_{s_r-1}, w_{t_r-1})$$
$$\leq f(\vartheta_{ERb}(w_{s_r-1}, w_{s_r})) + f(\vartheta_{ERb}(w_{s_r}, w_{t_r-2})) + f(\vartheta_{ERb}(w_{t_r-2}, w_{t_r-1}))$$
$$\leq f(\vartheta_{ERb}(w_{s_r-1}, w_{s_r})) + f(\varepsilon) + f(\vartheta_{ERb}(w_{t_r-2}, w_{t_r-1})). \tag{3.8}$$

Taking the limit to (3.8) as $r \to \infty$, by (3.4), the continuity of $f$ and $f(0) = 0$, we get

$$f(\varepsilon) \leq \lim_{r\to\infty} f(\vartheta_{ERb}(w_{s_r}, w_{t_r})) \leq f(\lim_{r\to\infty} \vartheta_{ERb}(w_{s_r-1}, w_{s_r})) + f(\varepsilon) + f(\lim_{r\to\infty} \vartheta_{ERb}(w_{t_r-2}, w_{t_r-1}))$$
$$= f(0) + f(\varepsilon) + f(0) = f(\varepsilon).$$

That is,

$$\lim_{r\to\infty} f(\vartheta_{ERb}(w_{s_r}, w_{t_r})) = \lim_{r\to\infty} \vartheta_{ERb}(w_{s_r-1}, w_{t_r-1}) = f(\varepsilon). \tag{3.9}$$

By $(NF3)$,

$$\eta(\vartheta_{ERb}(w_{s_r-1}, w_{t_r-1})) \leq \tilde{F}(\vartheta_{ERb}(w_{s_r-1}, w_{t_r-1})) - \tilde{F}(f(\vartheta_{ERb}(w_{s_r}, w_{t_r}))). \tag{3.10}$$

Taking the limit to (3.10) as $r \to \infty$and using (3.9) and the fact that $\tilde{F}$ is continuous at $f(\varepsilon)$, we have

$$\liminf_{\alpha\to f(\varepsilon)^+} \eta(\alpha) \leq \liminf_{r\to\infty} \eta(\vartheta_{ERb}(w_{s_r-1}, w_{t_r-1})) \leq \tilde{F}(f(\varepsilon)) - \tilde{F}(f(\varepsilon)) = 0.$$

This contradicts $(NF2)$, and we conclude that $\{w_t\}$ is Cauchy. Since $\aleph$is complete, there is $\hat{w} \in \aleph$ such that$lim_{t\to\infty} w_t = \hat{w}$.

**Step 4:** In the last step we will show the existence and uniqueness of the fixed point of $\Gamma$. Now, we prove that$\hat{w} = \Gamma\hat{w}$. Assume that$\hat{w} \neq \Gamma\hat{w}$, we can find a subsequence $\{w_{t_r}\}$ of $\{w_t\}$ with $\Gamma\hat{w} \neq \Gamma w_{t_r} = w_{t_r+1}$. From $(NF3)$, we get

$$\tilde{F}(f(\vartheta_{ERb}(\Gamma\hat{w}, \Gamma w_{t_r}))) \leq \tilde{F}(\vartheta_{ERb}(\hat{w}, w_{t_r})) - \eta(\vartheta_{ERb}(\hat{w}, w_{t_r})) \leq \tilde{F}(\vartheta_{ERb}(\hat{w}, w_{t_r})).$$

From $(F1)$, we obtain

$$f(\vartheta_{ERb}(\Gamma\hat{w}, \Gamma w_{t_r})) \leq \vartheta_{ERb}(\hat{w}, w_{t_r}). \tag{3.11}$$

By using (3.2) of Lemma 3.8 and (3.11), we have

$$0 < \vartheta_{ERb}(\hat{w}, \Gamma\hat{w}) = f(f^{-1}(\vartheta_{ERb}(\hat{w}, \Gamma\hat{w}))) \leq f\left(limsup_{r\to\infty}\vartheta_{ERb}\ \ (w_{t_r}, |\exists\hat{w})\right)$$

$$= limsup_{r\to\infty} f\ \ (\vartheta_{ERb}(w_{t_r}, \Gamma\hat{w})) \leq limsup_{r\to\infty}\vartheta_{ERb}\ \ (w_{t_r}, \hat{w}) = 0,$$

which is contradiction, so $\hat{w} = \Gamma\hat{w}$.
Next, for uniqueness of fixed point, suppose that $\hat{w} = \Gamma\hat{w} \neq \Gamma\hat{z} = \hat{z}$.
By $(NF3)$,

$$\tilde{F}(f(\vartheta_{ERb}(\hat{w}, \hat{z}))) = \tilde{F}(f(\vartheta_{ERb}(\Gamma\hat{w}, \Gamma\hat{z}))) \leq \tilde{F}(\vartheta_{ERb}(\hat{w}, \hat{z})) - \eta(\vartheta_{ERb}(\hat{w}, \hat{z})) < \tilde{F}(\vartheta_{ERb}(\hat{w}, \hat{z})).$$

Thus, by $(F1)$, we obtain that$f(\vartheta_{ERb}(\hat{w}, \hat{z})) < \vartheta_{ERb}(\hat{w}, \hat{z})$, which is a contradiction with the fact that $f(x) \geq x$ for all $x > 0$. This leads to the conclusion that the fixed point of $\Gamma$is unique. $\eta(\alpha) = ln(\ 1/k)$, it is proved. □

**Corollary 3.11.** *Let $(\aleph, \vartheta_{ERb})$ be a complete ERb-MS with a strictly increasing continuous function $f : [0, \infty) \to [0, \infty)$ such that $f(x) \geq x$, $\forall x > 0$ and $f(0) = 0$, and let $\Gamma : \aleph \to \aleph$ such that*

$$f(\vartheta_{ERb}(\Gamma w, \Gamma z)) \leq k\vartheta_{ERb}(w, z), \text{ for all } w, z \in \aleph \tag{3.12}$$

*where $k \in (0, 1)$. Then $\Gamma$has a unique fixed point.*

*Proof.* In Theorem 3.10, taking $\tilde{F}(\beta) = ln\beta$ and $\eta(\alpha) = ln(\ 1/k)$, it is proved. □

**Example 3.12.** *Let $\aleph = [0, \infty)$ Define $\vartheta_{ERb} : \aleph \times \aleph \to [0, \infty)$ as $\vartheta_{ERb}(w, z) = e^{|w-z|} - 1$, for all $w, z \in \aleph$. Given $f : [0, \infty) \to [0, \infty)$ and $\Gamma : \aleph \to \aleph$ by $f(x) = e^{3x} - 1$ and $\Gamma w = \frac{1}{6}ln(\ w + 1)$.*
*From Example 3.5, it is easy to show that $(\aleph, \vartheta_{ERb})$ is a complete ERb-MS with f. Taking a function $\psi : [0, \infty) \to [0, \infty)$ by $\psi(x) = e^{\alpha x} - 1$ for $\alpha \geq 1$, we have $\psi \in \Psi$. Thus, by Lemma 1.1 in [7], it holds that for all $x, y \in [0, \infty)$, $r \in (0, 1)$,*

$$\psi(rx) \leq r\psi(x), \tag{3.13}$$

$$\left|\psi^{-1}(x) - \psi^{-1}(y)\right| \leq \psi^{-1}|x - y|, i.e, |\Gamma x - \Gamma y| \leq \Gamma(|x - y|) \tag{3.14}$$

*By using (3.13) and (3.14), we have that for all $w, z \in \aleph$,*

$$\vartheta_{ERb}(\Gamma w, \Gamma z) = e^{|\Gamma w - \Gamma z|} - 1 \leq e^{\Gamma(|w-z|)} - 1 = e^{\frac{1}{6}ln(|w-z|+1)}\ - 1 \leq \frac{1}{6}\left(e^{ln(|w-z|+1)}\ - 1\right) = \frac{1}{6}(|w - z|).$$

*So, we get*

$$f(\vartheta_{ERb}(\Gamma w, \Gamma z)) = e^{3\vartheta_{ERb}(\Gamma w, \Gamma z)} - 1 \leq e^{\frac{1}{2}|w-z|} - 1 \leq \frac{1}{2}\left(e^{|w-z|} - 1\right) = \frac{1}{2}\vartheta_{ERb}(w, z) \leq k\vartheta_{ERb}(w, z),$$

*for all $k \in [1/2, 1)$. So, from Corollary 3.11 $\Gamma$has a unique fixed point $w = 0$.*

*Remark* 3.13.

1. *In Theorem 3.10, taking by $f(x) = x$, we obtain fixed point result for $(\eta, F)$ -contractions by using only $(F1)$ without $(F2')$ in complete RMS and complete metric space. (Compare with Theorem 2.1 in [19])*
2. *Taking $\eta(\alpha) = \tau$ in Theorem 3.10, we obtain a fixed point result for $\tilde{F}$-contractions in ERb-MS. Also, taking by $f(x) = x$, we obtain fixed point results for F-contractions in RMS and MS as corollaries.*
3. *Taking by $f(x) = x$ in Corollary 3.11, we obtain Banach's fixed point result in complete RMS.*
4. *From Theorem 3.10, we obtain Corollary 2.2, Corollary 2.3 and Corollary 2.4 in [10] in complete Eb-MS.*

## 4. Application

We apply Theorem 3.10 to consider the existence of solution for Fredholm type integral equation as follows:

$$w(u) = h(u) + \int_a^b K(u, v, w(v))dv, forallu, v \in [a, b] \tag{4.1}$$

where $K : [a, b] \times [a, b]\times\mathbb{R}\to\mathbb{R}$ and $h : [a, b]\to\mathbb{R}$ are continuous.
Let $\aleph$=$C([a,\ b])$. Define $\vartheta_{ERb} : \aleph \times \aleph \to [0, \infty)$ and $f : [0, \infty) \to [0, \infty)$ by
$\vartheta_{ERb}(w, z) = sinh\left(max_{u\in[a,b]} |w(u) - z(u)|\ \right)$ and $f(x) = sinh(\ 3x)$.
Then, $(\aleph, \vartheta_{ERb})$ is a complete *ERb-MS* with *f*.

**Theorem 4.1.** *Suppose that for all* $w, z \in \aleph$ *and* $u, v \in [a, b]$

$$|K(u, v, w(v)) - K(u, v, z(v))| \le \frac{e^{-1/sinh|w(v)-z(v)|}}{3(b - a)} sinh^{-1} |w(v) - z(v)|\ .$$

*Then, the integral equation* (4.1) *has a unique solution.*

*Proof.* Define the integral operator $\Gamma : \aleph \to \aleph$ by

$$\Gamma w(u) = h(u) + \int_a^b K(u, v, w(v))dv, forallu, v \in [a, b] \tag{4.2}$$

For all $w, z \in \aleph$ and $u, v \in [a, b]$, by the assumption of the theorem we have

$$\begin{aligned}
|\Gamma w(u) - \Gamma z(u)| &= \left|h(u) + \textstyle\int_a^b K(u, v, w(v))dv - h(u) - \int_a^b K(u, v, z(v))dv\right| \\
&= \left|\textstyle\int_a^b K(u, v, w(v))dv - \int_a^b K(u, v, z(v))dv\right| \\
&\le \textstyle\int_a^b |K(u, v, w(v)) - K(u, v, z(v))|\, dv \\
&\le \textstyle\int_a^b \frac{e^{-1/sinh|w(v)-z(v)|}}{3(b-a)} sinh^{-1} |w(v) - z(v)|\ dv \\
&\le \frac{e^{-1/sinh\left(max_{v\in[a,b]}|w(v)-z(v)|\ \right)}}{3(b-a)} sinh^{-1}\left(max_{v\in[a,b]} |w(v) - z(v)|\ \right)\ \textstyle\int_a^b dv \\
&= \frac{e^{-1/\vartheta_{ERb}(w,z)}}{3} sinh^{-1}\left(max_{v\in[a,b]} |w(v) - z(v)|\ \right) \\
&= \frac{e^{-1/\vartheta_{ERb}(w,z)}}{3} sinh^{-1}\left(max_{u\in[a,b]} |w(u) - z(u)|\ \right)\ .
\end{aligned}$$

Since the above inequality holds for all $u \in [a, b]$, we have

$$max_{u\in[a,b]} |\Gamma w(u) - \Gamma z(u)|\ \le \frac{e^{-1/\vartheta_{ERb}(w,z)}}{3} sinh^{-1}\left(max_{u\in[a,b]} |w(u) - z(u)|\ \right)\ .$$

Since the function $y = sinh(\ x)$ is strictly increasing and $sinh(\ kx) \le ksinh(\ x)$ for all $x \ge 0$ and $k \in (0, 1)$, we obtain

$$\begin{aligned}
sinh\left(max_{u\in[a,b]} |\Gamma w(u) - \Gamma z(u)|\ \right) &\le sinh\left(\frac{e^{-1/\vartheta_{ERb}(w,z)}}{3} sinh^{-1}\left(max_{u\in[a,b]} |w(u) - z(u)|\ \right)\right) \\
&\le \frac{e^{-1/\vartheta_{ERb}(w,z)}}{3} sinh\left(sinh^{-1}\left(max_{u\in[a,b]} |w(u) - z(u)|\ \right)\ \right)
\end{aligned}$$

$$= \frac{e^{-1/\vartheta_{ERb}(w,z)}}{3} max_{u\in[a,b]} \left|w(u) - z(u)\right| ,$$

that is,

$$3\vartheta_{ERb}(\Gamma w, \Gamma z) \le e^{-1/\vartheta_{ERb}(w,z)} max_{u\in[a,b]} \left|w(u) - z(u)\right| .$$

Again we get

$$\begin{aligned} sinh\left(3\vartheta_{ERb}(\Gamma w, \Gamma z)\right) &= sinh\left(e^{-1/\vartheta_{ERb}(w,z)} max_{u\in[a,b]} \left|w(u) - z(u)\right| \right) \\ &\le e^{-1/\vartheta_{ERb}(w,z)} sinh\left(max_{u\in[a,b]} \left|w(u) - z(u)\right| \right) , \end{aligned}$$

which implies that

$$f(\vartheta_{ERb}(\Gamma w, \Gamma z)) \le e^{-1/\vartheta_{ERb}(w,z)} \vartheta_{ERb}(w, z). \tag{4.3}$$

Taking logarithms on (4.3), we get

$$ln\left(f(\vartheta_{ERb}(\Gamma w, \Gamma z))\right) \le -\frac{1}{\vartheta_{ERb}(w, z)} + ln\left(\vartheta_{ERb}(w, z)\right) ,$$

that is,

$$\frac{1}{\vartheta_{ERb}(w, z)} + ln\left(f(\vartheta_{ERb}(\Gamma w, \Gamma z))\right) \le ln\left(\vartheta_{ERb}(w, z)\right) . \tag{4.4}$$

If we take $\tilde{F} : (0, \infty) \to (-\infty, +\infty)$ and $\eta : (0, \infty) \to (0, \infty)$ by $\tilde{F}(\beta) = ln\beta$ and $\eta(\alpha) = 1/\alpha$, then (4.4) implies that

$$\eta(\vartheta_{ERb}(w, z)) + \tilde{F}(f(\vartheta_{ERb}(\Gamma w, \Gamma z))) \le \tilde{F}(\vartheta_{ERb}(w, z)).$$

Thus, by Theorem 3.10 $\Gamma$has a unique fixed point, which means that (4.1) has a unique solution. □

## 5. Conclusion

In this paper, we introduced a new concept, namely, *ERb-MS* in the field of fixed point theory. We highlighted the significance of our new concept by presenting examples showing that *ERb-MS* is a new generalization of several abstract spaces such as *MS*, *b-MS*, *Eb-MS*, *RMS* and *Rb-MS*. We also established fixed point result for nonlinear $F$-contractions under weaker assumptions in *ERb-MS*, where we used only $(F1)$ of three properties associated with $F$-contractions. Our results improve and generalize many previous fixed point results related to $F$-contractions. Finally, we investigated the existence of solutions to Fredholm type integral equations by using obtained results.

Future research directions will include generalization from the expanded rectangular $b$-metric space to expanded $b_v(s)$-metric space and establishment of the related fixed point results, applications to fractals and fractional differential equations.

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