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Derivations of BiHom-Jacobi-Jordan Superalgebras and BJJYB Equations

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Abstract

This paper explores the derivations of BiHom-Jacobi-Jordan algebras, building on classical structures with BiHom-operations and investigating their algebraic properties. We extend this analysis to BiHom-Jacobi-Jordan superalgebras, focusing on the derivations and the complexities introduced by their graded structure. Furthermore, we construct solutions to the BiHom-Jacobi-Jordan Yang-Baxter Equation (BJJYBE) within the context of BiHom-Jacobi-Jordan superalgebras.

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1. Introduction

The development of algebraic structures such as Jacobi-Jordan algebras, BiHom-associative algebras, BiHom-Jacobi-Jordan algebras, and BiHom-associative superalgebras has significantly enriched both theoretical and applied mathematics. The motivation for studying Jacobi-Jordan algebras, sometimes referred to as Jordan-Lie or mock-Lie algebras, arises from their structural relationship with Lie algebras. In these algebras, the skew-symmetric condition inherent to Lie algebras is relaxed, giving rise to symmetric commutative operations while still preserving certain Lie-like properties [15, 9]. Historically, Jacobi-Jordan algebras trace their origins to Jordan's attempts to formalize the mathematical foundations of quantum mechanics in the 1930s [15]. These algebras were initially explored by physicists such as Pascual Jordan, John von Neumann, and Eugene Wigner, who sought non-associative algebras that generalize associative algebras and better fit quantum observables [15, 9]. Later developments led to connections with Lie theory and the broader family of Jordan algebras, forming a framework for analyzing symmetries and conserved quantities [14, 16].

In parallel, the advent of Hom-algebras and their generalization to BiHom-algebras opened new avenues in the study of non-associative structures [16, 12]. BiHom-algebras, introduced to extend Hom-algebraic

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structures by incorporating two commuting twisting maps, have found applications in diverse areas, including deformations, homotopy algebras, and representation theory[12, 1]. The foundational works of Makhlouf and Silvestrov, among others, laid the groundwork for BiHom-algebra theory, enabling further exploration of their associative and Lie-type generalizations [16, 12, 11].

The BiHom-associative algebra, which forms a natural extension of associative algebras through BiHom structures, has facilitated the generalization of classical associative operations [16, 12]. By imposing two commuting maps, BiHom-associative algebras encompass both hom-associative and classical associative frameworks, providing an elegant platform for studying deformation theory and quantum algebraic structures [5, 3, 4, 11, 10]. The BiHom-associative superalgebra, a graded generalization, introduces a compatible structure on superalgebras, linking them to supersymmetry and other graded systems inmathematical physics [5, 3, 4, 13, 18].

The BiHom-Jacobi-Jordan algebra, a natural extension of Jacobi-Jordan algebras within the BiHom framework, is a core subject of this work. This algebraic structure maintains symmetry while incorporating twisting maps, resulting in enriched algebraic and combinatorial properties. Applications of these algebras span from integrable systems to algebraic combinatorics and representation theory, offering a flexible toolset for modern mathematical analysis [12, 17, 8]. Recent studies have highlighted connections to operator algebras and related frameworks [6, 5, 3, 4, 7, 2, 19].

Several authors have contributed extensively to these topics. Among them, Makhlouf and Silvestrov are recognized for their pioneering contributions to hom-algebra and BiHom-algebra theory [16, 12, 11]. The works of Hartwig, Larsson, and Richard have provided significant insights into the algebraic deformations and structural properties of hom-type algebras [1]. Additionally, studies on Jacobi-Jordan algebras, notably by Jacobson and others, have paved the way for applications in various fields [14, 8, 2]. Collectively, these contributions form a robust mathematical edifice that inspires contemporary exploration of BiHom-structures [16, 11, 19].

The main purpose of this paper is to introduce and study the concept of BiHom-Jacobi-Jordan superalgebras, defining their related properties and investigating derivations of BiHom-Jacobi-Jordan algebras and superalgebras. We also focus on constructing solutions to the BiHom-Jacobi-Jordan Yang-Baxter Equation (BJJYBE) from BiHom-Jacobi-Jordan superalgebras.
Across this paper, all vector spaces are finite-dimensional and defined over a field K with characteristic zero.

2. BiHom-associative Superalgebras and BiHom-Jacobi-Jordan Superalgebras

Definition 2.1. An algebra $(\mathcal{V}, \langle ., .\rangle)$ is considered a Jacobi-Jordan algebra if it meets these two conditions:

1. **Symmetry:** $\langle x, y\rangle = \langle y, x\rangle$
2. **Jacobi Identity:** $\langle x, \langle y, z\rangle\rangle + \langle y, \langle z, x\rangle\rangle + \langle z, \langle x, y\rangle\rangle = 0$

for any elements $x, y, z \in \mathcal{V}$.

Example 2.2. Consider the space V_n with basis elements
$\{x_1, \ldots, x_n, y_1, \ldots, y_n, z\}$
This structure is a Jacobi Jordan algebra, referred to as the Heisenberg Jacobi Jordan algebra, if it satisfies the bilinear map $\langle x_k, y_k\rangle = \langle y_k, x_k\rangle = z$ for each $k = 1, \ldots, n$.

Definition 2.3. Let $(\mathcal{V}, \langle ., .\rangle)$ be a Jacobi Jordan algebra, and let $\mathcal{W}$ be a vector space. If there exists a linear mapping
$\theta :\rightarrow \operatorname{End}(\mathcal{W})$ that satisfies the condition

$$\theta(x)\theta(y) + \theta(y)\theta(x) = -\theta(\langle x, y\rangle),$$

for any $x, y \in \mathcal{V}$, then $(\theta, \mathcal{W})$ is defined as a representation of $(\mathcal{V}, \langle ., .\rangle)$.

Example 2.4. A Jacobi Jordan algebra $\mathcal{V}$ is considered as a representation of itself relative to its own bilinear map $\langle ., .\rangle$. This specific case is referred to as the adjoint representation of $\mathcal{V}$.

Definition 2.5. Let $(\mathcal{V}, \langle ., . \rangle)$ be a Jacobi Jordan algebra, and let Δ be a linear transformation on $\mathcal{V}$. We say that Δ is an anti-derivation of $\mathcal{V}$ if it satisfies the following condition for all $x, y \in \mathcal{V}$:

$$\Delta(\langle x, y \rangle) = -\langle \Delta(x), y \rangle - \langle x, \Delta(y) \rangle.$$

We denote the space of anti-derivations of $\mathcal{V}$ as Ader $(\mathcal{V})$.

Definition 2.6. A Bihom-associative algebra over a field $\mathbb{K}$ is defined as a 4-tuple $(\mathcal{B}, \iota, f, g)$, where:

- $\mathcal{B}$ is a $\mathbb{K}$-linear vector space,
- $f : \mathcal{B} \to \mathcal{B}, g : \mathcal{B} \to \mathcal{B}$, and $\iota : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$ are linear mappings,

with the notation $\iota\left(a_1 \otimes a_2\right) = a_1 a_2$. These maps satisfy the following properties for any elements $a_1, a_2, a_3 \in \mathcal{B}$:

1. $f \circ g = g \circ f$,
2. $f\left(a_1\right)\left(a_2 a_3\right) = \left(a_1 a_2\right) g\left(a_3\right)$. (Bihom-associativity)

If, in addition, $f\left(a_1 a_2\right) = f\left(a_1\right) f\left(a_2\right)$ and $g\left(a_1 a_2\right) = g\left(a_1\right) g\left(a_2\right)$, the algebra is termed a **multiplicative Bihom-associative algebra.** In this context, f and g are known as the **structure maps** of $\mathcal{B}$. Additionally, a **Hom-associative algebra** $(\mathcal{B}, \iota, f)$ can be viewed as a specific case of a Bihomassociative algebra $(\mathcal{B}, \iota, f, f)$.

Definition 2.7. A **Bihom-Jacobi-Jordan algebra** over $\mathbb{K}$ is a 4-tuple $(\mathcal{V}, \langle \cdot, \cdot \rangle, f, g)$, where:

- $\mathcal{V}$ is a $\mathbb{K}$-linear vector space,
- $f : \mathcal{V} \to \mathcal{V}, g : \mathcal{V} \to \mathcal{V}$, and $\langle \cdot, \cdot \rangle : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$ are linear mappings,

and we use the notation $\langle \cdot, \cdot \rangle \left(a_1 \otimes a_2\right) = \langle a_1, a_2 \rangle$. These mappings satisfy the following conditions for any $a_1, a_2, a_3 \in \mathcal{V}$:

1. $f \circ g = g \circ f$,
2. $\langle g\left(a_1\right), f\left(a_2\right) \rangle = \langle g\left(a_2\right), f\left(a_1\right) \rangle$, (Bihom-symmetry)
3. $\left\langle g^2\left(a_1\right), \langle g\left(a_2\right), f\left(a_3\right) \rangle \right\rangle + \left\langle g^2\left(a_2\right), \langle g\left(a_3\right), f\left(a_1\right) \rangle \right\rangle + \left\langle g^2\left(a_3\right), \langle g\left(a_1\right), f\left(a_2\right) \rangle \right\rangle = 0$. (Bihomjacobi identity)

A Bihom-Jacobi-Jordan algebra is referred to as a **multiplicative Bihom-Jacobi-Jordan algebra** if f and g are algebraic homomorphisms, meaning that for any $a_2, a_3 \in \mathcal{V}$.

1. $f\left(\langle a_2, a_3 \rangle\right) = \langle f\left(a_2\right), f\left(a_3\right) \rangle$,
2. $g\left(\langle a_2, a_3 \rangle\right) = \langle g\left(a_2\right), g\left(a_3\right) \rangle$.

A **Hom-Jacobi-Jordan algebra** $(\mathcal{V}, \langle \cdot, \cdot \rangle, f)$ is a special instance of a Bihom-Jacobi-Jordan algebra, specifically $(\mathcal{V}, \langle \cdot, \cdot \rangle, f, f)$. On the other hand, any Bihom-Jacobi-Jordan algebra $(\mathcal{V}, \langle \cdot, \cdot \rangle, f, f)$ where f is an isomorphism can be viewed as a Hom-Jacobi-Jordan algebra $(\mathcal{V}, \langle \cdot, \cdot \rangle, f)$.

Definition 2.8. A **Bihom-associative algebra** over a field $\mathbb{K}$ is defined as a 4-tuple $(\mathcal{B}, \iota, f, g)$, where $\mathcal{B}$ represents a superspace, and $f : \mathcal{B} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{B}$ are even homomorphisms. The mapping $\iota : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$ is an even bilinear operation denoted by $\iota(a_1 \otimes a_2) = a_1 a_2$. The following conditions must be satisfied for all homogeneous elements $a_1, a_2, a_3 \in \mathcal{B}$:

$$f \circ g = g \circ f,$$
$$f(a_1)(a_2 a_3) = (a_1 a_2) g(a_3). \quad \text{(Bihom-associativity)}$$

If $f(a_1 a_2) = f(a_1) f(a_2)$ and $g(a_1 a_2) = g(a_1) g(a_2)$, the structure is termed a **multiplicative Bihom-associative superalgebra.**

Definition 2.9. A Bihom-Jacobi-Jordan superalgebra over a field $\mathbb{K}$ is represented by a 4-tuple $(\mathcal{V}, \langle\cdot,\cdot\rangle, f, g)$, where $\mathcal{V}$ is a superspace, $f : \mathcal{V} \to \mathcal{V}$ and $g : \mathcal{V} \to \mathcal{V}$ are even homomorphisms, and $\langle\cdot,\cdot\rangle : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$ denotes an even bilinear mapping, expressed as $\langle a_1, a_2\rangle = \langle\cdot,\cdot\rangle(a_1 \otimes a_2)$. The following properties must hold for all homogeneous elements $a_1, a_2, a_3 \in \mathcal{V}$:

$$f \circ g = g \circ f,$$

$$\langle g(a_1), f(a_2)\rangle = (-1)^{|a_1||a_2|} \langle g(a_2), f(a_1)\rangle. \text{ (Bihom-super-symmetry)}$$

$$\circlearrowleft_{a_1,a_2,a_3} (-1)^{|a_1||a_3|} \left\langle g^2(a_1), \langle g(a_2), f(a_3)\rangle\right\rangle = 0. \text{ (Bihom-super-Jacobi identity)}$$

Definition 2.10.

1. A Bihom-Jacobi-Jordan superalgebra is identified as a multiplicative Bihom-Jacobi-Jordan superalgebra if f and g are algebraic morphisms. For any elements $a_2, a_3 \in \mathcal{V}$, the following must be true:
$$f(\langle a_2, a_3\rangle) = \langle f(a_2), f(a_3)\rangle, \quad g(\langle a_2, a_3\rangle) = \langle g(a_2), g(a_3)\rangle.$$
2. A Bihom-Jacobi-Jordan superalgebra $(\mathcal{V}, \langle\cdot,\cdot\rangle, f, g)$ is termed regular if both f and g are algebra automorphisms.
3. A sub-vector space $\mathcal{I} \subset \mathcal{V}$ is defined as a Bihom subalgebra of $(\mathcal{V}, \langle\cdot,\cdot\rangle, f, g)$ if it satisfies $f(\mathcal{I}) \subset \mathcal{I}, g(\mathcal{I}) \subset \mathcal{I}$, and
$$\langle u, v\rangle \in \mathcal{I}, \quad \forall u, v \in \mathcal{I}$$
4. A sub-vector space $\mathcal{I} \subset \mathcal{V}$ is considered a Bihom ideal of $(\mathcal{V}, \langle\cdot,\cdot\rangle, f, g)$ if $f(\mathcal{I}) \subset \mathcal{I}, g(\mathcal{I}) \subset \mathcal{I}$ and
$$\langle u, v\rangle \in \mathcal{I}, \quad \forall u \in \mathcal{I}, v \in \mathcal{V}$$

Example 2.11. Consider a 3-dimensional superspace $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$, where $\mathcal{V}_0$ is induced by the elements x_1 and x_2, and $\mathcal{V}_1$ is induced by x_3. We define a bracket product $\langle\cdot,\cdot\rangle$ on $\mathcal{V}$ as follows:

$$\begin{aligned}
&\langle x_1, x_1\rangle = 0, \quad \langle x_1, x_2\rangle = x_1, \quad \langle x_1, x_3\rangle = 0\\
&\langle x_2, x_1\rangle = x_1, \quad \langle x_2, x_2\rangle = 0, \quad \langle x_2, x_3\rangle = 0\\
&\langle x_3, x_1\rangle = 0, \quad \langle x_3, x_2\rangle = 0, \quad \langle x_3, x_3\rangle = 0
\end{aligned}$$

Let ζ and η be two nonzero scalars from the field $\mathbb{K}$. We then define the mappings $f, g : \mathcal{V} \to \mathcal{V}$ on the basis elements as follows:

$$\begin{aligned}
&f(x_1) = \eta x_1, \quad f(x_2) = x_2, \quad f(x_3) = \zeta x_3,\\
&g(x_1) = \eta x_1, \quad g(x_2) = x_2, \quad g(x_3) = -\zeta x_3.
\end{aligned}$$

It is easy to demonstrate that the maps f and g constitute two BiHom-Jacobi-Jordan superalgebra homomorphisms, and that they commute: $f \circ g = g \circ f$. Furthermore, one can verify that the bracket product $\langle\cdot,\cdot\rangle$ and the structure maps f and g fulfill the conditions for Bihom-super-symmetry and the Bihom-super-Jacobi identity. Consequently, $(\mathcal{V}, \langle\cdot,\cdot\rangle, f, g)$ forms a BiHom-Jacobi-Jordan superalgebra.

Proposition 2.12. *Let $(\mathcal{B}, \iota, f, g)$ represent a multiplicative Bihom-associative superalgebra with bijective homomorphisms f and g. For any elements $a_1, a_2 \in \mathcal{B}$, define*

$$\langle a_1, a_2\rangle = a_1a_2 + (-1)^{|a_1||a_2|} \left(f^{-1}(g(a_2))\right)\left(f\left(g^{-1}(a_1)\right)\right).$$

Then $(\mathcal{B}, \langle\cdot,\cdot\rangle, f, g)$ forms a Bihom-Jacobi-Jordan superalgebra, denoted by $\mathcal{V}(\mathcal{B})$.

Proposition 2.13. *Let $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ denote a Jacobi-Jordan superalgebra over a field $\mathbb{K}$, with two even commuting algebra homomorphisms $f, g : \mathcal{V} \to \mathcal{V}$ such that for all $a_2, a_3 \in \mathcal{V}$, we have $f(\langle a_2, a_3\rangle) = \langle f(a_2), f(a_3)\rangle$ and $g(\langle a_2, a_3\rangle) = \langle g(a_2), g(a_3)\rangle$.Define $\mathcal{V}_{(f,g)} := (\mathcal{V}, \langle\cdot,\cdot\rangle_{f,g}, f, g)$ where*

$$\langle a_1, a_2\rangle_{f,g} = \langle f(a_1), g(a_2)\rangle.$$

Then $\mathcal{V}_{(f,g)}$ is a multiplicative Bihom-Jacobi-Jordan superalgebra, known as the Yau twist of $(\mathcal{V}, \langle\cdot,\cdot\rangle)$.

Theorem 2.14. *Suppose $(\mathcal{V}, \langle\cdot,\cdot\rangle, f, g)$ is a Bihom-Jacobi-Jordan superalgebra, with two even algebra homomorphisms $f_1, g_1 : \mathcal{V} \to \mathcal{V}$ satisfying $f_1(\langle a_2, a_3\rangle) = \langle f_1(a_2), f_1(a_3)\rangle$ and $g_1(\langle a_2, a_3\rangle) = \langle g_1(a_2), g_1(a_3)\rangle$ for all $a_2, a_3 \in \mathcal{V}$, and with any two of the maps f, g, f_1, g_1 commuting. Then*

$$(\mathcal{V}, \langle\cdot,\cdot\rangle_{f_1,g_1} := \langle\cdot,\cdot\rangle \circ (f_1 \otimes g_1), f \circ f_1, g \circ g_1)$$

is also a Bihom-Jacobi-Jordan superalgebra.

Proof. To demonstrate the compatibility of the bracket product $\langle\cdot,\cdot\rangle_{f_1,g_1}$ with the structure maps $f \circ f_1$ and $g \circ g_1$, we consider any elements $a_1, a_2 \in \mathcal{V}$: ☐

$$\begin{aligned}
\langle g \circ g_1(a_1), f \circ f_1(a_2)\rangle_{f_1,g_1} &= \langle f_1 \circ g \circ g_1(a_1), g_1 \circ f \circ f_1(a_2)\rangle \\
&= f_1 \circ g_1(\langle g(a_1), f(a_2)\rangle) \\
&= (-1)^{|a_1||a_2|} f_1 \circ g_1(\langle g(a_2), f(a_1)\rangle) \\
&= (-1)^{|a_1||a_2|} \langle f_1 \circ g_1 \circ g(a_2), f_1 \circ g_1 \circ f(a_1)\rangle \\
&= (-1)^{|a_1||a_2|} \langle g \circ g_1(a_2), f \circ f_1(a_1)\rangle_{f_1,g_1}.
\end{aligned}$$

Thus, the Bihom-super symmetry holds.

Next, we verify the Bihom-Jacobi condition. For any elements $a_1, a_2, a_3 \in \mathcal{V}$, we have:

$$\begin{aligned}
&(-1)^{|a_1||a_3|} \left\langle (g \circ g_1)^2(a_1), \langle g \circ g_1(a_2), f \circ f_1(a_3)\rangle_{f_1,g_1} \right\rangle_{f_1,g_1} \\
&= (-1)^{|a_1||a_3|} \left\langle (g \circ g_1)^2(a_1), f_1 \circ g_1(\langle g(a_2), f(a_3)\rangle) \right\rangle_{f_1,g_1} \\
&= (-1)^{|a_1||a_3|} \left\langle f_1 \circ (g \circ g_1)^2(a_1), f_1 \circ (g_1)^2(\langle g(a_2), f(a_3)\rangle) \right\rangle \\
&= (-1)^{|a_1||a_3|} f_1 \circ (g_1)^2 \left(\langle g^2(a_1), \langle g(a_2), f(a_3)\rangle\rangle\right).
\end{aligned}$$

Similarly, we obtain:

$$\begin{aligned}
&(-1)^{|a_1||a_2|} \left\langle (g \circ g_1)^2(a_2), \langle g \circ g_1(a_3), f \circ f_1(a_1)\rangle_{f_1,g_1} \right\rangle_{f_1,g_1} \\
&= (-1)^{|a_1||a_2|} f_1 \circ (g_1)^2 \left(\langle g^2(a_2), \langle g(a_3), f(a_1)\rangle\rangle\right),
\end{aligned}$$

and

$$\begin{aligned}
&(-1)^{|a_3||a_2|} \left\langle (g \circ g_1)^2(a_3), \langle g \circ g_1(a_1), f \circ f_1(a_2)\rangle f_{f_1,g_1} \right\rangle f_{f_1,g_1} \\
&= (-1)^{|a_3||a_2|} f_1 \circ (g_1)^2 \left(\langle g^2(a_3), \langle g(a_1), f(a_2)\rangle\rangle\right).
\end{aligned}$$

Summing these results gives:

$$\begin{aligned}
&(-1)^{|a_1||a_3|} \left\langle (g \circ g_1)^2(a_1), \langle g \circ g_1(a_2), f \circ f_1(a_3)\rangle_{f_1,g_1} \right\rangle_{f_1,g_1} \\
&+(-1)^{|a_1||a_2|} \left\langle (g \circ g_1)^2(a_2), \langle g \circ g_1(a_3), f \circ f_1(a_1)\rangle_{f_1,g_1} \right\rangle_{f_1,g_1} \\
&+(-1)^{|a_3||a_2|} \left\langle (g \circ g_1)^2(a_3), \langle g \circ g_1(a_1), f \circ f_1(a_2)\rangle_{f_1,g_1} \right\rangle_{f_1,g_1} = 0.
\end{aligned}$$

Then the Bihom-super Jacobi identity holds.

Corollary 2.15. *Let $(\mathcal{B}, \iota)$ be an associative superalgebra, and let $f, g : \mathcal{V} \to \mathcal{V}$ be two even algebra isomorphisms that commute. Then $\mathcal{V}\left(\mathcal{B}_{(f,g)}\right) = \mathcal{V}(\mathcal{B})_{(f,g)}$ forms a Bihom-Jacobi-Jordan superalgebra.*

Proposition 2.16. *Let* $(\mathcal{V}, \langle\cdot,\cdot\rangle, f, g)$ *and* $(\mathcal{V}', \langle\cdot,\cdot\rangle', f', g')$ *be two Bihom-Jacobi-Jordan superalgebras. Thus, we can establish a new Bihom-Jacobi-Jordan superalgebra* $(\mathcal{V}\oplus\mathcal{V}', \langle\cdot,\cdot\rangle_{\mathcal{V}\oplus\mathcal{V}'}, f+f', g+g')$*, where the even bilinear map* $\langle\cdot,\cdot\rangle_{\mathcal{V}\oplus\mathcal{V}'} : \wedge^2(\mathcal{V}\oplus\mathcal{V}') \to \mathcal{V}\oplus\mathcal{V}'$ *is determined by*

$$\langle (x_1, y_1), (x_2, y_2)\rangle\rangle_{\mathcal{V}\oplus\mathcal{V}'} = \left(\langle x_1, x_2\rangle, \langle y_1, y_2\rangle'\right), \quad \forall x_1 \in \mathcal{V}_\tau, x_2 \in \mathcal{V}_{\tau'}, y_1 \in \mathcal{V}'_\tau, y_2 \in \mathcal{V}'_{\tau'}$$

and the two even linear maps $f+f'$ *and* $g+g' : \mathcal{V}\oplus\mathcal{V}' \to \mathcal{V}\oplus\mathcal{V}'$ *are defined as*

$$\left(f+f'\right)(x,y) = \left(f(x), f'(y)\right), \quad \left(g+g'\right)(x,y) = \left(g(x), g'(y)\right), \quad \forall x \in \mathcal{V}_\tau, y \in \mathcal{V}'_\tau$$

Proof. For any $x_k \in \mathcal{V}_{\tau_k}$ and $y_k \in \mathcal{V}'_{\tau_k}$ we maintain:

$$\begin{aligned}
\left(f+f'\right)\circ\left(g+g'\right)(x_1, y_1) &= \left(f+f'\right)\left(g(x_1), g'(y_1)\right)\\
&= \left(f\circ g(x_1), f'\circ g'(y_1)\right)\\
&= \left(g\circ f(x_1), g'\circ f'(y_1)\right)\\
&= \left(g+g'\right)\circ\left(f+f'\right)(x_1, y_1).
\end{aligned}$$

$$\begin{aligned}
&\left\langle \left(g+g'\right)(x_1, y_1), \left(f+f'\right)(x_2, y_2)\right\rangle_{\mathcal{V}\oplus\mathcal{V}'}\\
&= \left\langle \left(g(x_1), g'(y_1)\right), \left(f(x_2), f'(y_2)\right)\right\rangle_{\mathcal{V}\oplus\mathcal{V}'}\\
&= \left(\langle g(x_1), f(x_2)\rangle, \langle g'(y_1), f'(y_2)\rangle'\right)\\
&= \left((-1)^{|\tau_1||\tau_2|}\langle g(x_2), f(x_1)\rangle, (-1)^{|\tau_1||\tau_2|}\langle g'(y_2), f'(y_1)\rangle'\right)\\
&= (-1)^{|\tau_1||\tau_2|}\left(\langle g(x_2), f(x_1)\rangle, \langle g'(y_2), f'(y_1)\rangle'\right)\\
&= (-1)^{|\tau_1||\tau_2|}\left\langle \left(g+g'\right)(x_2, y_2), \left(f+f'\right)(x_1, y_1)\right\rangle\Big\rangle_{\mathcal{V}\oplus\mathcal{V}'},
\end{aligned}$$

and

$$\begin{aligned}
&\circlearrowleft_{(x_1,y_1),(x_2,y_2),(x_3,y_3)} (-1)^{|\tau_1||\tau_3|}\left\langle \left(g+g'\right)^2(x_1, y_1), \left\langle \left(g+g'\right)(x_2, y_2), \left(f+f'\right)(x_3, y_3)\right\rangle_{\mathcal{V}\oplus\mathcal{V}'\rangle}\right\rangle \mathcal{V}^{|c\mathcal{V}'}\\
&=\circlearrowleft_{(x_1,y_1),(x_2,y_2),(x_3,y_3)} (-1)^{|\tau_1||\tau_3|}\left\langle \left(g^2(x_1), g'^2(y_1)\right), \left(\langle g(x_2), f(x_3)\rangle, \langle g'(y_2), f'(y_3)\rangle'\right)\right\rangle_{\mathcal{V}\oplus\mathcal{V}'}\\
&= \left(\circlearrowleft_{x_1,x_2,x_3} (-1)^{|\tau_1||\tau_3|}\left\langle g^2(x_1), \langle g(x_2), f(x_3)\rangle\right\rangle, \circlearrowleft_{y_1,y_2,y_3} (-1)^{|\tau_1||\tau_3|}\left\langle g'^2(y_1), \langle g'(y_2), f'(y_3)\rangle'\right\rangle'\right)\\
&= 0.
\end{aligned}$$

Therefore, the Bihom-super symmetry and Bihom-super Jacobi identity are verified as desired. □

Definition 2.17. Consider two Bihom-Jacobi-Jordan superalgebras, $(\mathcal{V}, \langle\cdot,\cdot\rangle, f, g)$ and $(\mathcal{V}', \langle\cdot,\cdot\rangle', f', g')$. An even homomorphism $\chi : \mathcal{V} \to \mathcal{V}'$ is defined as a morphism of Bihom-Jacobi-Jordan superalgebras if it satisfies the following conditions:

$$\chi\langle x, y\rangle = \langle \chi(x), \chi(y)\rangle', \quad \forall x, y \in \mathcal{V}$$
$$\chi\circ f = f'\circ\chi, \quad \chi\circ g = g'\circ\chi.$$

The set $\Gamma_\chi = \{(\xi, \chi(\xi)) \mid \xi \in \mathcal{V}\} \subset \mathcal{V}\oplus\mathcal{V}'$ represents the graph of the linear map $\chi : \mathcal{V} \to \mathcal{V}'$.

Proposition 2.18. *The even homomorphism* $\chi : (\mathcal{V}, \langle\cdot,\cdot\rangle, f, g) \to (\mathcal{V}', \langle\cdot,\cdot\rangle', f', g')$ *is a morphism of Bihom-Jacobi-Jordan superalgebras if and only if its graph* $\Gamma_\chi \subset \mathcal{V}\oplus\mathcal{V}'$ *forms a Bihom-subalgebra of the combined structure* $(\mathcal{V}\oplus\mathcal{V}', \langle\cdot,\cdot\rangle_{\mathcal{V}\oplus\mathcal{V}'}, f+f', g+g')$*.*

Proof. Let $\chi : (\mathcal{V}, \langle\cdot,\cdot\rangle, f, g) \to (\mathcal{V}', \langle\cdot,\cdot\rangle', f', g')$ be a morphism of Bihom-Jacobi-Jordan superalgebras. For any elements $x_1, x_2 \in \mathcal{V}$, we find that

$$\langle (x_1, \chi(x_1)), (x_2, \chi(x_2)) \rangle_{\mathcal{V}\oplus\mathcal{V}'} = \left(\langle x_1, x_2 \rangle, \langle \chi(x_1), \chi(x_2) \rangle' \right) = \left(\langle x_1, x_2 \rangle, \chi \langle x_1, x_2 \rangle \right)$$

This indicates that the graph Γ_χ is closed under the bracket $\langle\cdot,\cdot\rangle_{\mathcal{V}\oplus\mathcal{V}'}$. Moreover, we have

$$\left(f + f' \right)(x_1, \chi(x_1)) = \left(f(x_1), f' \circ \chi(x_1) \right) = \left(f(x_1), \chi \circ f(x_1) \right)$$

implying that

$$\left(f + f' \right)(\Gamma_\chi) \subset \Gamma_\chi.$$

Correspondingly,

$$\left(g + g' \right)(\Gamma_\chi) \subset \Gamma_\chi$$

Therefore, Γ_χ is a Bihom-subalgebra of $(\mathcal{V} \oplus \mathcal{V}', \langle\cdot,\cdot\rangle_{\mathcal{V}\oplus\mathcal{V}'}, f + f', g + g')$.

On the other hand, if the graph Γ_χ is a Bihom-subalgebra of $(\mathcal{V} \oplus \mathcal{V}', \langle\cdot,\cdot\rangle_{\mathcal{V}\oplus\mathcal{V}'}, f + f', g + g')$, we have

$$\langle (x_1, \chi(x_1)), (x_2, \chi(x_2)) \rangle_{\mathcal{V}\oplus\mathcal{V}'} = \left(\langle x_1, x_2 \rangle, \langle \chi(x_1), \chi(x_2) \rangle' \right) \in \Gamma_\chi$$

indicating that

$$\langle \chi(x_1), \chi(x_2) \rangle' = \chi \langle x_1, x_2 \rangle$$

Additionally, since $(f + f')(\Gamma_\chi) \subset \Gamma_\chi$, we get

$$\left(f + f' \right)(x_1, \chi(x_1)) = \left(f(x_1), f' \circ \chi(x_1) \right) \in \Gamma_\chi$$

which corresponds to the condition $f' \circ \chi(x_1) = \chi \circ f(x_1)$, or equivalently, $f' \circ \chi = \chi \circ f$. In the same way, we have $g' \circ \chi = \chi \circ g$. Thus, χ is a morphism of Bihom-Jacobi-Jordan superalgebras. ☐

3. Derivations of BiHom-Jacobi-Jordan algebras

This section explores the derivations of Bihom-Jacobi-Jordan algebras. Let $(\mathcal{V}, \langle\cdot,\cdot\rangle_\mathcal{V}, f, g)$ be a multiplicative Bihom-Jacobi-Jordan algebra. For any nonnegative integers m and n, the composition of f applied m-times is denoted by f^m, and similarly, g^n represents the composition of g applied n times:

$$\begin{aligned} f^m = f \circ \cdots \circ f \quad & | \quad \{(m\text{-times })\} \\ g^n = g \circ \cdots \circ g \quad & | \quad \{(n\text{-times })\} \end{aligned}$$

Since f and g commute, their combined action is expressed as:

$$f^m g^n = f \circ \cdots \circ f \quad | \quad \{(m\text{-times })\} \circ g \circ \cdots \circ g \quad | \quad \{(n\text{-times })\}$$

In particular, we have $f^0 g^0 = \mathrm{Id}$, $\quad f^1 g^1 = fg$, and $f^{-m} g^{-n}$ as the inverse of $f^m g^n$. In regular Bihom-Jacobi-Jordan algebras, f^{-m} denotes the inverse of f^m, constructed as the composition of f^{-1} repeated m-times.

Definition 3.1. Let m and n be nonnegative integers. A linear map $\Delta : \mathcal{V} \to \mathcal{V}$ is called an $f^m g^n$ derivation of the multiplicative Bihom-Jacobi-Jordan algebra $(\mathcal{V}, \langle\cdot,\cdot\rangle_\mathcal{V}, f, g)$, provided it satisfies the following conditions:

1. **Commutativity with f :**

$$\langle \Delta, f \rangle = 0 \quad \text{meaning} \quad \Delta \circ f = f \circ \Delta$$

2. **Commutativity with:** g

$$\langle \Delta, g\rangle = 0 \quad \text{meaning} \quad \Delta \circ g = g \circ \Delta$$

3. **Compatibility:**

$$\Delta\langle p, q\rangle_{\mathcal{V}} = (-1)^m \left(\langle \Delta(p), f^m g^n(q)\rangle_{\mathcal{V}} + \langle f^m g^n(p), \Delta(q)\rangle_{\mathcal{V}}\right), \quad \forall p, q \in \mathcal{V}$$

In regular Bihom-Jacobi-Jordan algebras, $f^{-m}g^{-n}$-derivations are defined similarly.
Additionally, if f and g are bijective, the Bihom-symmetry condition leads to the identity:

$$\langle p, q\rangle = \left\langle f^{-1}g(q), fg^{-1}(p)\right\rangle_{\mathcal{V}}, \quad \forall p, q \in \mathcal{V}$$

We define the set of all $f^t g^n$-derivations of the multiplicative Bihom-Jacobi-Jordan algebra $(\mathcal{V}, \langle\cdot,\cdot\rangle_{\mathcal{V}}, f, g)$ as $\mathrm{Der}_{f^t g^n}(\mathcal{V})$. For any $p \in \mathcal{V}$ such that $f(p) = p$ and $g(p) = p$, the map $\Delta_{m,n}(p) : \mathcal{V} \to \mathcal{V}$ is given by:

$$\Delta_{m,n}(p)(q) = \langle f^m g^n(q), p\rangle_{\mathcal{V}}, \quad \forall q \in \mathcal{V}$$

we have:

$$\Delta_{m,n}(p)(q) = \langle f^m g^n(q), p\rangle_{\mathcal{V}} = -\left\langle f^{-1}g(p), fg^{-1}\left(f^m g^n(q)\right)\right\rangle_{\mathcal{V}} = -\left\langle p, f^{m+1}g^{n-1}(q)\right\rangle_{\mathcal{V}}.$$

Consequently, we determine that $\Delta_{m,n}(p)$ considered as an $f^{m+1}g^n$-derivation, which we identify as an inner $f^{m+1}g^n$-derivation. Specifically, we observe that:

$$\begin{aligned}
\Delta_{m,n}(p)(f(q)) &= \left\langle f^{m+1}g^n(q), p\right\rangle_{\mathcal{V}} = f\left(\langle f^m g^n(q), p\rangle_{\mathcal{V}}\right) = f \circ \Delta_{m,n}(p)(q)\\
\Delta_{m,n}(p)(g(q)) &= \left\langle f^m g^{n+1}(q), p\right\rangle_{\mathcal{V}} = g\left(\langle f^m g^n(q), p\rangle_{\mathcal{V}}\right) = g \circ \Delta_{m,n}(p)(q)
\end{aligned}$$

Conversely, we have:

$$\begin{gathered}
\Delta_{m,n}(p)\left(\left\langle q, q'\right\rangle_{\mathcal{V}}\right) = \left\langle f^m g^n\left(\left\langle q, q'\right\rangle_{\mathcal{V}}\right), p\right\rangle_{\mathcal{V}}\\
= \left\langle\left\langle g f^m g^{n-1}(q), f f^m g^{n-1}\left(q'\right)\right\rangle_{\mathcal{V}}, g^2(p)\right\rangle_{\mathcal{V}}\\
= -\left\langle g^2(p), \left\langle g f^m g^{n-1}(q), f f^m g^{n-1}\left(q'\right)\right\rangle_{\mathcal{V}}\right\rangle_{\mathcal{V}}\\
= \left\langle f^{m+1}g^n(q), \left\langle f^m g^n\left(q'\right), f(p)\right\rangle_{\mathcal{V}}\right\rangle_{\mathcal{V}} + \left\langle f^m g^{n+1}\left(q'\right), \left\langle g(p), f^{m+2}g^{n-2}(q)\right\rangle_{\mathcal{V}}\right\rangle_{\mathcal{V}}\\
= \left\langle f^{m+1}g^n(q), \left\langle f^m g^n\left(q'\right), f(p)\right\rangle_{\mathcal{V}}\right\rangle_{\mathcal{V}} + \left\langle f^m g^{n+1}\left(q'\right), \left\langle g(p), f^{m+2}g^{n-2}(q)\right\rangle_{\mathcal{V}}\right\rangle_{\mathcal{V}}\\
= -(-1)^{m+1}\left\langle f^{m+1}g^n(q), -\left\langle f^m g^n\left(q'\right), p\right\rangle_{\mathcal{V}}\right\rangle_{\mathcal{V}} - (-1)^{m+1}\left\langle -\left\langle p, f^{m+1}g^{n-1}(q)\right\rangle_{\mathcal{V}}, f^{m+1}g^n\left(q'\right)\right\rangle_{\mathcal{V}}\\
= (-1)^{m+1}\left\langle f^{m+1}g^n(q), \Delta_{m,n}(p)\left(q'\right)\right\rangle_{\mathcal{V}} + \left\langle \Delta_{m,n}(p)(q), f^{m+1}g^n\left(q'\right)\right\rangle_{\mathcal{V}}
\end{gathered}$$

Thus, $\Delta_{m,n}(p)$ behaves as an $f^{m+1}g^n$-derivation. We denote the collection of inner $f^m g^n$-derivations as $\mathrm{Inn}_{f^m,g^n}(\mathcal{V})$, expressed as:

$$\mathrm{Inn}_{f^m,g^n}(\mathcal{V}) = \left\{\left\langle f^{m-1}g^n(\cdot), p\right\rangle_{\mathcal{V}} \mid p \in \mathcal{V}, f(p) = p, g(p) = p, (-1)^m = 1\right\}$$

For any $\Delta \in \mathrm{Der}_{f^m,g^n}(\mathcal{V})$ and $\Delta' \in \mathrm{Der}_{f^t,g^r}(\mathcal{V})$, we define their commutator $\langle\Delta, \Delta'\rangle$ as follows:

$$\left\langle \Delta, \Delta'\right\rangle = \Delta \circ \Delta' - \Delta' \circ \Delta.$$

Lemma 3.2. *For any $\Delta \in \mathrm{Der}_{f^m,g^n}(\mathcal{V})$ and $\Delta' \in \mathrm{Der}_{f^t,g^r}(\mathcal{V})$, it follows that:*

$$\left\langle \Delta, \Delta'\right\rangle \in \mathrm{Der}_{f^{m+t},g^{n+r}}(\mathcal{V})$$

Proof. For any $p, q \in \mathcal{V}$, we derive:

$$\begin{aligned}\left\langle \Delta, \Delta' \right\rangle (\langle p, q\rangle_{\mathcal{V}}) &= \Delta \circ \Delta' (\langle p, q\rangle_{\mathcal{V}}) - \Delta' \circ \Delta (\langle p, q\rangle_{\mathcal{V}}) \\ &= (-1)^t \Delta \left(\left\langle \Delta'(p), f^t g^r(q)\right\rangle_{\mathcal{V}} + \left\langle f^t g^r(p), \Delta'(q)\right\rangle_{\mathcal{V}}\right) \\ &\quad -(-1)^m \Delta' \left(\langle \Delta(p), f^m g^n(q)\rangle_{\mathcal{V}} + \langle f^m g^n(p), \Delta(q)\rangle_{\mathcal{V}}\right) \\ &= (-1)^{m+t} \left(\left\langle \Delta \circ \Delta'(p), f^{m+t} g^{n+r}(q)\right\rangle_{\mathcal{V}} + \left\langle f^m g^n \circ \Delta'(p), \Delta \circ f^t g^r(q)\right\rangle_{\mathcal{V}} \right. \\ &\quad \left. + \left\langle \Delta \circ f^t g^r(p), f^m g^n \circ \Delta'(q)\right\rangle_{\mathcal{V}} + \left\langle f^{m+t} \circ \Delta(p), \Delta \circ \Delta'(q)\right\rangle_{\mathcal{V}}\right) \\ &\quad -(-1)^{m+t} \left(\left\langle \Delta' \circ \Delta(p), f^{m+t}(q)\right\rangle_{\mathcal{V}} + \left\langle f^t \circ \Delta(p), \Delta' \circ f^m(q)\right\rangle_{\mathcal{V}}\right) \\ &\quad -(-1)^{m+t} \left(\left\langle \Delta' \circ f^m(p), f^t \circ \Delta(q)\right\rangle_{\mathcal{V}} + \left\langle f^{m+t} g^{n+r}(p), \Delta' \circ \Delta(q)\right\rangle_{\mathcal{V}}\right)\end{aligned}$$

Given that any pair of the maps Δ, Δ', f, g commute, we can establish the following equations:

$$\begin{aligned}\Delta \circ f^t &= f^t \circ \Delta, \\ \Delta' \circ f^m &= f^m \circ \Delta', \\ \Delta \circ g^r &= g^r \circ \Delta, \\ \Delta' \circ g^n &= g^n \circ \Delta.\end{aligned}$$

As a result, we derive

$$\begin{aligned}\left\langle \Delta, \Delta' \right\rangle (\langle p, q\rangle_{\mathcal{V}}) &= (-1)^{m+t} \left(\left\langle \Delta \circ \Delta'(p) - \Delta' \circ \Delta(p), f^{m+t} g^{n+r}(q)\right\rangle_{\mathcal{V}} \right. \\ &\quad \left. + \left\langle f^{m+t} g^{n+r}(p), \Delta \circ \Delta'(p) - \Delta' \circ \Delta(q)\right\rangle_{\mathcal{V}}\right) \\ &= (-1)^{m+t} \left(\left\langle \left\langle \Delta, \Delta' \right\rangle (p), f^{m+t} g^{n+r}(q)\right\rangle_{\mathcal{V}} \right. \\ &\quad \left. + \left\langle f^{m+t} g^{n+r}(p), \left\langle \Delta, \Delta' \right\rangle (q)\right\rangle_{\mathcal{V}}\right).\end{aligned}$$

Additionally, it is evident that

$$\begin{aligned}\left\langle \Delta, \Delta' \right\rangle \circ f &= \Delta \circ \Delta' \circ f - \Delta \circ \Delta' \circ f \\ &= f \circ \Delta \circ \Delta' - f \circ \Delta \circ \Delta' \\ &= f \circ \left\langle \Delta, \Delta' \right\rangle\end{aligned}$$

and

$$\begin{aligned}\left\langle \Delta, \Delta' \right\rangle \circ g &= \Delta \circ \Delta' \circ g - \Delta \circ \Delta' \circ g \\ &= g \circ \Delta \circ \Delta' - g \circ \Delta \circ \Delta' \\ &= g \circ \left\langle \Delta, \Delta' \right\rangle\end{aligned}$$

Consequently, we have $\langle \Delta, \Delta' \rangle \in \operatorname{Der}_{f^{m+t} g^{n+r}}(\mathcal{V})$.
For any integers m and n, we define $\operatorname{Der}(\mathcal{V}) = \bigoplus_{m \geq 0, n \geq 0} \operatorname{Der}_{f^m g^n}(\mathcal{V})$. Clearly, $\operatorname{Der}(\mathcal{V})$ forms a Jacobi-Jordan algebra, characterized by the Jacobi-Jordan bracket defined as:

$$\left\langle \Delta, \Delta' \right\rangle = \Delta \circ \Delta' - \Delta' \circ \Delta.$$

Next, we examine the derivation extension of the regular Bihom-Jacobi-Jordan algebra $(\mathcal{V}, \langle \cdot, \cdot \rangle_{\mathcal{V}}, f, g)$ and explore an application of the $f^0 g^1$-derivation $\operatorname{Der}_{f^0 g^1}(\mathcal{V})$.
Let $\Delta, f, g : \mathcal{V} \to \mathcal{V}$ be any linear mappings, where f and g are inverses. Consider the vector space $\mathcal{V} \oplus \mathcal{V}'\Delta$. We define a symmetric bilinear bracket operation $\langle \cdot, \cdot \rangle_{\Delta}$ on $\mathcal{V} \oplus \mathcal{V}'\Delta$ as follows:

$$\langle p, q \rangle_{\Delta} = \langle p, q \rangle_{\mathcal{V}}, \quad \langle \Delta, p \rangle_{\Delta} = \left\langle f^{-1} g(p), f g^{-1} \Delta \right\rangle_{\Delta} = \Delta(p), \quad \forall p, q \in \mathcal{V}$$

Next, we define two linear mappings $f_{\Delta}, g_{\Delta} : \mathcal{V} \oplus \mathcal{V}'\Delta \to \mathcal{V} \oplus \mathcal{V}'\Delta$ by:

$$f_{\Delta}(p, \Delta) = (f(p), \Delta), \quad g_{\Delta}(p, \Delta) = (g(p), \Delta)$$

The linear maps f and g used in the bracket definition $\langle\cdot,\cdot\rangle_\Delta$ must be multiplicative, meaning:

$$f\circ\langle\Delta,p\rangle_\Delta=\langle f\circ\Delta,f(p)\rangle_\Delta,\quad g\circ\langle\Delta,p\rangle_\Delta=\langle g\circ\Delta,g(p)\rangle_\Delta.$$

□

Thus, we conclude:

$$\langle p,\Delta\rangle_\Delta=\left\langle f^{-1}g\Delta,fg^{-1}(p)\right\rangle_\Delta=f^{-1}g\left\langle\Delta,f^2g^{-2}(p)\right\rangle_\Delta=f^{-1}g\Delta\left(f^2g^{-2}(p)\right)=fg^{-1}\Delta(p)$$

Theorem 3.3. *Using the notations established,* $(\mathcal{V}\oplus\mathcal{V}'\Delta,\langle\cdot,\cdot\rangle_\Delta,f_\Delta,g_\Delta)$ *constitutes a multiplicative Bihom-JacobiJordan algebra if and only if* Δ *serves as a* f^0g^1*-derivation of the multiplicative Bihom-Jacobi-Jordan algebra* $(\mathcal{V},\langle\cdot,\cdot\rangle_\mathcal{V},f,g)$*.*

Proof. For any elements $p,q\in\mathcal{V}$ and $p_1,q_1\in\mathcal{V}'$, we find that:

$$f_\Delta\circ g_\Delta\left(p,p_1\Delta\right)=f_\Delta\left(g(p),p_1\Delta\right)=\left(f\circ g(p),p_1\Delta\right)$$

and

$$g_\Delta\circ f_\Delta\left(p,p_1\Delta\right)=g_\Delta\left(f(p),p_1\Delta\right)=\left(g\circ f(p),p_1\Delta\right)$$

Consequently, we have

$$f_\Delta\circ g_\Delta=g_\Delta\circ f_\Delta\Longleftrightarrow f\circ g=g\circ f$$

Conversely,

$$\begin{aligned}f_\Delta\left\langle\left(p,p_1\Delta\right),\left(q,q_1\Delta\right)\right\rangle&=f_\Delta\left(\langle p,q\rangle_\mathcal{V}+\langle p,q_1\Delta\rangle_\Delta+\langle p_1\Delta,q\rangle_\Delta\right)\\&=f_\Delta\left(\langle p,q\rangle_\mathcal{V}+q_1\circ fg^{-1}(p)+p_1\Delta(q)\right)\\&=f\left(\langle p,q\rangle_\mathcal{V}\right)+q_1f\circ\Delta\circ fg^{-1}(p)+p_1f\circ\Delta(q)\\\left\langle f_\Delta\left(p,p_1\Delta\right),f_\Delta\left(q,q_1\Delta\right)\right\rangle_\Delta&=\left\langle\left(f(p),p_1\Delta\right),\left(f(q),q_1\Delta\right)\right\rangle_\Delta\\&=\langle f(p),f(q)\rangle_\mathcal{V}+\langle f(p),q_1\Delta\rangle_\Delta+\langle p_1\Delta,f(q)\rangle_\Delta\\&=\langle f(p),f(q)\rangle_\mathcal{V}+q_1\circ fg^{-1}(f(p))+p_1\Delta(f(q))\end{aligned}$$

Given that $f\left(\langle p,q\rangle_\mathcal{V}\right)=\langle f(p),f(q)\rangle_\mathcal{V}$,

$$f_\Delta\left\langle\left(p,p_1\Delta\right),\left(q,q_1\Delta\right)\right\rangle_\Delta=\left\langle f_\Delta\left(p,p_1\Delta\right),f_\Delta\left(q,q_1\Delta\right)\right\rangle_\Delta$$

if and only if

$$\Delta\circ f=f\circ\Delta,\quad\Delta\circ g=g\circ\Delta$$

In a similar manner,

$$g_\Delta\left\langle\left(p,p_1\Delta\right),\left(q,q_1\Delta\right)\right\rangle_\Delta=\left\langle g_\Delta\left(p,p_1\Delta\right),g_\Delta\left(q,q_1\Delta\right)\right\rangle_\Delta$$

if and only if

$$\Delta\circ f=f\circ\Delta,\quad\Delta\circ g=g\circ\Delta.$$

Next, we consider

$$\begin{aligned}\left\langle g_\Delta\left(q,q_1\Delta\right),f_\Delta\left(p,p_1\Delta\right)\right\rangle_\Delta&=\left\langle\left(g(q),q_1\Delta\right),\left(f(p),p_1\Delta\right)\right\rangle_\Delta\\&=\langle g(q),f(p)\rangle_\mathcal{V}+\langle g(q),p_1\Delta\rangle_\Delta+\langle q_1\Delta,f(p)\rangle_\Delta\\&=\langle g(q),f(p)\rangle_\mathcal{V}+p_1fg^{-1}\circ\Delta\circ(g(q))+q_1\Delta(f(p))\\&=\langle g(p),f(q)\rangle_\mathcal{V}+p_1fg^{-1}\circ\Delta\circ(g(q))+q_1\Delta(f(p)),\\\left\langle g_\Delta\left(p,p_1\Delta\right),f_\Delta\left(q,q_1\Delta\right)\right\rangle_\Delta&=\left\langle\left(g(p),p_1\Delta\right),\left(f(q),q_1\Delta\right)\right\rangle_\Delta\\&=\langle g(p),f(q)\rangle_\mathcal{V}+\langle g(p),q_1\Delta\rangle_\Delta+\langle p_1\Delta,f(q)\rangle_\Delta\\&=\langle g(p),f(q)\rangle_\mathcal{V}+q_1fg^{-1}\circ\Delta\circ(g(p))+p_1\Delta(f(q)),\end{aligned}$$

therefore

$$\langle g_\Delta (q, q_1\Delta), f_\Delta (p, p_1\Delta)\rangle_\Delta = \langle g_\Delta (p, p_1\Delta), f_\Delta (q, q_1\Delta)\rangle_\Delta$$

if and only if

$$\Delta \circ f = f \circ \Delta, \quad \Delta \circ g = g \circ \Delta.$$

Conversely, we find that

$$\begin{aligned}
&\langle g^2\Delta (p, p_1\Delta), \langle g\Delta (q, q_1\Delta), f (q', n\Delta)\rangle_\Delta\rangle_\Delta + \langle g^2\Delta (q, q_1\Delta), \langle g\Delta (q', n\Delta), f (p, p_1\Delta)\rangle_\Delta\rangle_\Delta \\
&+ \langle g^2\Delta (q', n\Delta), \langle g\Delta (p, p_1\Delta), f (q, q_1\Delta)\rangle_\Delta\rangle_\Delta \\
&= \langle (g^2(p), p_1\Delta), \langle (g(q), q_1\Delta), (f (q'), n\Delta)\rangle_\Delta\rangle_\Delta + \langle (g^2(q), q_1\Delta), \langle (g (q'), n\Delta), (f(p), p_1\Delta)\rangle_\Delta\rangle_\Delta \\
&+ \langle (g^2 (q'), n\Delta), \langle (g(p), p_1\Delta), (f(q), q_1\Delta)\rangle_\Delta\rangle_\Delta \\
&= \langle (g^2(p), p_1\Delta), \langle (g(q), f (q')) + nf \circ \Delta(q) + q_1\Delta \circ f (q')\rangle_\Delta \\
&+ \langle (g^2(q), q_1\Delta), \langle (g (q'), f(p)) + q_1 f \circ \Delta (q') + p_1\Delta \circ f(p)\rangle_\Delta \\
&+ \langle (g^2 (q'), n\Delta), \langle (g(p), f(q)) + p_1 f \circ \Delta(p) + p_1\Delta \circ f(q)\rangle_\Delta \\
&= \langle (g^2(p), p_1\Delta), \langle (g(q), f (q'))\rangle\rangle + \langle (g^2(p), nf \circ \Delta(q)\rangle + \langle (g^2(p), q_1\Delta \circ f (q')\rangle \\
&+ \langle p_1\Delta, \langle (g(q), f (q'))\rangle\rangle + \langle p_1\Delta, nf \circ \Delta(q)\rangle + \langle p_1\Delta, q_1\Delta \circ f (q')\rangle \\
&+ \langle g^2(q), \langle (g (q'), f(p)\rangle\rangle + \langle (g^2(q), pf \circ \Delta (q')\rangle + \langle (g^2(q), n\Delta \circ f(p)\rangle \\
&+ \langle q_1\Delta, \langle (g (q'), f(p)\rangle\rangle + \langle (q_1\Delta, pf \circ \Delta (q')\rangle + \langle (q_1\Delta, n\Delta \circ f(p)\rangle \\
&+ \langle g^2 (q'), \langle (g(p), f(q)\rangle\rangle + \langle (g^2 (q'), q_1 f \circ \Delta(p)\rangle + \langle (g^2 (q'), p_1\Delta \circ f(q)\rangle \\
&+ \langle n\Delta, \langle (g(p), f (q')\rangle\rangle + \langle (n\Delta, q_1 f \circ \Delta(p)\rangle + \langle (n\Delta, p_1\Delta \circ f(q)\rangle \\
&= \langle (g^2(p), \langle g(q), f (q')\rangle\rangle + \langle p_1\Delta, nf \circ \Delta(q)\rangle + \langle (g^2(p), q_1\Delta \circ f (q')\rangle \\
&+ \langle p_1\Delta, \langle g(q), f (q')\rangle\rangle + p_1 nf \circ \Delta^2(q) + p_1 q_1\Delta^2 \circ f (q') \\
&+ \langle g^2(q), \langle g (q'), f(p)\rangle\rangle + \langle (g^2(q), pf \circ \Delta (q')\rangle + \langle (g^2(q), n\Delta \circ f(p)\rangle \\
&+ \langle q_1\Delta, \langle g (q'), f(p)\rangle\rangle + p_1 nf \circ \Delta^2 (q') + nq_1\Delta^2 \circ f (q') \\
&+ \langle g^2 (q'), \langle g(p), f(q)\rangle\rangle + \langle (g^2 (q'), q_1 f \circ \Delta(p)\rangle + \langle (g^2 (q'), p_1\Delta \circ f(q)\rangle \\
&+ \langle n\Delta, \langle g(p), f (q')\rangle\rangle + \langle (n\Delta, q_1 f \circ \Delta(p)\rangle + \langle (n\Delta, p_1\Delta \circ f(q)\rangle.
\end{aligned}$$

If Δ is an f^0g^1-derivation of the multiplicative Bihom-Jacobi-Jordan algebra $(\mathcal{V}, \langle \cdot, \cdot\rangle_\mathcal{V}, f, g)$, then

$$\langle p_1\Delta, \langle g(q), f (q')\rangle_\Delta = p_1\Delta \langle g(q), f (q')\rangle$$

$$\begin{aligned}
&= -\langle p_1\Delta \circ g(q), f^0g^1 (f (q'))\rangle + \langle f^0g^2(q), p_1\Delta \circ f (q')\rangle \\
&= \langle\langle f^0g^2 (q'), p_1\Delta \circ f(q)\rangle + \langle f^0g^2(q), p_1\Delta \circ f (q')\rangle \\
&= \langle g^2 (q'), p_1\Delta \circ f(q)\rangle + \langle g^2(q), p_1\Delta \circ f (q')\rangle.
\end{aligned}$$

In a similar manner,

$$\langle q_1\Delta, \langle g (q'), f(p)\rangle_\Delta = \langle g^2(p), q_1\Delta \circ f (q')\rangle + \langle g^2 (q'), q_1 f \circ \Delta(p)\rangle$$

Also,

$$\langle n\Delta, \langle g(p), f(q)\rangle_\Delta = \langle g^2(q), n\Delta \circ f(p)\rangle + \langle g^2 (q'), nf \circ \Delta (q')\rangle$$

Thus, the Bihom-Jacobi identity holds true if and only if Δ is an f^0g^1-derivation of $(\mathcal{V}, \langle \cdot, \cdot\rangle_\mathcal{V}, f, g)$. Therefore, $(\mathcal{V} \oplus \mathcal{V}'\Delta, \langle \cdot, \cdot\rangle_\Delta, f_\Delta, g_\Delta)$ constitutes a multiplicative Bihom-Jacobi-Jordan algebra if and only if Δ is considered as an f^0g^1-derivation of $(\mathcal{V}, \langle \cdot, \cdot\rangle_\mathcal{V}, f, g)$. □

4. Derivations of BiHom-Jacobi-Jordan superalgebras

This section introduces the derivations of a BiHom-Jacobi-Jordan superalgebra $\mathcal{V}$, establishing that the set of such derivations inherits a BiHom-Jacobi-Jordan superalgebra structure.

Let $\mathcal{V} = (\mathcal{V}, \langle \cdot, \cdot \rangle, f, g)$ represent a BiHom-Jacobi-Jordan superalgebra. For any nonnegative integer m, the notation f^m refers to the m-fold composition of the map f, meaning:

$$f^m = f \circ \cdots \circ f \qquad \text{(applied } m \text{ times)}.$$

By convention, $f^{-1} = 0, f^0 = \mathrm{id}$, and $f^1 = f$. The same notation applies to g^m.

Definition 4.1. For any integer $m \geq -1$, a homogeneous linear map $\Delta : \mathcal{V} \to \mathcal{V}$ of degree $|\Delta|$ is called a g^m-derivation of the BiHom-Jacobi-Jordan superalgebra $(\mathcal{V}, \langle \cdot, \cdot \rangle, f, g)$ if it satisfies the following conditions:

$$\begin{gathered} \Delta \circ f = f \circ \Delta, \quad \Delta \circ g = g \circ \Delta, \\ \Delta \langle p, q \rangle = \langle \Delta(p), g^m(q) \rangle + (-1)^{|p||\Delta|} \langle g^m(p), \Delta(q) \rangle , \end{gathered}$$

where p and q are homogeneous elements of $\mathcal{V}$.

We denote the space of all g^m-derivations as $\mathrm{Der}_{g^m}(\mathcal{V}) = (\mathrm{Der}_{g^m}(\mathcal{V}))_k \oplus (\mathrm{Der}_{g^m}(\mathcal{V}))_l$, and define the total set of derivations as $\mathrm{Der}(\mathcal{V}) = \bigoplus_{m \geq -1} \mathrm{Der}_{g^m}(\mathcal{V})$. The maps $\tilde{f}$ and $\tilde{g}$ are endomorphisms on $\mathrm{Der}(\mathcal{V})$, defined by:

$$\tilde{f}(\Delta) = f \circ \Delta, \quad \tilde{g}(\Delta) = g \circ \Delta$$

For two derivations $\Delta, \Delta' \in \mathrm{Der}(\mathcal{V})$, we define their commutator $\langle \Delta, \Delta' \rangle$ by:

$$\left\langle \Delta, \Delta' \right\rangle = \Delta \circ \Delta' + (-1)^{|\Delta||\Delta'|} \Delta' \circ \Delta.$$

Lemma 4.2. *Suppose $(\mathcal{V}, \langle \cdot, \cdot \rangle, f, g)$ is a BiHom-Jacobi-Jordan superalgebra. If $\Delta \in (\mathrm{Der}_{g^m}(\mathcal{V}))_k$ and $\Delta' \in \left(\mathrm{Der}_{g^t}(\mathcal{V})\right)_l$, where $m+t \geq -1$ and $(k, l) \in \mathbb{Z}_2 \times \mathbb{Z}_2$, then their commutator $\langle \Delta, \Delta' \rangle$ lies in $\left(\mathrm{Der}_{g^{m+t}}(\mathcal{V})\right)_l$.*

Proof. Let $p, q \in \mathcal{V}$, we obtain

$$\begin{aligned} &\left\langle \Delta, \Delta' \right\rangle (\langle p, q \rangle) = \left(\Delta \circ \Delta' + (-1)^{|\Delta||\Delta'|} \Delta' \circ \Delta \right) (\langle p, q \rangle) \\ &= \Delta \left(\left\langle \Delta'(p), g^t(q) \right\rangle + (-1)^{|p||\Delta|} \left\langle g^t(p), \Delta'(q) \right\rangle \right) \\ &+ (-1)^{|\Delta||\Delta'|} \Delta' \left(\langle \Delta(p), g^m(q) \rangle + (-1)^{|p||\Delta|} \langle g^m(p), \Delta(q) \rangle \right) \\ &= \left\langle \Delta\Delta'(p), g^{t+m}(q) \right\rangle \\ &+ (-1)^{|\Delta||\Delta'(p)|} \left\langle \Delta' \left(g^m(p) \right), \Delta \left(g^t(q) \right) \right\rangle \\ &+ (-1)^{|p||\Delta'|} \left(\left\langle \Delta \left(g^t(p) \right), \Delta \left(g^m(q) \right) \right\rangle + (-1)^{|p||\Delta|} \left\langle g^{t+m}(p), \Delta\Delta'(q) \right\rangle \right) \\ &+ (-1)^{|\Delta||\Delta'|} \left(\left\langle \Delta'\Delta(p), g^{t+m}(q) \right\rangle + (-1)^{|\Delta'||\Delta(p)|} \left\langle \Delta \left(g^t(p) \right), \Delta' \left(g^m(q) \right) \right\rangle \right) \\ &+ (-1)^{|\Delta|(|\Delta'|+|p|)} \left(\left\langle \Delta' \left(g^m(p) \right), \Delta \left(g^t(q) \right) \right\rangle + (-1)^{|p||\Delta'|} \left\langle g^{t+m}(p), \Delta'\Delta(q) \right\rangle \right) \\ &= \left\langle \Delta\Delta'(p) + (-1)^{|\Delta||\Delta'|} \Delta'\Delta(p), g^{t+m}(q) \right\rangle \\ &+ (-1)^{|p|(|\Delta|+|\Delta'|)} \left\langle g^{t+m}(p), \left(\Delta\Delta' + (-1)^{|\Delta||\Delta'|} \Delta'\Delta \right) (q) \right\rangle \\ &= \left\langle \left\langle \Delta, \Delta' \right\rangle (p), g^{t+m}(q) \right\rangle \\ &+ (-1)^{|p||\langle \Delta, \Delta' \rangle|} \left\langle g^{t+m}(p), \left\langle \Delta, \Delta' \right\rangle (q) \right\rangle . \end{aligned}$$

It is straightforward to verify that

$$\begin{aligned} \left\langle \Delta, \Delta' \right\rangle \circ f = f \circ \left\langle \Delta, \Delta' \right\rangle , \\ \left\langle \Delta, \Delta' \right\rangle \circ g = g \circ \left\langle \Delta, \Delta' \right\rangle , \end{aligned}$$

which results in

$$\left\langle \Delta, \Delta' \right\rangle \in \mathrm{Der}_{f^{m+t}}(\mathcal{V}).$$

□

Remark 4.3. Clearly, we have

$$\mathrm{Der}_{g^{-1}}(\mathcal{V}) = \{\Delta \in \mathrm{End}(\mathcal{V}) \mid \Delta \circ f = f \circ \Delta, \Delta \circ g = g \circ \Delta, \Delta\langle p, q\rangle = 0, \forall p, q \in \mathcal{V}\}$$

Thus, for any $\Delta, \Delta' \in \mathrm{Der}_{g^{-1}}(\mathcal{V})$, it follows that

$$\left\langle \Delta, \Delta' \right\rangle \in \mathrm{Der}_{g^{-1}}(\mathcal{V})$$

Proposition 4.4. *Consider* $(\mathcal{V}, \langle\cdot,\cdot\rangle, f, g)$ *as a BiHom-Jacobi-Jordan superalgebra. Then* $(\mathrm{Der}(\mathcal{V}), \langle\cdot,\cdot\rangle, f, g)$ *is also a BiHom-Jacobi-Jordan superalgebra.*

Proof. We demonstrate that the bracket $\langle\cdot,\cdot\rangle$ defined on $\mathrm{Der}(\mathcal{V})$ meets the criteria outlined in the definition of a BiHom-Jacobi-Jordan superalgebra. Let $\Delta \in \mathrm{Der}_{f^m}(\mathcal{V})_k, \Delta' \in \mathrm{Der}_{f^t}(\mathcal{V})_l, \Delta'' \in \mathrm{Der}_{f^r}(\mathcal{V})_n$, and $p \in \mathcal{V}$. We then have

$$(\tilde{f} \circ \tilde{g})(\Delta) = \Delta \circ f \circ g = \Delta \circ g \circ f = (\tilde{g} \circ \tilde{f})(\Delta).$$

□

This establishes commutativity, and similar reasoning applies for the homomorphic property. Regarding BiHom super-symmetry, we find

$$\begin{aligned}
\left\langle \tilde{g}(\Delta), \tilde{f}\left(\Delta'\right) \right\rangle &= \langle \Delta \circ g, \Delta' \circ f\rangle \\
&= (\Delta \circ g) \circ \left(\Delta' \circ f\right) + (-1)^{|\Delta||\Delta'|}\left(\Delta' \circ f\right) \circ (\Delta \circ g) \\
&= \left(\Delta \circ \Delta' + (-1)^{|\Delta||\Delta'|}\Delta' \circ \Delta\right) \circ (fg) \\
&= (-1)^{|\Delta||\Delta'|}\left(\Delta' \circ \Delta + (-1)^{|\Delta||\Delta'|}\Delta \circ \Delta'\right) \circ (fg) \\
&= (-1)^{|\Delta|\Delta'|}\left\langle \tilde{g}\left(\Delta'\right), \tilde{f}(\Delta)\right\rangle.
\end{aligned}$$

For the BiHom super-Jacobi identity, we compute

$$(-1)^{|\Delta||\Delta''|}\left\langle \tilde{g}^2(\Delta), \left\langle \tilde{g}\left(\Delta'\right), \tilde{f}\left(\Delta''\right)\right\rangle\right\rangle$$

$$\begin{aligned}
&= (-1)^{|\Delta||\Delta''|}\left\langle \Delta \circ g^2, \langle \Delta' \circ g, \Delta'' \circ f\rangle\right\rangle \\
&= (-1)^{|\Delta||\Delta''|}\left\langle \Delta \circ g^2, \left(\Delta' \circ \Delta''\right) \circ (gf) + (-1)^{|\Delta'||\Delta''|}\left(\Delta'' \circ \Delta'\right) \circ (gf)\right\rangle \\
&= (-1)^{|\Delta||\Delta''|}\left\{\left(\Delta \circ \left(\Delta' \circ \Delta''\right)\right) + (-1)^{|\Delta||\Delta'|}\left(\left(\Delta' \circ \Delta''\right) \circ \Delta\right)\right\} \circ \left(g^3 f\right) \\
&+(-1)^{|\Delta''|(|\Delta|+|\Delta'|)}\left\{\left(\Delta \circ \left(\Delta'' \circ \Delta'\right)\right) + (-1)^{|\Delta'|(|\Delta|+|\Delta''|)}\left(\left(\Delta'' \circ \Delta'\right) \circ \Delta\right)\right\} \circ \left(g^3 f\right).
\end{aligned}$$

Thus, one can verify that

$$\circlearrowleft_{\Delta, \Delta', \Delta''} \quad (-1)^{|\Delta||\Delta''|}\left\langle \tilde{g}^2(\Delta), \left\langle \tilde{g}\left(\Delta'\right), \tilde{f}\left(\Delta''\right)\right\rangle\right\rangle = 0, \text{ as required.}$$

This concludes the proof.
For every homogeneous element $h \in \mathcal{V}$ that meets the condition $f(h) = h = g(h)$, we introduce $ad_m(h) \in \mathrm{End}(\mathcal{V})$ defined by:

$$ad_m(h)(p) = \langle h, g^m(p)\rangle, \forall p \in \mathcal{V}.$$

Proposition 4.5. *Suppose $(\mathcal{V}, \langle\cdot,\cdot\rangle, f, g)$ is a BiHom-Jacobi-Jordan superalgebra, and suppose h is a homogeneous element in $\mathcal{V}$. If the maps f and g are bijective, therefore $ad_m(h)$ is a g^{m+1}-derivation, referred to as an inner g^{m+1}-derivation.*

Proof. For any homogeneous elements p and q in $\mathcal{V}$, we first observe that:

$$\begin{aligned}
ad_m(h)\langle p,q\rangle &= \langle h, g^m\langle p,q\rangle\rangle = \left\langle g^2(h), \langle g^m(p), g^m(q)\rangle\right\rangle \\
&= (-1)^{|h||q|}(-1)^{|p|h|}\left\langle g^{m+1}(p), \left\langle g^{m+1}f^{-1}(q), f(h)\right\rangle\right\rangle \\
&\quad +(-1)^{|h||q|}(-1)^{|p||q|}\left\langle g^{m+2}f^{-1}(q), \left\langle g(h), fg^{m-1}(p)\right\rangle\right\rangle \\
&= (-1)^{|h||q|}(-1)^{|p||h|}\left\langle g^{m+1}(p), \left\langle g^{m+1}f^{-1}(q), f(h)\right\rangle\right\rangle \\
&\quad +(-1)^{|h||q|}(-1)^{|p||q|}\left\langle g^{m+2}f^{-1}(q), \left\langle h, fg^{m-1}(p)\right\rangle\right\rangle.
\end{aligned}$$

Next, we calculate:

$$\begin{aligned}
\left\langle ad_m(h)(p), g^{m+1}(q)\right\rangle &= \langle h, g^m(p)\rangle = \left\langle g\left\langle h, g^{m-1}(p)\right\rangle, g^{m+1}(q)\right\rangle \\
&= (-1)^{(|h|+|p|)|q|}\left\langle gg^{m+1}f^{-1}(q), f\left\langle h, g^{m-1}(p)\right\rangle\right\rangle \\
&= (-1)^{(|h|+|p|)|q|}\left\langle g^{m+2}f^{-1}(q), \left\langle h, fg^{m-1}(p)\right\rangle\right\rangle.
\end{aligned}$$

Similarly, we also have:

$$\begin{aligned}
\left\langle g^{m+1}(p), ad_m(h)(q)\right\rangle &= \left\langle g^{m+1}(p), h, g^m(q)\right\rangle = \left\langle g^{m+1}(p), \left\langle g(h), fg^m f^{-1}(q)\right\rangle\right\rangle \\
&= (-1)^{|h||q|}\left\langle g^{m+1}(p), \langle gg^m(q), f(h)\rangle\right\rangle \\
&= (-1)^{|h||q|}\left\langle g^{m+1}(p), \left\langle g^{m+1}f^{-1}(q), f(h)\right\rangle\right\rangle.
\end{aligned}$$

As a result, we conclude that:

$$ad_m(h)\langle p,q\rangle = \left\langle ad_m(h)(p), g^{m+1}(q)\right\rangle + (-1)^{|p||h|}\left\langle g^{m+1}(p), ad_m(h)(q)\right\rangle$$

which completes the proof. □

5. Constructing Solutions of the BJJYBE from BiHom-Jacobi-Jordan superalgebras

In this section, we introduce a method for constructing the BiHom-Jacobi-Jordan-Yang-Baxter Equation (BJJYBE) from BiHom-Jacobi-Jordan superalgebras. It is well known that for any representation of a Jacobi-Jordan algebra, one can form a semidirect product.

Proposition 5.1. *Let $(\mathcal{V}, \langle\cdot,\cdot\rangle, f, g)$ be a regular BiHom-Jacobi-Jordan superalgebra, and $(\mathcal{W}, \theta, f_{\mathcal{W}}, g_{\mathcal{W}})$ a representation of $\mathcal{V}$, where $\theta : \mathcal{V}\otimes\mathcal{W} \to \mathcal{W}, p\otimes u \mapsto p\cdot u$. Then, the semidirect product $\mathcal{V}\ltimes\mathcal{W} := (\mathcal{V}\oplus\mathcal{W}, \langle\cdot,\cdot\rangle_\theta, f+f_{\mathcal{W}}, g+g_{\mathcal{W}})$ forms a BiHom-Jacobi-Jordan superalgebra. The maps $f+f_{\mathcal{W}}$ and $g+g_{\mathcal{W}} : \mathcal{V}\oplus\mathcal{W} \to \mathcal{V}\oplus\mathcal{W}$ are defined as $(f+f_{\mathcal{W}})(p,u) = (f(p), f_{\mathcal{W}}(u))$ and $(g+g_{\mathcal{W}})(p,u) = (g(p), g_{\mathcal{W}}(u))$ for all $p\in\mathcal{V}$ and $u\in\mathcal{W}$. The bilinear map $\langle\cdot,\cdot\rangle_\theta$ is given by*

$$\langle (p,u),(q,v)\rangle_\theta = \Big(\langle p,q\rangle, p\cdot v + (-1)^{|p||q|}f^{-1}g(q)\cdot f_{\mathcal{W}}g_{\mathcal{W}}^{-1}(u)\Big)$$

where $p,q\in\mathcal{V}$ and $u,v\in\mathcal{W}$.

Theorem 5.2. *Let $(\mathcal{V}, \langle\cdot,\cdot\rangle, f, g)$ be a regular BiHom-Jacobi-Jordan superalgebra. Define $\mathcal{V}' = k\oplus\mathcal{V}$, with $f(\zeta,p) = (\zeta, f(p))$ and $g(\zeta,p) = (\zeta, g(p))$ for all $(\zeta,p)\in\mathcal{V}'$. Introduce an even bilinear map $J : \mathcal{V}'\otimes\mathcal{V}' \to \mathcal{V}'\otimes\mathcal{V}'$ defined by*

$$J((\zeta,p)\otimes(\eta,q)) = (\eta, g(q))\otimes(\zeta, f(p)) + (1,0)\otimes\Big(0, (-1)^{|p||q|}\langle p,q\rangle\Big)$$

Then J solves the BJJYBE:

$$\begin{aligned}&(fg\otimes(g\otimes f)J)\circ((g\otimes f)J\otimes fg)\circ(fg\otimes(g\otimes f)J)\\=&((g\otimes f)J\otimes fg)\circ(fg\otimes(g\otimes f)J)\circ((g\otimes f)J\circ fg)\end{aligned}$$

for the BiHom-Jacobi-Jordan superalgebra $(\mathcal{V}', f, g)$.

Proof. We begin by proving that J is compatible with f and g, i.e., $(f\otimes f)\circ J = J\circ(f\otimes f)$ and $(g\otimes g)\circ J = J\circ(g\otimes g)$. To establish this, consider $(\zeta,p),(\eta,q)\in\mathcal{V}'$ and compute:

$$\begin{aligned}(f\otimes f)\circ J((\zeta,p)\otimes(\eta,q)) &= (f\otimes f)((\eta,g(q))\otimes(\zeta,f(p))) + (1,0)\otimes\Big(0,(-1)^{|p||q|}\langle p,q\rangle\Big)\\&=(\eta,fg(q))\otimes\big(\zeta,f^2(p)\big)+(1,0)\otimes\Big(0,(-1)^{|p||q|}f(\langle p,q\rangle)\Big),\\J\circ(f\otimes f)((\zeta,p)\otimes(\eta,q)) &= J((\zeta,f(p))\otimes(\eta,f(q)))\\&=(\eta,fg(q))\otimes\big(\zeta,f^2(p)\big)+(1,0)\otimes\Big(0,(-1)^{|p||q|}f(\langle p,q\rangle)\Big).\end{aligned}$$

Thus, we have $(f\otimes f)\circ J = J\circ(f\otimes f)$. Correspondingly, we can show that $(g\otimes g)\circ J = J\circ\,(g\otimes g)$. Next, we confirm that J satisfies the BJJYBE. Specifically, for any $(\zeta,p),(\eta,q),(\xi,r)\in\mathcal{V}'$, we calculate...

$$\begin{aligned}&((g\otimes f)J\otimes fg)\circ(fg\otimes(g\otimes f)J)\circ((g\otimes f)J\circ fg)\\&((\zeta,p)\otimes(\eta,q)\otimes(\xi,r)) = ((g\otimes f)J\otimes fg)\circ(fg\otimes(g\otimes f)J)\\&(g\otimes f)\Big\{(\eta,g(q))\otimes(\zeta,f(p))+(1,0)\otimes\Big(0,(-1)^{|p||q|}f\langle p,q\rangle\Big)\Big\}\otimes(\xi,fg(r)) = ((g\otimes f)J\otimes fg)\circ(fg\otimes(g\otimes f)J)\\&\Big\{\big(\eta,g^2(q)\big)\otimes\big(\zeta,f^2(p)\big)\otimes(\xi,fg(r))+(1,0)\otimes\Big(0,(-1)^{|p||q|}\langle f(p),f(q)\rangle\Big)\otimes(\xi,fg(r))\Big\}\end{aligned}$$

$$\begin{aligned}&=((g\otimes f)J\otimes fg)\big\{\big(\eta,fg^3(q)\big)\otimes(g\otimes f)\big(\big(\xi,fg^2(r)\big)\otimes\big(\zeta,f^3(p)\big)\big)\\&+\big(\eta,fg^3(q)\big)\otimes(g\otimes f)\Big(\Big(0,(-1)^{|p|r|}\left\langle f^2(p),fg(r)\right\rangle\Big)\Big)\\&+(1,0)\otimes(g\otimes f)\Big(\big(\xi,fg^2(r)\big)\otimes\Big(0,(-1)^{|p||q|}\left\langle f^2(p),f^2(q)\right\rangle\Big)\Big)\\&+(1,0)\otimes(g\otimes f)\Big((1,0)\otimes\Big(0,(-1)^{|p||q|+|q||r|+|p|r|}\langle f(p),f(q),fg(r)\rangle\Big)\Big)\Big\}\end{aligned}$$

$$\begin{aligned}&=((g\otimes f)J\otimes fg)\big\{\big(\eta,fg^3(q)\big)\otimes\big(\xi,fg^3(r)\big)\otimes\big(\zeta,f^4(p)\big)\\&+\big(\eta,fg^3(q)\big)\otimes(1,0)\otimes\Big(0,(-1)^{|p|r|}\left\langle f^3(p),f^2g(r)\right\rangle\Big)\\&+(1,0)\otimes\big(\xi,fg^3(r)\big)\otimes\Big(0,(-1)^{|p||q|}\left\langle f^3(p),f^3(q)\right\rangle\Big)\\&+(1,0)\otimes(1,0)\otimes\Big(0,(-1)^{|p||q|+|q||r|+|p||r|}\left\langle f^2(p),f^2(q),f^2g(r)\right\rangle\Big)\Big\}\end{aligned}$$

$$\begin{aligned}&=\big(\xi,fg^5(r)\big)\otimes\big(\eta,f^3g^3(q)\big)\otimes\big(\zeta,f^5g(p)\big)\\&+(1,0)\otimes\Big(0,(-1)^{|q||r|}\left\langle f^2g^3(q),f^2g^3(r)\right\rangle\Big)\otimes\big(\zeta,f^5g(p)\big)\\&+(1,0)\otimes\big(\eta,f^3g^3(q)\big)\otimes\Big(0,(-1)^{|p|r|}\left\langle f^4g(p),f^3g^2(r)\right\rangle\Big)\\&+\big(\xi,fg^5(r)\big)\otimes(1,0)\otimes\Big(0,(-1)^{|p||q|}\left\langle f^4g(p),f^4g(q)\right\rangle\Big)\\&+(1,0)\otimes(1,0)\otimes\Big(0,(-1)^{|p||q|+|q|r|+|p|r|}\left\langle f^3g(p),f^3g(q),f^3g^2(r)\right\rangle\Big).\end{aligned}$$

Equivalently, we obtain:

$$(fg \otimes (g \otimes f)J) \circ ((g \otimes f)J \otimes fg) \circ (fg \otimes (g \otimes f)J)$$

$$\begin{aligned} ((\zeta, p) \otimes (\eta, q) \otimes (\xi, r)) &= \big(\xi, fg^5(r)\big) \otimes \big(\eta, f^3g^3(q)\big) \otimes \big(\zeta, f^5g(p)\big) \\ &+ \big(\xi, fg^5(r)\big) \otimes (1,0) \otimes \Big(0, (-1)^{|p||q|} \left\langle f^4g(p), f^4g(q)\right\rangle\Big) . \\ + (1,0) \otimes \big(\eta, f^3g^3(q)\big) &\otimes \Big(0, (-1)^{|p|r|} \left\langle f^4g(p), f^3g^2(r)\right\rangle\Big) \\ + (1,0) \otimes (1,0) &\otimes \Big(0, (-1)^{|p|q||||||r|} \left\langle \left\langle f^3g(p), f^2g^2(r)\right\rangle , f^4g(q)\right\rangle\Big) \\ + (1,0) \otimes (1,0) &\otimes \Big(0, (-1)^{|p|q|+|p||r|}| + |q||r \mid f^4g(p), \left\langle f^3g(q), f^3g(r)\right\rangle\Big) \\ + (1,0) &\otimes \Big(0, (-1)^{qq|||l|} \left\langle f^2g^3(q), f^2g^3(r)\right\rangle\Big) \otimes \big(\zeta, f^5g(p)\big) . \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} &(1,0) \otimes (1,0) \otimes \Big(0, (-1)^{|p||q|+|q||r|+|p||r|} \left\langle \left\langle f^3g(p), f^3g(q)\right\rangle , f^3g^2(r)\right\rangle\Big) \\ &= (1,0) \otimes (1,0) \otimes \Big(0, (-1)^{|p||q|+|p||r|} \left\langle \left\langle f^3g(p), f^2g^2(r)\right\rangle , f^4g(q)\right\rangle\Big) \\ &+ (1,0) \otimes (1,0) \otimes \Big(0, (-1)^{|p||q|+|p|r|+|q||r|} \left\langle f^4g(p), \left\langle f^3g(q), f^3g(r)\right\rangle\right\rangle\Big) . \end{aligned}$$

Thus, it is enough to confirm the following equality

$$(-1)^{|q|r|} \left\langle \left\langle f^3g(p), f^2g^2(r)\right\rangle , f^4g(q)\right\rangle + \left\langle f^4g(p), \left\langle f^3g(q), f^3g(r)\right\rangle\right\rangle = \left\langle \left\langle f^3g(p), f^3g(q)\right\rangle , f^3g^2(r)\right\rangle .$$

Applying the symmetry in $\mathcal{V}$, we have

$$\left\langle \left\langle f^3g(p), f^3g(q)\right\rangle , f^3g^2(r)\right\rangle = (-1)^{|q||r|+|p|r|} \left\langle f^2g^3(r), \left\langle f^4(p), f^4(q)\right\rangle\right\rangle$$

and

$$\begin{aligned} \left\langle \left\langle f^3g(p), f^2g^2(r)\right\rangle , f^4g(q)\right\rangle &= (-1)^{|q||p|+|q||r|} \left\langle f^3g^2(q), \left\langle f^4(p), f^3g(r)\right\rangle\right\rangle \\ &= (-1)^{|p||q|+|q||r|+|p|r|} \left\langle f^3g^2(q), \left\langle f^2g^2(r), f^5g^{-1}(p)\right\rangle\right\rangle \end{aligned}$$

Therefore, we find

$$\begin{aligned} &(-1)^{|q||r|} \left\langle \left\langle f^3g(p), f^2g^2(r)\right\rangle , f^4g(q)\right\rangle + \left\langle f^4g(p), \left\langle f^3g(q), f^3g(r)\right\rangle\right\rangle + \left\langle \left\langle f^3g(p), f^3g(q)\right\rangle , f^3g^2(r)\right\rangle \\ &= (-1)^{|p||q|+|p||r|} \left\langle f^3g^2(q), \left\langle f^2g^2(r), f^5g^{-1}(p)\right\rangle\right\rangle + \left\langle f^4g(p), \left\langle f^3g(q), f^3g(r)\right\rangle\right\rangle \\ &+ (-1)^{|q||r|+|p|r|} \left\langle f^2g^3(r), \left\langle f^4(p), f^4(q)\right\rangle\right\rangle \\ &= (-1)^{|p||r|} \Big\{ (-1)^{|p|q|} \left\langle f^3g^2(q), \left\langle f^2g^2(r), f^5g^{-1}(p)\right\rangle\right\rangle + (-1)^{|p|r|} \left\langle f^4g(p), \left\langle f^3g(q), f^3g(r)\right\rangle\right\rangle \\ &+ (-1)^{|q||r|} \left\langle f^2g^3(r), \left\langle f^4(p), f^4(q)\right\rangle\right\rangle \Big\} \\ &= (-1)^{|\hat{p}||\hat{r}|} \Big\{ (-1)^{|\hat{p}|\hat{q}|} \left\langle g^2(\hat{q}), \langle g(\hat{r}), f(\hat{p})\rangle\right\rangle + (-1)^{|\hat{p}|\hat{r}|} \left\langle g^2(\hat{p}), \langle g(\hat{q}), f(\hat{r})\rangle\right\rangle \\ &+ (-1)^{|\hat{q}|\hat{r}|} \left\langle g^2(\hat{r}), \langle g(\hat{p}), f(\hat{q})\rangle\right\rangle \Big\} = 0, \end{aligned}$$

as intended, where $\hat{p} = f^4g^{-1}(p), \hat{q} = f^3(q), \hat{r} = f^2g(r)$. Thus, the proof is finalized. □

Corollary 5.3. *We establish that if J is a solution to the BJJYBE, then it is not necessarily invertible.*

Proof. We claim that J^{-1} is expressed as follows:

$$J^{-1}((\zeta,p)\otimes(\eta,q)) = \big(\eta, f^{-1}(q)\big)\otimes\big(\zeta, g^{-1}(p)\big) + \big(0, \big\langle f^{-2}(p), f^{-1}g^{-1}(q)\big\rangle\big)\otimes(1,0),$$

for all $(\zeta,p),(\eta,q)\in\mathcal{V}'$. Additionally, J^{-1} is not necessarily a solution to the BJJYBE for the BiHom-Jacobi-Jordan superalgebra $(\mathcal{V}', f, g)$. □

To demonstrate that $J\circ J^{-1}\neq \mathrm{id}_{\mathcal{V}'\otimes\mathcal{V}'}$, consider arbitrary elements (ζ,p) and (η,q) in $\mathcal{V}'$:
$\big(J\circ J^{-1}\big)((\zeta,p)\otimes(\eta,q)) = J\left(\big(\eta, f^{-1}(q)\big)\otimes\big(\zeta, g^{-1}(p)\big)\right) + J\left(\big(0,\big\langle f^{-2}(p), f^{-1}g^{-1}(q)\big\rangle\big)\otimes(1,0)\right).$

$$= (\zeta,p)\otimes(\eta,q) + (1,0)\otimes\Big(0,(-1)^{|p||q|}\left\langle f^{-1}(q), g^{-1}(p)\right\rangle\Big)$$

$$+(1,0)\otimes\big(0,\big\langle f^{-1}(p), g^{-1}(q)\big\rangle\big) + (1,0)\otimes\big(0,(-1)^{(|p|+|q|)\cdot 0}\left\langle\left\langle f^{-2}(q), f^{-1}g^{-1}(p)\right\rangle, 0\right\rangle\big).$$

$$\begin{aligned}&= (\zeta,p)\otimes(\eta,q) + (1,0)\otimes\Big(0,(-1)^{|p||q|}\left\langle f^{-1}(q), g^{-1}(p)\right\rangle\Big)\\ &\quad +(1,0)\otimes\Big(0,(-1)^{|p||q|}\left\langle f^{-1}(q), g^{-1}(p)\right\rangle\Big)\\ &= (\zeta,p)\otimes(\eta,q) + 2\Big((1,0)\otimes\Big(0,(-1)^{|p||q|}\left\langle f^{-1}(q), g^{-1}(p)\right\rangle\Big)\Big)\\ &\neq (\zeta,p)\otimes(\eta,q)\end{aligned}$$

This confirms that $J\circ J^{-1}\neq \mathrm{id}_{\mathcal{V}'\otimes\mathcal{V}'}$. A similar verification shows that $J^{-1}\circ J\neq \mathrm{id}_{\mathcal{V}'\otimes\mathcal{V}'}$, establishing that J^{-1} is not necessarily the inverse of J. Furthermore, J^{-1} is not necessarily a solution to the BJJYBE for $(\mathcal{V}', f, g)$, which follows from a straightforward but lengthy computation.

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