Journal of Prime Research in Mathematics Vol. **3***(2007), 169-177*

ANALYTICAL SOLUTIONS TO BRAJINSKII'S EQUATIONS IN ONE DIMENSION BY USING LAPLACE TRANSFORM TECHNIQUE

S.N.HOSSEINIMOTLAGH*,M.ROOSTAIE **,H.KAZEMIFARD***

Abstract. Brajinskii's equations are the fundamental relations governing the behavior of the plasma produced during a fusion reaction, especially ICF plasma. These equations contains six partial differential coupled together. In this paper we have tried to give analytical solutions to these equations using a one dimensional method. Laplace transform technique is the main tool to do that with an arbitrary boundary and initial conditions for some special cases.

Key words : laplace,equation,Brajinskii ,one dimensional,transform.
AMS SUBJECT:16W60, 16W22, 16W20.

1. Introduction

Fusion is divided into cold and warm fusions .One of the important branches of warm fusion is known as inertial confinement fusion(ICF). ICF by a laser is called laser fusion.Our studies on the ICF show that Brajinskii's equations can describe the physics of ICF very well .Therefore ,in this work the authors survey the major areas of research. In section I, the basic equations for ICF (Brajinskii's equations)are given,while section II describes the analytical solutions for these equations in one dimension ,for the first time ,by using Laplace transform technique. The Laplace transform is just a theoretical and mathematical concept (outside of the real world) . The Laplace transforms are considered as a tool to make mathematical calculations easier.It is pointed out that the method of the Laplace transform has found an

S.N.Hosseinimotlagh. * S.N.Hosseinimotlagh ,Iran University Of Science and Technology,Tehran- Iran. E-mail: hosseinimotlagh@iust.ac.ir, **M.Roostaie, Iran University Of Science and Technology,Tehran- Iran. E-mail: roostaie@yahoo.com ,***H.Kazemifard,Iran University Of Science and Technology,Tehran- Iran. E-mail: Hkazemifard@gmail.com.

increasing number of applications in the fields of physics and technology.The term transform refers to a mathematical operation that takes a given function and returns a new function.The transformation is often done by means of an integral formula.Commonly used transforms and named after Laplace and Fourier.Transforms are frequently used to change a complicated problem into a simpler one.The simpler problem is then solved, usually using the inverse transform.A standard example is the use of the Laplace transform to solve a differential equation.A brief review of the Laplace transform is given here. The Laplace transform is useful in the modeling of a linear, time-invariant analog system as a transfer function. The Laplace transform may also be used to obtain the response of the system. If $s > 0$, and the integral $\int_0^\infty f(x)e^{-sx}dx$ exists, we call $F(s) = \int_0^\infty f(x)e^{-sx}dx$, the Laplace transform of $f(x)$ and denote it by $L[f(x)]$ .Laplace transform can be defined for functions with more variables than one.The inverse Laplace transform is given by the complex integral $f(x) = L^{-1}(F(s)) = \frac{1}{2\pi j}\int_{c+j\infty}^{c-j\infty} F(x)e^{-xs}ds$, where c is any real constant that is greater than the real parts of all the poles of $F(s)$ .Laplace transform and inverse Laplace transform , form the Laplace transform pair.Properties of the Laplace Transform are given in **Appendix A**.

2. Basic equations for ICF physics

The Brajinskii's equations terms which describe a very wide range of physical phenomena .For this reason ,they are exteremely complicated. Properties of produced plasma through ICF method are determined from Brajinskii's equations. This system includes the following equations (*see 1,2*).

$$\frac{d\rho}{dt} = -\rho\nabla.\overrightarrow{u} \tag{1}$$

$$\rho\frac{d\overrightarrow{u}}{dt} = -\nabla P \tag{2}$$

$$\rho\frac{d\varepsilon_i}{dt} = -P_i\nabla.\overrightarrow{u} - \nabla.\overrightarrow{q_i} + Q_{ei} + s_\alpha^i \tag{3}$$

$$\rho\frac{d\varepsilon_e}{dt} = -P_e\nabla.\overrightarrow{u} - \nabla.\overrightarrow{q_e} - Q_{ei} + s_l + s_r + s_\alpha^e \tag{4}$$

$$\frac{1}{c}\frac{\partial I^\nu}{\partial t} + \overrightarrow{\Omega}.\nabla I^\nu = \eta^\nu - \chi^\nu I^\nu \tag{5}$$

$$\frac{\partial}{\partial t}\Phi^\alpha + \overrightarrow{\nu}.\nabla\Phi^\alpha = S_{DT} + (\frac{\partial\varphi^\alpha}{\partial t})_{coll} \tag{6}$$

In equation (1) ρ is the mass density and $\overrightarrow{u}$ the flow velocity. The electron density can be neglected compared to the ion density and $\rho = m_i n_i$, where m_i and n_i are the average ion mass and the number density of ions, respectively. In equation (2), P is the total pressure in the form:

$$P = P_e(\rho, T_e) + P_i(\rho, T_i) \tag{7}$$

where $P_e(T_e)$ and $P_i(T_e)$ are the electron and ion pressures, respectively. In the case of fully ionized plasmas, those pressures should tend to the forms $P_e = Zk_Bn_iT_e$ and ,$P_i = k_Bn_iT_i$, where Z is the charge number of nuclei. In equation (3),ε_i is the ion internal energy per unit mass and is a function of ρ and T_i under the assumption of local thermal equilibrium. Using chain rule, we have

$$\frac{d\varepsilon_i}{dt} = \frac{\partial\varepsilon_i}{\partial T_i}\frac{dT_i}{dt} + \frac{\partial\varepsilon_i}{\partial\rho}\frac{d\rho}{dt} \tag{8}$$

The first term on the right-hand side in equation (3) is the pressure work term. In equation (3), $\overrightarrow{q_i}$ is the ion heat flux,it can be given by the diffusion type expression :

$$\overrightarrow{q_i} = -\frac{1}{3}l_iv_i\nabla T_i = -k_i\nabla T_i \tag{9}$$

In equation (9),l_i , v_i and k_i are the ion mean-free-path, thermal velocity, and heat conductivity, respectively. In equation (3), Q_{ei} is the ion-electron energy relaxation term proportional to $(T_e - T_i)$. The term s^i_α in equation (3) represents the energy source to the ions through the alpha particle heating. Equation (4) is the equation for the electron internal energy ε_e , which is a function of ρ and T_e . Similar to the case of equation (3), equation (4) is solved by replacing the left-hand side with

$$\frac{d\varepsilon_e}{dt} = \frac{\partial\varepsilon_e}{\partial\rho}\frac{d\rho}{dt} + \frac{\partial\varepsilon_e}{\partial T_e}\frac{dT_e}{dt} \tag{10}$$

In equation (4), $\overrightarrow{q_e}$ is the electron heat flux, which is the most dominant energy transport term similar to the case of the ions, the electron heat flux can also be given by the diffusion type expression

$$\overrightarrow{q_e} = -\frac{1}{3}l_ev_e\nabla T_e = -k_e\nabla T_e \tag{11}$$

Where l_e , v_e and k_e are the electron mean-free-path, thermal velocity, and heat conductivity, respectively. In the ideal plasmas, $l_e \propto T_e^2$ and $v_e \propto \sqrt{T_e}$ and k_e has the well known form of Spitzer-Härm thermal conductivity . In equation (4), Q_{ei} is the electron-ion relaxation term and s_l , s_r and s^e_α are the energy sources by laser heating, radiation and alpha particles, respectively .It is noted that the energy source term s_r could be positive or negative depending on the dominance of photon absorption or emission, respectively. Equation (5) represents the kinetic equation of the radiation with the energy of $h\nu$. In equation (5), I^ν is the spectral radiation intensity, c is the speed of light, and $\overrightarrow{\Omega}$ represents the unit vector of the direction of radiation propagation. On the right-hand side of equation (5), $\eta^\nu$is the spontaneous emission rate per unit volume, unit time, unit spectral interval of ν , and unit solid angle. The absorption coefficient χ^ν consists of pure absorption minus the induced emission contribution. The effective absorption and emission coefficients $\chi^\nu$and η^ν are determined after specifying the atomic state of the plasmas. In what follows, we call χ^ν and η^ν (spectral) opacity and emissivity,respectively. When we assume a quasi steady state, the atomic state is uniquely specified for a given density ρ and temperature T_e .As a result, we can represent the emissivity and opacity with the form

$$\eta^\nu = \eta^\nu(\rho, T_e) \tag{12}$$

$$\chi^\nu = \chi^\nu(\rho, T_e) \tag{13}$$

Equation (6) is a kinetic equation of particles generated by nuclear reactions, where we only discuss the alpha particles produced by the DT fusion reaction (*see 3,4,5,6*). The right-hand side of equation (6) represents the source term S_{DT} and the term for the slowing down by collisions with background ions and electrons $(\frac{\partial \varphi^\alpha}{\partial t})_{coll}$.

3. Analytical solutions for Brajinskii's equations

Brajinskii's equations can be simplified using some assumptions. If we apply these assumptions, we can obtain exact solutions for this set of differential equations. Our basic assumption is that the system is considered for one-dimensional case. In our solutions, we have also that ion temperature and electron temperature are the same approximately ,therefore, every where that the term $(T_i - T_e)$ is present, we neglect it. Also, we suppose that the temperature is almost constant throughout the plasma. By these assumptions , we try to give an analytical solution to each set Brajinskii's equations.Let us begin with the first two equations coupled each other(equations 1 and 2) .In one dimensional case these equations takes the following form:

$$\rho \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \tag{14}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{2k_B T}{m_i \rho} \frac{\partial \rho}{\partial x} \tag{15}$$

We have made use of:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \overrightarrow{v}.\nabla, \rho = m_i n, P = k_B(Z n_i T_e + n_i T_i) \tag{16}$$

Suppose that $T_i \approx T_e =$ constant and Z=1. So, (14),(15) can be written as:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \tag{17}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{2k_B T}{m_i} \frac{\partial}{\partial x} \ln \rho \tag{18}$$

The set of equations (17),(18) includes partial differential equations to solve them we apply the interesting trick: u and ρ are functions of x and t. We expect that for long times, i.e. when $t \longrightarrow \infty, \rho \longrightarrow 0$. On the other hand, expansion of plasma hints us that ρ should be diminish when x increased.

If ρ decreases exponentially with x, the simplest form for ρ could be $\rho = \rho_o \exp(-\frac{\alpha x}{t})$,in which ρ_0 and α are constant values that should be determined .Also ,the equation of (18) suggest that ρ must be in the form of exponentially. By replacing $\rho = \rho_o \exp(-\frac{\alpha x}{t})$ into the equation of (17) we obtain:

$$\frac{\partial u}{\partial x} - \frac{\alpha}{t}u = \frac{\alpha}{t^2}x \tag{19}$$

This equation has a solution in the following form :

$$u = c_1 e^{\frac{\alpha x}{t}} + \alpha(1 + \frac{x}{\alpha t}) \tag{20}$$

Since u should be finite when $x \longrightarrow \infty$ and $t \longrightarrow \infty$,therefore $c_1 = 0$. Hence, $u = \alpha(1+\frac{x}{\alpha t})$. Since the speed of pressure wave should be equal to the speed of the sound at $x = 0$, thus $\alpha = c_s$.ρ_0 is the density at $x = 0$ which is equal to the density of solid fuel (ρ_s)(*see 6,7,8*). Therefore, we have

$$\rho = \rho_s \exp(-\frac{x}{c_s t}) \tag{21}$$

$$u = c_s(1 + \frac{x}{c_s t}) \tag{22}$$

The second set of equations which should be solved is related to the variation of ion energy and electron energy(equations (3) and (4)). In order to reduce of the complexity of the problem, suppose that the terms s_α^i , s_l ,s_r and s_α^e are so small that one could neglect them. Moreover, if we consider the system with constant temperature for both ions and electrons and $T_i \approx T_e$,Q_{ei} will vanish, because this term is proportional to $(T_i - T_e)$ .As a result of the above assumptions the system of equations (3)and (4) becomes:

$$\rho\frac{d\varepsilon_i}{dt} = -P_i\nabla.\overrightarrow{u} - \nabla.\overrightarrow{q_i} \tag{23}$$

$$\rho\frac{d\varepsilon_e}{dt} = -P_e\nabla.\overrightarrow{u} - \nabla.\overrightarrow{q_e} \tag{24}$$

In one dimensional case the above set equations can be written as

$$\rho\frac{d\varepsilon_i}{dt} = -P_i\frac{\partial u}{\partial x} - \frac{\partial q_i}{\partial x} \tag{25}$$

$$\rho\frac{d\varepsilon_e}{dt} = -P_e\frac{\partial u}{\partial x} - \frac{\partial q_e}{\partial x} \tag{26}$$

Using $P_i = k_B n_i T$ and $P_e = Zk_B n_i T$ and $Z = 1$, we have

$$\rho\frac{d\varepsilon_i}{dt} = -k_B n_i T\frac{\partial u}{\partial x} - \frac{\partial q_i}{\partial x} \tag{27}$$

$$\rho\frac{d\varepsilon_e}{dt} = -k_B n_i T\frac{\partial u}{\partial x} - \frac{\partial q_e}{\partial x} \tag{28}$$

Also, in one-dimensional case, q_i and q_e have the following forms:

$$q_i = -k_i \frac{\partial T_i}{\partial x} = 0 \longrightarrow since, T_i = constant \tag{29}$$

$$q_e = -k_e \frac{\partial T_e}{\partial x} = 0 \longrightarrow since, T_e = constant \tag{30}$$

By using equations (29) and (30), equations(27)and(28) are changed into the following set of equations:

$$\frac{d\varepsilon_i}{dt} = -\frac{k_B T}{m} \frac{\partial u}{\partial x} \tag{31}$$

$$\frac{d\varepsilon_e}{dt} = -\frac{k_B T}{m} \frac{\partial u}{\partial x} \tag{32}$$

Using equations (8) and (10) for ε_i and ε_e respectively, we have

$$\frac{d\varepsilon_i}{dt} = \frac{\partial \varepsilon_i}{\partial \rho} \frac{d\rho}{dt} + \frac{\partial \varepsilon_e}{\partial T} \frac{dT_i}{dt} = \frac{\partial \varepsilon_i}{\partial \rho} \frac{d\rho}{dt} \tag{33}$$

$$\frac{d\varepsilon_e}{dt} = \frac{\partial \varepsilon_e}{\partial \rho} \frac{d\rho}{dt} + \frac{\partial \varepsilon_e}{\partial T_e} \frac{dT_e}{dt} = \frac{\partial \varepsilon_e}{\partial \rho} \frac{d\rho}{dt} \tag{34}$$

considering equations (21) and (22) and with doing some simplification we obtain:

$$\varepsilon_i = \frac{k_B T}{m} \tag{35}$$

$$\varepsilon_e = \frac{k_B T}{m} \tag{36}$$

This is a result that we could expect from physical considerations. Because the temperature is constant for both ions and electrons. So far, we have given the solutions for four Brajinskii's equations. They included two sets of coupled equations. The last two equations are two separate partial differential equations(equations (5) and (6)), one for the intensity of light (laser) propagation in plasma and the other for the flux of alpha particles. To solve these two equations, we apply Laplace transform technique.

In the fifth equation of Brajinskii's system of equations we choose $\overrightarrow{\Omega}$ in the direction of x therefore the equation(5) has the following form in one-dimensional case:

$$\frac{1}{c} \frac{\partial I}{\partial t} + \frac{\partial I}{\partial x} = \eta - \chi I \tag{37}$$

Suppose that the initial condition is $I(Z, 0) = f(z)$ and the boundary condition is given by $I(0, t) \approx 0$ .Taking Laplace transform from the both sides of equation (3), we have(*see 9,10,11*):

$$\frac{\partial}{\partial t} L[I] + sL[I] = \frac{\eta}{s_i} - \chi L[I] \tag{38}$$

Note that we have made use of the following theorem in Laplace transform technique:

$$L[\frac{df(x)}{dx}] = sL[f(x)] - f(0) \tag{39}$$

For writing the equation (38), we have supposed that χ and η are constant. Equation (38) is a differential equation. Solving this equation in terms of t, we have

$$L[I] = ke^{-c(\chi+s)t} + \frac{\eta}{s(s+\chi)} \tag{40}$$

In which k is a constant that should be determined using initial condition $I(Z,0) = f(z)$,

$$k = L[f(\chi)] - \frac{\eta}{s(s+\chi)} \tag{41}$$

Now we have

$$L[I(\chi,t)] = \{L[f(\chi)] - \frac{\eta}{s(s+\chi)}\}e^{-c(\chi+s)t} + \frac{\eta c}{s(s+\chi)} \tag{42}$$

Using inverse Laplace transform, we have

$$I(x,t) = e^{-\chi ct}f(x-ct)u(x-ct) - (1-\exp(-x(x-ct))) + \frac{\eta c}{\chi}(1-e^{-\chi z}) \tag{43}$$

In which, $u(x-ct) = \begin{cases} 0, & \text{x< ct} \\ 1, & \text{x> ct} \end{cases}$ and we have made use of the following theorem:

$$L^{-1}[e^{-sa}F(s)] = u(x-a)e^{(x-a)} \tag{44}$$

The last equation of Brajinskii's equations describes the variation of alpha particles flux(equation(6)). We assume that the right-hand side of this equation is constant. If alpha particles move with constant velocity $\overrightarrow{v}$, the equation (6) for one dimensional case takes the following form:

$$\frac{\partial\varphi}{\partial t} + v\frac{\partial\varphi}{\partial x} = k = constant \tag{45}$$

Suppose that the initial condition is $\varphi(x,0) = f(x)$ and $\varphi(0,t) = g(t)$. Taking Laplace transform from both sides of equation (45)(*see 12,13,14*), we have

$$L[\frac{\partial\varphi}{\partial t}] + L[v\frac{\partial\varphi}{\partial x}] = \frac{k}{s} \tag{46}$$

or

$$\frac{\partial}{\partial t}L[\varphi] + vsL[\varphi] = k + vg(t) \tag{47}$$

The above differential equation has a solution as

$$L[\varphi] = C_1e^{-sVt} + \frac{1}{vs}(k + g(t)) \tag{48}$$

C_1 is a constant determined from initial condition such that by using $\Phi(x,0) = f(x)$ we will find:

$$C_1 = L[F(x)] - \frac{1}{vs}(k + vg(0)) \quad (49)$$

Inserting C_1 into equation (48), we obtain

$$L[\phi(x,t)] = \{L[f(x)] - \frac{1}{vs}(k + vg(0))\}e^{-sVt} + \frac{1}{vs}(k + g(t)) \quad (50)$$

Using inverse Laplace transform,

$$\phi(x,t) = u(x,t)[f(x - ct) - \frac{k + vg(0)}{v}] + \frac{k + vg(t)}{v} \quad (51)$$

this is the exact solution for differential equation describing variation of ϕ (flux of alpha particles).Therefore, the Laplace transform is basically mathematics but mathematics can be interpreted and in the physical world, the experiments and empirical data, etc. are represent with a model to analyze it and that data became abstract. It is not the real world; its a description of kind of process as it might take place. The theoretical process might take place in the real world but, is the solution that you get with the computing using the Laplace transform, is the solution in the process that we find in the real world.(see 15,16)

4. APPENDIX A

Properties of the Laplace Transform are given in the below:
We assume, for these properties, that $F(s) = L(f(t))$,$G(s) = L(g(t))$.

1. Linearity
$L[af(t) + bg(t)] = a[L(f(t)] + b[L(g(t)] = aF(s) + bG(s)$
a,b- real constants

2. Differentiation
$L[\frac{d^n f(t)}{dt^n}] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f(0) - ... - sf^{n-2}(0) - f^{n-1}(0)$

3. Integration
$L[\int_0^t \int_0^{t_{n-1}} ... \int_0^{t_2} f(t_1)dt_1 dt_2 ...dt_{n-1}] = \frac{F(s)}{s^n}$

4.Time Delay
$L[f(t-\tau)u_s(t-\tau)] = e^{-s\tau}F(s)$

5.Shifting Property
$L[f(t)\delta(t)] = f(0)$,$L[f(t-a)\delta(t-\tau)] = f(\tau - a)e^{-s\tau}$
where $\delta(t)$ is the Dirac delta function.

6.Exponentially Weighted Function
$L[e^{-at}f(t)] = F(s+a)$

7. Time Weighted Function
$L[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}$

8. Real Convolution
$L[\int_0^t f(t-\tau)g(\tau)d\tau] = L[\int_0^t f(\tau)g(t-\tau)d\tau] = F(s)G(s)$

9.Initial Value Theorem

$lim_{t\to 0}f(t) = lim_{s\to\infty}sF(s)$,if the limit exists

10.Final Value Theorem

$lim_{t\to\infty}f(t) = lim_{s\to 0}sF(s)$,if $sF(s)$ has all its poles in the LHP

5. Acknowledgment

Thanks are due to Research Council of Iran University of Science and Technology for financial support of this research.

References

1. S.Nakai and H.Takabe Principles of inertial confinement fusion - physics of implosion and the confinement of inertial fusion energy, Institute of laser engineering, Osaka university, Suita, Osaka 565. Japan,1997
2. R.Fitzpatrick The Brajinskii equations, 2006.
3. F.Chen Introduction to plasma physics, Electrical Seience and Engineering Departments School of Engineering and Appliced Science, University of Colifornia, PLENUM PRESS, Second Edition,1974.
4. N.Keishiro Nuclear Fusion, Tokyoinstitue of Technology, English Edition. CAM BRIDGE UNIVERSITY PRESST,1986.
5. L.Kruer ,The Physics of Laser-pplasma Interactions , Lawrence National Laboratory, 1988.
6. P.Chatelier and P.Magaud ,Les Perspectives de la fusion thermonucleaire, dEcole National Superieur de Grenoble,2003 .
7. S.Eliezer , Z.Henis , J.M.Martinez , et al. Nuclear Fusion, 40(2), 195,2000.
8. S.N.Hosseinimotlagh ,et al.Effect of tritium reduction in determining energy gain by using R-matrix method direct laser fusion in D-T reaction,Nuclear Science and Techniqus , 18(3),1,2007 .
9. F.Ayers F.Differential Equations ,SHAUM outline Series,1990
10. M.Siegel , Advanced Mathematics for scientists and engineers, SHAUM outline Series ,1990
11. N. Rajabov, An introduction to Partial differential equations ,1976
12. E. Zeidler,Teubner-Taschenbuch der Mathematik . B. G. Teubner Verlag, Stuttgart and Leipzig, 1996.
13. Wu W.-T.,Mathematics Mechanization, Math. and its Applications, vol. 489. Science Press, Beijing and Kluwer Acad. Publ., Dordrecht, 2000.
14. D. Zwillinger, Standard Mathematical Tables and Formulae , 30th edition. CRC Press, Boca Raton, 1996.
15. E. Kreyzig, Advanced Engineering Mathematics, John Wiley and Sons,8th Edition, 2001
16. D. G. Duffy, Advanced Engineering Mathematics with MATLAB, 2nd Edition, Chapman and Hill,CRC, 2003