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Geometric Properties of Janowski-type Harmonic Functions Involving Bessel Functions

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Abstract

In this paper, we characterize geometric properties and inclusion relationships of particular subclasses of harmonic univalent functions in $U = \{\xi \in C : |\xi| < 1\}$. By employing a convolution operator constructed via Bessel functions of generalized type, we establish new relationships between Janowski-type harmonic starlike functions and several well-known harmonic subclasses, including harmonic convex, starlike, close-to-convex, and Ruscheweyh-type classes. Sufficient coefficient criterias are derived to guarantee that the introduced operator preserves membership in the Janowski-type harmonic class $\mathcal{S}^*_{\mathcal{H}}(\mathcal{E}, \mathcal{W})$. Furthermore, we obtain some inclusion conclusions. The obtained criteria are expressed in terms of the values and derivatives of the associated generalized Bessel functions at unity. Our conclusions extend and refine several earlier works in the literature concerning harmonic mappings associated with special functions. In particular, they provide a unified framework for studying Janowski-type harmonic functions involving generalized Bessel functions and open the way for further investigations via other special function operators.

Keywords: Harmonic functions; Bessel function; Generalized Bessel functions; harmonic univalent functions; Janowski functions; convolution; analytic functions.

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1. Motivation and Background

We represent by $\mathcal{A}$ the class consisting of functions n given by:

$$n(\xi) = \xi + \sum_{\daleth=2}^{\infty} a_{\daleth} \xi^{\daleth},$$

which are analytic in $U = \{\xi \in C : |\xi| < 1\}$, and are subject to the normalization criterion $n(0) = n^{'}(0) - 1 = 0$. Let $\mathcal{H}$ represent the class of all harmonic functions represented by

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$$m = n + \bar{v}, \tag{1.1}$$

where

$$n(\xi) = \xi + \sum_{\daleth=2}^{\infty} a_{\daleth}\xi^{\daleth}, \quad v(\xi) = \sum_{\daleth=1}^{\infty} b_{\daleth}\xi^{\daleth}, \quad (\xi \in U) \tag{1.2}$$

are in the class $\mathcal{A}$ and then $m(\xi)$ is given by,

$$m(\xi) = \xi + \sum_{\daleth=2}^{\infty} a_{\daleth}\xi^{\daleth} + \overline{\sum_{\daleth=1}^{\infty} b_{\daleth}\xi^{\daleth}}, \quad (\xi \in U). \tag{1.3}$$

We represent by $S_{\mathcal{H}}$ the subclass of $\mathcal{H}$ whose elements are univalent and sense-preserving in U. In this setting, the sense-preserving criteria immediately yields $|b_1| < 1$. Note that $\frac{m-\overline{b_1 m}}{1-|b_1|^2} \in S_{\mathcal{H}}$ whenever $m \in S_{\mathcal{H}}$. In accordance with Clunie and Sheil-Small [1] we represent by $S^0_{\mathcal{H}}$ of $S_{\mathcal{H}}$ the subclass,

$$S^0_{\mathcal{H}} = \{m = n + \bar{v} \in S_{\mathcal{H}} : v^{'}(0) = b_1 = 0\}.$$

Further, let are subclasses of $S^0_{\mathcal{H}}$ whose elements are harmonic convex, harmonic starlike, and harmonic close-to-convex functions in U correspondingly. Also $T^0_{\mathcal{H}}$, represents the class of typically real harmonic functions; for further details, see ([1],[2],[3],[4]).

Let $\mathcal{T}_H$ represent the family of all harmonic functions of the form $m = n + \bar{v}$, where

$$n(\xi) = \xi - \sum_{\daleth=2}^{\infty} |a_{\daleth}|\xi^{\daleth}, \quad v(\xi) = \sum_{\daleth=1}^{\infty} |b_{\daleth}|\xi^{\daleth}, \quad (\xi \in U),$$

where

$$\mathcal{T}_H = \left\{ m(\xi) = \xi + \sum_{\daleth=2}^{\infty} |a_{\daleth}|\xi^{\daleth} - \sum_{\daleth=1}^{\infty} |b_{\daleth}|\xi^{\daleth}, 0 \le b_1 < 1 \right\}. \tag{1.4}$$

2. Preliminaries

Lemma 2.1. *([1],[2]) If $m \in K^0_{\mathcal{H}}$ and $f = n + \bar{v}$ where n and v are given by (1.2) with $b_1 = 0$, then*

$$|a_{\daleth}| \le \frac{\daleth + 1}{2}, \quad |b_{\daleth}| \le \frac{\daleth - 1}{2} \qquad (\daleth \ge 1).$$

Lemma 2.2. *([1],[2]) If $m \in \mathcal{S}^{*,0}_{\mathcal{H}}$ (or $m \in C^0_{\mathcal{H}}$) and $m = n + \bar{v}$ where n and v are given by (1.2) with $b_1 = 0$, then*

$$|a_{\daleth}| \le \frac{(2\daleth + 1)(\daleth + 1)}{6}, \quad |b_{\daleth}| \le \frac{(2\daleth - 1)(\daleth - 1)}{6} \qquad (\daleth \ge 1).$$

Recently, for $m \in \mathcal{S}_{\mathcal{H}}$ whose functions are represented as in (1.3), Dziok [5] introduced a new class $\mathcal{S}^*_{\mathcal{H}}(\mathcal{E}, \mathcal{W})$ $(-\mathcal{W} \le \mathcal{E} < \mathcal{W} \le 1)$ by

$$\mathcal{S}^*_{\mathcal{H}}(\mathcal{E}, \mathcal{W}) = \left\{ m = n + \bar{v} \in \mathcal{H} : \frac{\mathcal{D}_{\mathcal{H}} m(\xi)}{m(\xi)} \prec \frac{1 + \mathcal{E}\xi}{1 + \mathcal{W}\xi} \right\},$$

where $\mathcal{D}_{\mathcal{H}} m(\xi) = \xi n^{'}(\xi) - \overline{\xi v^{'}(\xi)}$ and $\xi \in U$. Also let

$$\mathcal{K}_{\mathcal{H}}(\mathcal{E}, \mathcal{W}) := m \in \mathcal{S}_{\mathcal{H}} : \mathcal{D}_{\mathcal{H}} m \in \mathcal{S}^*_{\mathcal{H}}(\mathcal{E}, \mathcal{W}).$$

More specifically, the class $\mathcal{R}_{\mathcal{H}}(\mathcal{E},\mathcal{W})$, introduced by Dziok [6] is given by

$$\mathcal{R}_{\mathcal{H}}(\mathcal{E},\mathcal{W}) = \left\{ m = n + \bar{v} \in \mathcal{H} : \frac{\mathcal{D}_{\mathcal{H}} m(\xi)}{\xi'} \prec \frac{1+\mathcal{E}\xi}{1+\mathcal{W}\xi} \right\}.$$

Remark 2.3. We note that

1.$\mathcal{S}^*_{\mathcal{H}}(\mathcal{E},\mathcal{W}) := \mathcal{S}^*_{\mathcal{H}}(\mathcal{E},\mathcal{W}) \cap \mathcal{A}$ was introduced by Janowski [7].
2.For $0 \le \vartheta < 1$ the classes $S^*_{\mathcal{H}}(\vartheta) := \mathcal{S}^*_{\mathcal{H}}(2\vartheta - 1, 1)$ and $\mathcal{K}_{\mathcal{H}}(\vartheta) := \mathcal{K}_{\mathcal{H}}(2\vartheta - 1, 1)$ were examined by Jahangiri ([8],[9]).
3.Also $\mathcal{S}^*_{\mathcal{H}} := \mathcal{S}^*_{\mathcal{H}}(0)$ and $\mathcal{K}_{\mathcal{H}} = \mathcal{K}_{\mathcal{H}}(0)$ represent the classes of harmonic starlike and convex harmonic functions in U and $\mathcal{S}^*_{\mathcal{H}}(\mathcal{E},\mathcal{W}) \subset \mathcal{S}^*_{\mathcal{H}}$, $\mathcal{K}_{\mathcal{H}}(\mathcal{E},\mathcal{W}) \subset \mathcal{K}_{\mathcal{H}}$.

Recently, Dizok [5] established the following coefficient criterias:

Lemma 2.4. *[5] Let $m \in \mathcal{H}$ be described in (1.3), then $m \in \mathcal{S}^*_{\mathcal{H}}(\mathcal{E},\mathcal{W})$ if*

$$\sum_{\daleth=2}^{\infty} \left(\frac{\daleth(1+\mathcal{W}) - (1+\mathcal{E})}{\mathcal{W}-\mathcal{E}} |a_\daleth| + \frac{\daleth(1+\mathcal{W}) + (1+\mathcal{E})}{\mathcal{W}-\mathcal{E}} |b_\daleth| \right) \le 1,$$

where $a_1 = 1; b_1 = 0$ and $-\mathcal{W} \le \mathcal{E} < \mathcal{W} \le 1$.

Lemma 2.5. *[5] Let $m \in \mathcal{T}_H$ be considered as defined in (1.4) and*

$$m \in \mathcal{ST}^*_{\mathcal{H}}(\mathcal{E},\mathcal{W}) \Leftrightarrow \sum_{\daleth=2}^{\infty} \left(\frac{\daleth(1+\mathcal{W}) - (1+\mathcal{E})}{\mathcal{W}-\mathcal{F}} |a_\daleth| + \frac{\daleth(1+\mathcal{W}) + (1+\mathcal{E})}{\mathcal{W}-\mathcal{E}} |b_\daleth| \right) \le 1,$$

where $a_1 = 1; b_1 = 0$ and $-\mathcal{W} \le \mathcal{E} < \mathcal{W} \le 1$.

Remark 2.6. *If* $m \in \mathcal{S}^*_{\mathcal{H}}(\mathcal{E},\mathcal{W})$, then

$$|a_\daleth| \le \frac{\mathcal{W}-\mathcal{E}}{\daleth(1+\mathcal{W}) - (1+\mathcal{E})} \quad and \quad |b_\daleth| \le \frac{\mathcal{W}-\mathcal{E}}{\daleth(1+\mathcal{W}) + (1+\mathcal{E})}, \quad \daleth \ge 2.$$

Lemma 2.7. *[5] Let $m \in \mathcal{T}_H$ be considered as defined in (1.4) and*

$$m \in \mathcal{KT}_{\mathcal{H}}(\mathcal{E},\mathcal{W}) \Leftrightarrow \sum_{\daleth=2}^{\infty} \left(\frac{\daleth(1+\mathcal{W}) - (1+\mathcal{E})}{\mathcal{W}-\mathcal{E}} |a_\daleth| + \frac{\daleth(1+\mathcal{W}) + (1+\mathcal{E})}{\mathcal{W}-\mathcal{E}} |b_\daleth| \right) \le 1,$$

where $a_1 = 1; b_1 = 0$ and $-\mathcal{W} \le \mathcal{E} < \mathcal{W} \le 1$.

Lemma 2.8. *[6] Let $m \in \mathcal{H}$ be considered as defined in (1.3) and $f \in \mathcal{R}_{\mathcal{H}}(\mathcal{E},\mathcal{W})$ if*

$$\sum_{\daleth=2}^{\infty} \left(\frac{\daleth(1+\mathcal{W})}{\mathcal{W}-\mathcal{E}} |a_\daleth| + \frac{\daleth(1+\mathcal{W})}{\mathcal{W}-\mathcal{E}} |b_\daleth| \right) \le 1,$$

where $a_1 = 1; b_1 = 0$ and $-\mathcal{W} \le \mathcal{E} < \mathcal{W} \le 1$.

Lemma 2.9. *[6] Let $m \in \mathcal{T}_H$ be considered as defined in (1.4) and*

$$m \in \mathcal{RT}_{\mathcal{H}}(\mathcal{E},\mathcal{W}) \Leftrightarrow \sum_{\daleth=2}^{\infty}\left(\frac{\daleth(1+\mathcal{W})}{\mathcal{W}-\mathcal{E}}|a_\daleth| + \frac{\daleth(1+\mathcal{W})}{\mathcal{W}-\mathcal{E}}|b_\daleth|\right) \leq 1,$$

where $a_1 = 1; b_1 = 0$ and $-\mathcal{W} \leq \mathcal{E} < \mathcal{W} \leq 1$.

Remark 2.10. If $m \in \mathcal{RT}_{\mathcal{H}}(\mathcal{E},\mathcal{W})$, then $|a_\daleth| \leq \frac{\mathcal{W}-\mathcal{E}}{\daleth(1+\mathcal{W})}$ and $|b_\daleth| \leq \frac{\mathcal{W}-\mathcal{E}}{\daleth(1+\mathcal{W})}$, $\daleth \geq 2$.

The following second-order linear homogeneous differential equation admits the classical Bessel function of the first kind, denoted by $w = w_{p,b,e}$, as one of its solutions:

$$\xi^2\omega''(\xi) + b\xi\omega'(\xi) + [e\xi^2 - p^2 + (1-b)p]\omega(\xi) = 0, \tag{2.1}$$

where $p, b, e \in C$. The solutions corresponding to this generalized Bessel differential equation are frequently expressible through special functions given below:

$$\omega(\xi) = \omega_{p,b,e}(\xi) = \sum_{\daleth=0}^{\infty}\frac{(-1)^\daleth e^\daleth}{\daleth!\Gamma(p+\daleth+\frac{b+1}{2})}\left(\frac{\xi}{2}\right)^{2\daleth+p}, (\xi \in C). \tag{2.2}$$

The generalized Bessel equation in (2.1) may be extended through suitable modifications so as to characterize various classes of special functions, including the different types of Bessel functions [20]. Within this framework, the generalized Bessel function of order p naturally arises as a family of solutions to the associated generalized differential equation. The particular solution presented in (2.2) is referred to as the generalized Bessel function of the first kind of order p. Although the series introduced above converges throughout the complex plane, the function $\omega_{p,b,e}(\xi)$ is, in general, not univalent in U. It should also be observed that, for the special choice $b = e = 1$, one obtains the classical Bessel function $\omega_{p,1,1}(\xi) = j_p$, whereas for $e = -1, b = 1$ the function $\omega_{p,1,-1}$ reduces to the modified Bessel function I_p. To facilitate our analysis, we define the function $u_{p,b,e}$ by the transformation:

$$u_{p,b,e}(\xi) = 2^p\Gamma\left(p + \frac{b+1}{2}\right)\xi^{-p/2}\omega_{p,b,e}(\xi^{1/2}).$$

Let $(a)_\daleth$ denote the Pochhammer (or Appell) symbol, defined in terms of the Euler gamma function for $a \neq 0, -1, -2, ...$ by :

$$(a)_\daleth = \frac{\Gamma(a+\daleth)}{\Gamma(a)} = \begin{cases} 1, & if \quad \daleth = 0 \\ a(a+1)...(a+\daleth-1), & if \quad \daleth = 1,2,3,... \end{cases}.$$

This leads to the following representation of the function $u_{p,b,c}$:

$$u_{p,b,e}(\xi) = \sum_{\daleth=0}^{\infty}\frac{(-e/4)^\daleth}{(p+\frac{b+1}{2})_\daleth}\frac{\xi^\daleth}{\daleth!}, \tag{2.3}$$

where $p + (b+1)/2 \neq 0, -1, -2, ...$. The function is analytic in the complex plane C and fulfills the following second-order linear differential equation:

$$4\xi^2u''(\xi) + 2(2p+b+1)\xi u'(\xi) + e\xi u(\xi) = 0.$$

For notational simplicity, we write

$$u_{p,b,e} = u_p, \quad t = p + \frac{b+1}{2}.$$

For complex parameters $e_1, e_2, t_1, t_2 (t_1, t_2 \neq 0, -1, -2, ...)$, we introduce the functions defined by $\eth_1(\xi) = \xi u_{p_1}(\xi)$ and $\eth_2(\xi) = \xi u_{p_2}(\xi)$.

Inspired by the above definitions, we now introduce the following convolution operator:

$$\Upsilon \equiv \Upsilon\begin{pmatrix} t_1, & e_1 \\ t_2, & e_2 \end{pmatrix} : \mathcal{H} \to \mathcal{H}$$

defined by

$$\Upsilon\begin{pmatrix} t_1, & e_1 \\ t_2, & e_2 \end{pmatrix} m = m * (\eth_1 + \overline{\eth_2}) = n(\xi) * \eth_1(\xi) + \overline{v(\xi) * \eth_2(\xi)}$$

for any function $m = n + \overline{v}$ in $\mathcal{H}$.
Letting

$$\Upsilon\begin{pmatrix} t_1, & e_1 \\ t_2, & e_2 \end{pmatrix} m(\xi) = H(\xi) + \overline{G(\xi)}, \tag{2.4}$$

where

$$H(\xi) = \xi + \sum_{\daleth=2}^{\infty} \frac{(-e_1/4)^{\daleth-1}}{(t_1)_{\daleth-1}(\daleth-1)!} a_{\daleth} \xi^{\daleth}, \quad G(\xi) = \sum_{\daleth=1}^{\infty} \frac{(-e_2/4)^{\daleth-1}}{(t_2)_{\daleth-1}(\daleth-1)!} b_{\daleth} \xi^{\daleth}. \tag{2.5}$$

The study of inclusion relationships among various subclasses of analytic and univalent functions has attracted considerable attention, particularly in association with special functions such as hypergeometric functions.([10],[11],[12]), the Mittag-Leffler function ([13],[14],[15]), generalized Bessel function ([16],[17],[18][19][20]), Janowski-type harmonic function ([21],[22]) and, more recently, distribution series ([23],[24],[25],[26],[27],[28]), were studied in the literature.

The following notation will be used throughout this paper.

$$\Upsilon(m) = \Upsilon\begin{pmatrix} t_1, & e_1 \\ t_2, & e_2 \end{pmatrix} m.$$

Let

$$\eth(\xi) = \xi u_p(\xi) = \xi + \sum_{\daleth=2}^{\infty} \frac{(-e/4)^{\daleth-1}}{(t)_{\daleth-1}(\daleth-1)!} \xi^{\daleth}$$

$$\eth(1) = u_p(1) = 1 + \sum_{\daleth=2}^{\infty} \frac{(-e/4)^{\daleth-1}}{(t)_{\daleth-1}(\daleth-1)!}$$

and

$$\eth'(\xi) = \xi u_p'(\xi) + u_p(\xi) = 1 + \sum_{\daleth=2}^{\infty} \daleth \frac{(-e/4)^{\daleth-1}}{(t)_{\daleth-1}(\daleth-1)!} \xi^{\daleth-1}$$

$$\eth'(1) = u_p'(1) + u_p(1) - 1 = \sum_{\daleth=2}^{\infty} \daleth \frac{(-e/4)^{\daleth-1}}{(t)_{\daleth-1}(\daleth-1)!}$$

$$\eth''(\xi) = \xi u_p''(\xi) + 2u_p'(\xi) = \sum_{\daleth=2}^{\infty} \daleth(\daleth-1) \frac{(-e/4)^{\daleth-1}}{(t)_{\daleth-1}(\daleth-1)!} \xi^{\daleth-2}$$

$$\eth''(1) = u_p''(1) + 2u_p'(1) = \sum_{\daleth=2}^{\infty} \daleth(\daleth-1) \frac{(-e/4)^{\daleth-1}}{(t)_{\daleth-1}(\daleth-1)!}$$

$$\eth'''(\xi) = \xi u_p'''(\xi) + 3u_p''(\xi) = \sum_{\daleth=2}^{\infty} \daleth(\daleth-1)(\daleth-2) \frac{(-e/4)^{\daleth-1}}{(t)_{\daleth-1}(\daleth-1)!} \xi^{\daleth-3}$$

$$\eth'''(1) = u_p'''(1) + 3u_p''(1) = \sum_{\daleth=2}^{\infty} \daleth(\daleth-1)(\daleth-2)\frac{(-e/4)^{\daleth-1}}{(t)_{\daleth-1}(\daleth-1)!},$$

More generally, we have

$$\eth^{(j)}(1) = u_p^{(j)}(1) + ju_p^{(j-1)}(1) = \sum_{\daleth=2}^{\infty} \daleth(\daleth-1)(\daleth-2)...(\daleth-(j-1))\frac{(-e/4)^{\daleth-1}}{(t)_{\daleth-1}(\daleth-1)!}, \quad j = 1, 2, ... \quad .$$

3. Convergence conclusions

In this section, we establish several inclusion relationships involving the harmonic class $\mathcal{S}^*_{\mathcal{H}}(\mathcal{E}, \mathcal{W})$ with the classes $K^0_{\mathcal{H}}$, $\mathcal{S}^{*,0}_{\mathcal{H}}$, $\mathcal{RT}_{\mathcal{H}}(\beta, \mathcal{W})$ and $\mathcal{ST}^*_{\mathcal{H}}(\mathcal{E}, \mathcal{W})$, correspondingly. These relations are obtained by employing the mapping properties of Janowski-type harmonic functions associated with the Bessel function. Unless otherwise stated, we assume that $t_1, t_2 > 0$, $e_1, e_2 < 0$, $-\mathcal{W} \le \mathcal{E} < \mathcal{W} \le 1$, $\beta < \mathcal{W}$ and $\beta \in [0, 1)$.

Theorem 3.1. *If*

$$\begin{aligned}(1+\mathcal{W})u''_{p_1}(1) + (3+4\mathcal{W}-\mathcal{E})u'_{p_1}(1) + 2(\mathcal{W}-\mathcal{E})u_{p_1}(1)&\\ + (1+\mathcal{W})u''_{p_2}(1) + (3+2\mathcal{W}+\mathcal{E})u'_{p_2}(1) \le 4(\mathcal{W}-\mathcal{E}),&\end{aligned} \tag{3.1}$$

then

$$\Upsilon(K^0_{\mathcal{H}}) \subset \mathcal{S}^*_{\mathcal{H}}(\mathcal{E}, \mathcal{W}).$$

Proof. Let $m = n + \overline{v} \in K^0_{\mathcal{H}}$ where h and g are represented in (1.2) with $b_1 = 0$. It remains to prove that $\Upsilon(m)=H+\overline{G} \in \mathcal{S}^*_{\mathcal{H}}(\mathcal{E}, \mathcal{W})$, which given by (2.5) with $b_1 = 0$. Lemma 2.4 reduces the proof to verifying that

$$\begin{aligned}P_1 = \sum_{\daleth=2}^{\infty} \left[\daleth(1+\mathcal{W}) - (1+\mathcal{E})\right] \frac{(-e_1/4)^{\daleth-1}}{(t_1)_{\daleth-1}(\daleth-1)!}|a_\daleth|&\\ + \sum_{\daleth=2}^{\infty} \left[\daleth(1+\mathcal{W}) + (1+\mathcal{E})\right] \frac{(-e_2/4)^{\daleth-1}}{(t_2)_{\daleth-1}(\daleth-1)!}|b_\daleth| \le \mathcal{W}-\mathcal{E}.&\end{aligned} \tag{3.2}$$

By using Lemma 2.1

$$\begin{aligned}P_1 \le \frac{1}{2}\Bigg\{\sum_{\daleth=2}^{\infty}(\daleth+1)\left[\daleth(1+\mathcal{W}) - (1+\mathcal{E})\right] \frac{(-e_1/4)^{\daleth-1}}{(t_1)_{\daleth-1}(\daleth-1)!}&\\ + \sum_{\daleth=2}^{\infty}(\daleth-1)\left[\daleth(1+\mathcal{W}) + (1+\mathcal{E})\right] \frac{(-e_2/4)^{\daleth-1}}{(t_2)_{\daleth-1}(\daleth-1)!}\Bigg\}&\end{aligned}$$

$$
= \frac{1}{2}\left\{\sum_{\daleth=2}^{\infty}\left[(\daleth^2+\daleth)(1+\mathcal{W})-(\daleth+1)(1+\mathcal{E})\right]\frac{(-e_1/4)^{\daleth-1}}{(t_1)_{\daleth-1}(\daleth-1)!}\right.
$$

$$
\left.+\sum_{\daleth=2}^{\infty}\left[(\daleth^2-\daleth)(1+\mathcal{W})+(\daleth-1)(1+\mathcal{E})\right]\frac{(-e_2/4)^{\daleth-1}}{(t_2)_{\daleth-1}(\daleth-1)!}\right\}.
$$

Expressing $\daleth^2=\daleth(\daleth-1)+\daleth$, we get

$$
= \frac{1}{2}\left\{\sum_{\daleth=2}^{\infty}[\daleth(\daleth-1)(1+\mathcal{W})+\daleth(1+2\mathcal{W}-\mathcal{E})-(1+\mathcal{E})]\frac{(-e_1/4)^{\daleth-1}}{(t_1)_{\daleth-1}(\daleth-1)!}\right.
$$

$$
\left.+\sum_{\daleth=2}^{\infty}[\daleth(\daleth-1)(1+\mathcal{W})+\daleth(1+\mathcal{E})-(1+\mathcal{E})]\frac{(-e_2/4)^{\daleth-1}}{(t_2)_{\daleth-1}(\daleth-1)!}\right\}
$$

$$
= \frac{1}{2}\left\{(1+\mathcal{W})\sum_{\daleth=2}^{\infty}\frac{\daleth(\daleth-1)(-e_1/4)^{\daleth-1}}{(t_1)_{\daleth-1}(\daleth-1)!}+(1+2\mathcal{W}-\mathcal{E})\sum_{\daleth=2}^{\infty}\frac{\daleth(-e_1/4)^{\daleth-1}}{(t_1)_{\daleth-1}(\daleth-1)!}\right.
$$

$$
-(1+\mathcal{E})\sum_{\daleth=2}^{\infty}\frac{(-e_1/4)^{\daleth-1}}{(t_1)_{\daleth-1}(\daleth-1)!}+(1+\mathcal{W})\sum_{\daleth=2}^{\infty}\frac{\daleth(\daleth-1)(-e_2/4)^{\daleth-1}}{(t_2)_{\daleth-1}(\daleth-1)!}
$$

$$
\left.+(1+\mathcal{E})\sum_{\daleth=2}^{\infty}\frac{\daleth(-e_2/4)^{\daleth-1}}{(t_2)_{\daleth-1}(\daleth-1)!}-(1+\mathcal{E})\sum_{\daleth=2}^{\infty}\frac{(-e_2/4)^{\daleth-1}}{(t_2)_{\daleth-1}(\daleth-1)!}\right\}
$$

$$
= \frac{1}{2}\Big\{(1+\mathcal{W})\eth_1''(1)+(1+2\mathcal{W}-\mathcal{E})\eth_1'(1)-(1+\mathcal{E})\eth_1(1)
$$

$$
+(1+\mathcal{W})\eth_2''(1)+(1+\mathcal{E})\eth_2'(1)-(1+\mathcal{E})\eth_2(1)\Big\}
$$

$$
= \frac{1}{2}\Big\{(1+\mathcal{W})(u_{p_1}''(1)+2u_{p_1}'(1))+(1+2\mathcal{W}-\mathcal{E})(u_{p_1}'(1)+u_{p_1}(1)-1)-(1+\mathcal{E})(u_{p_1}(1)-1)
$$

$$
+(1+\mathcal{W})(u_{p_2}''(1)+2u_{p_2}'(1))+(1+\mathcal{E})(u_{p_2}'(1)+u_{p_2}(1)-1)-(1+\mathcal{E})(u_{p_2}(1)-1)\Big\}\le\mathcal{W}-\mathcal{E}.
$$

This completes the proof.

Remark 3.2. Setting $\mathcal{W}=1$ and $\mathcal{E}=\gamma$ in Theorem 3.1, we refine (Setting $\lambda=1$) the conclusion established in ([20], Theorem 3.3).

Theorem 3.3. *If*

$$\begin{aligned} &2(1+\mathcal{W})u_{p_1}^{'''}(1)+(15\mathcal{W}-2\mathcal{E}+13)u_{p_1}^{''}(1)+(24\mathcal{W}-9\mathcal{E}+15)u_{p_1}^{'}(1)+6(\mathcal{W}-\mathcal{E})u_{p_1}(1) \\ &\quad +2(1+\mathcal{W})u_{p_2}^{'''}(1)+(9\mathcal{W}+2\mathcal{E}+11)u_{p_2}^{''}(1)+(6\mathcal{W}+3\mathcal{E}+9)u_{p_2}^{'}(1)\leq 12(\mathcal{W}-\mathcal{E}), \end{aligned} \tag{3.3}$$

then

$$\Upsilon(\mathcal{S}_{\mathcal{H}}^{*,0})\subset \mathcal{S}_{\mathcal{H}}^{*}(\mathcal{E},\mathcal{W}) \quad and \quad \Upsilon(C_{\mathcal{H}}^{0})\subset \mathcal{S}_{\mathcal{H}}^{*}(\mathcal{E},\mathcal{W}).$$

Proof. Let $m=n+\overline{v}\in \mathcal{S}_{\mathcal{H}}^{*,0}$ where h and g are represented in (1.2) with $b_1=0$. It remains to prove that $\Upsilon(m)=H+\overline{G}\in \mathcal{S}_{\mathcal{H}}^{*}(\mathcal{E},\mathcal{W})$, which given by (2.5) with $b_1=0$. Lemma 2.4 reduces the proof to verifying that □

$$\begin{aligned} P_1 &= \sum_{\daleth=2}^{\infty}\left[\daleth(1+\mathcal{W})-(1+\mathcal{E})\right]\frac{(-e_1/4)^{\daleth-1}}{(t_1)_{\daleth-1}(\daleth-1)!}|a_\daleth| \\ &\quad +\sum_{\daleth=2}^{\infty}\left[\daleth(1+\mathcal{W})+(1+\mathcal{E})\right]\frac{(-e_2/4)^{\daleth-1}}{(t_2)_{\daleth-1}(\daleth-1)!}|b_\daleth|\leq \mathcal{W}-\mathcal{E}. \end{aligned} \tag{3.4}$$

By using Lemma 2.2

$$\begin{aligned} P_1 \leq \frac{1}{6}\Bigg\{&\sum_{\daleth=2}^{\infty}(2\daleth+1)(\daleth+1)\left[\daleth(1+\mathcal{W})-(1+\mathcal{E})\right]\frac{(-e_1/4)^{\daleth-1}}{(t_1)_{\daleth-1}(\daleth-1)!} \\ &+\sum_{\daleth=2}^{\infty}(2\daleth-1)(\daleth-1)\left[\daleth(1+\mathcal{W})+(1+\mathcal{E})\right]\frac{(-e_2/4)^{\daleth-1}}{(t_2)_{\daleth-1}(\daleth-1)!}\Bigg\} \end{aligned}$$

$$\begin{aligned} =\frac{1}{2}\Bigg\{&\sum_{\daleth=2}^{\infty}\left[2(1+\mathcal{W})\daleth^3+(3\mathcal{W}-2\mathcal{E}+1)\daleth^2+(\mathcal{W}-3\mathcal{E}-2)\daleth-(1+\mathcal{E})\right]\frac{(-e_1/4)^{\daleth-1}}{(t_1)_{\daleth-1}(\daleth-1)!} \\ &+\sum_{\daleth=2}^{\infty}\left[2(1+\mathcal{W})\daleth^3-(3\mathcal{W}-2\mathcal{E}+1)\daleth^2+(\mathcal{W}-3\mathcal{E}-2)\daleth+(1+\mathcal{E})\right]\frac{(-e_2/4)^{\daleth-1}}{(t_2)_{\daleth-1}(\daleth-1)!}\Bigg\}. \end{aligned}$$

Writing $\daleth^3=\daleth(\daleth-1)(\daleth-2)+3\daleth(\daleth-1)+\daleth$ and $\daleth^2=\daleth(\daleth-1)+\daleth$, we have

$$
\begin{aligned}
P_1 = \frac{1}{6}\Bigg\{ & 2(1+\mathcal{W})\sum_{\daleth=2}^{\infty}\frac{\daleth(\daleth-1)(\daleth-2)(-e_1/4)^{\daleth-1}}{(t_1)_{\daleth-1}(\daleth-1)!} + (9\mathcal{W}-2\mathcal{E}+7)\sum_{\daleth=2}^{\infty}\frac{\daleth(\daleth-1)(-e_1/4)^{\daleth-1}}{(t_1)_{\daleth-1}(\daleth-1)!} \\
& + (6\mathcal{W}-5\mathcal{E}+1)\sum_{\daleth=2}^{\infty}\frac{\daleth(-e_1/4)^{\daleth-1}}{(t_1)_{\daleth-1}(\daleth-1)!} - (1+\mathcal{E})\sum_{\daleth=2}^{\infty}\frac{(-e_1/4)^{\daleth-1}}{(t_1)_{\daleth-1}(\daleth-1)!} \\
& + 2(1+\mathcal{W})\sum_{\daleth=2}^{\infty}\frac{\daleth(\daleth-1)(\daleth-2)(-e_2/4)^{\daleth-1}}{(t_2)_{\daleth-1}(\daleth-1)!} + (3\mathcal{W}+2\mathcal{E}+5)\sum_{\daleth=2}^{\infty}\frac{\daleth(\daleth-1)(-e_2/4)^{\daleth-1}}{(t_2)_{\daleth-1}(\daleth-1)!} \\
& - (1+\mathcal{E})\sum_{\daleth=2}^{\infty}\frac{\daleth(-e_2/4)^{\daleth-1}}{(t_2)_{\daleth-1}(\daleth-1)!} + (1+\mathcal{E})\sum_{\daleth=2}^{\infty}\frac{(-e_2/4)^{\daleth-1}}{(t_2)_{\daleth-1}(\daleth-1)!}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{6}\Big\{ & 2(1+\mathcal{W})\eth_1^{'''}(1) + (9\mathcal{W}-2\mathcal{E}+7)\eth_1^{''}(1) + (6\mathcal{W}-5\mathcal{E}+1)\eth_1^{'}(1) - (1+\mathcal{E})\eth_1(1) \\
& + 2(1+\mathcal{W})\eth_2^{'''}(1) + (3\mathcal{W}+2\mathcal{E}+5)\eth_2^{''}(1) - (1+\mathcal{E})\eth_2^{'}(1) + (1+\mathcal{E})\eth_2(1)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{6}\Big\{ & 2(1+\mathcal{W})(u_{p_1}^{'''}(1) + 3u_{p_1}^{''}(1)) + (9\mathcal{W}-2\mathcal{E}+7)(u_{p_1}^{''}(1) + 2u_{p_1}^{'}(1)) \\
& + (6\mathcal{W}-5\mathcal{E}+1)(u_{p_1}^{'}(1) + u_{p_1}(1) - 1) - (1+\mathcal{E})(u_{p_1}(1) - 1) \\
& + 2(1+\mathcal{W})(u_{p_2}^{'''}(1) + 3u_{p_2}^{''}(1)) + (3\mathcal{W}+2\mathcal{E}+5)(u_{p_2}^{''}(1) + 2u_{p_2}^{'}(1)) \\
& -(1+\mathcal{E})(u_{p_2}^{'}(1) + u_{p_2}(1) - 1) + (1+\mathcal{E})(u_{p_2}(1) - 1)\Big\} \leq \mathcal{W} - \mathcal{E}
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{6}\Big\{ & 2(1+\mathcal{W})u_{p_1}^{'''}(1) + (15\mathcal{W}-2\mathcal{E}+13)u_{p_1}^{''}(1) + (24\mathcal{W}-9\mathcal{E}+15)u_{p_1}^{'}(1) + 6(\mathcal{W}-\mathcal{E})u_{p_1}(1) \\
& + 2(1+\mathcal{W})u_{p_2}^{'''}(1) + (9\mathcal{W}+2\mathcal{E}+11)u_{p_2}^{''}(1) + (6\mathcal{W}+3\mathcal{E}+9)u_{p_2}^{'}(1) - 6(\mathcal{W}-\mathcal{E})\Big\} \leq \mathcal{W} - \mathcal{E}.
\end{aligned}
$$

This completes the proof.

Remark 3.4. Setting $\mathcal{W} = 1$ and $\mathcal{E} = \gamma$ in Theorem 3.3, we refine (setting $\lambda = 1$) the conclusion established in ([20], Theorem 3.5).

To clarify the relationship between the classes $\mathcal{RT}_{\mathcal{H}}(\mathcal{E},\mathcal{W})$ and $\mathcal{S}^*_{\mathcal{H}}(\mathcal{E},\mathcal{W})$, we shall make use of the following conclusion:

Lemma 3.5. *[20] If $e < 0$ and $\daleth > 0$, then*

$$
\sum_{\daleth=0}^{\infty}\frac{(-e/4)^{\daleth}}{(t)_{\daleth}(\daleth+1)!} = \frac{-4(t-1)}{e}[u_{p-1}(1) - 1].
$$

Proof. We can write

$$\begin{aligned}\sum_{\daleth=0}^{\infty}\frac{(-e/4)^{\daleth}}{(t)_{\daleth}(\daleth+1)!}&=\frac{(t-1)}{(-e/4)}\sum_{\daleth=0}^{\infty}\frac{(-e/4)^{\daleth+1}}{(t-1)_{\daleth+1}(\daleth+1)!}\\&=\frac{-4(t-1)}{e}[u_{p-1}(1)-1].\end{aligned}$$

□

Theorem 3.6. *If*

$$\begin{aligned}(\mathcal{W}-\beta)\Bigg\{[u_{p_1}(1)-1]+\frac{4(1+\mathcal{E})}{(1+\mathcal{W})}\frac{(t_1-1)}{e_1}\left[u_{p_1-1}(1)-1-\frac{(-e_1/4)}{t_1-1}\right]\\+u_{p_2}(1)-\frac{4(1+\mathcal{E})}{(1+\mathcal{W})}\frac{(t_2-1)}{e_2}\left[u_{p_2-1}(1)-1\right]\Bigg\}\le\mathcal{W}-\mathcal{E},\end{aligned}$$

then

$$\Upsilon(\mathcal{RT}_{\mathcal{H}}(\beta,\mathcal{W}))\subset\mathcal{S}^{*}_{\mathcal{H}}(\mathcal{E},\mathcal{W}).$$

Proof. Let $m=n+\overline{v}\in\mathcal{RT}_{\mathcal{H}}(\beta,\mathcal{W})$ where n and v are of the (1.2). Lemma 2.4 allows us to complete the proof by verifying that $P_2\le\mathcal{W}-\mathcal{E}$, where □

$$\begin{aligned}P_2=&\sum_{\daleth=2}^{\infty}[\daleth(1+\mathcal{W})-(1+\mathcal{E})]\frac{(-e_1/4)^{\daleth-1}}{(t_1)_{\daleth-1}(\daleth-1)!}|a_{\daleth}|\\&+\sum_{\daleth=1}^{\infty}[\daleth(1+\mathcal{W})+(1+\mathcal{E})]\frac{(-e_2/4)^{\daleth-1}}{(t_2)_{\daleth-1}(\daleth-1)!}|b_{\daleth}|.\end{aligned}\tag{3.5}$$

Using Remark 2.10, we have

$$\begin{aligned}P_2\le(\mathcal{W}-\beta)\Bigg\{&\sum_{\daleth=2}^{\infty}\left(1-\frac{1+\mathcal{E}}{\daleth(1+\mathcal{W})}\right)\frac{(-e_1/4)^{\daleth-1}}{(t_1)_{\daleth-1}(\daleth-1)!}\\&+\sum_{\daleth=1}^{\infty}\left(1+\frac{1+\mathcal{E}}{\daleth(1+\mathcal{W})}\right)\frac{(-e_2/4)^{\daleth-1}}{(t_2)_{\daleth-1}(\daleth-1)!}\Bigg\}\end{aligned}$$

$$= (\mathcal{W}-\beta)\left\{\sum_{\daleth=0}^{\infty}\frac{(-e_1/4)^{\daleth+1}}{(t_1)_{\daleth+1}(\daleth+1)!}-\frac{1+\mathcal{E}}{1+\mathcal{W}}\sum_{\daleth=0}^{\infty}\frac{(-e_1/4)^{\daleth+1}}{(t_1)_{\daleth+1}(\daleth+1)!}\right.$$
$$\left.+\sum_{\daleth=0}^{\infty}\frac{(-e_2/4)^{\daleth}}{(t_2)_{\daleth}\daleth!}+\frac{1+\mathcal{E}}{1+\mathcal{W}}\sum_{\daleth=0}^{\infty}\frac{(-e_2/4)^{\daleth}}{(t_2)_{\daleth}(\daleth+1)!}\right\}$$

$$= (\mathcal{W}-\beta)\left\{[u_{p_1}(1)-1]-\frac{(1+\mathcal{E})}{(1+\mathcal{W})}\frac{(t_1-1)}{(-e_1/4)}\sum_{\daleth=0}^{\infty}\frac{(-e_1/4)^{\daleth+2}}{(t_1-1)_{\daleth+2}(\daleth+2)!}\right.$$
$$\left.+\,u_{p_2}(1)+\frac{(1+\mathcal{E})}{(1+\mathcal{W})}\frac{(t_2-1)}{(-e_2/4)}\sum_{\daleth=0}^{\infty}\frac{(-e_2/4)^{\daleth+1}}{(t_2-1)_{\daleth+1}(\daleth+1)!}\right\}$$

$$= (\mathcal{W}-\beta)\left\{[u_{p_1}(1)-1]+\frac{(1+\mathcal{E})}{(1+\mathcal{W})}\frac{4(t_1-1)}{e_1}\left[u_{p_1-1}(1)-1-\frac{(-e_1/4)}{t_1-1}\right]\right.$$
$$\left.+\,u_{p_2}(1)-\frac{(1+\mathcal{E})}{(1+\mathcal{W})}\frac{4(t_2-1)}{e_2}\left[u_{p_2-1}(1)-1\right]\right\}\le \mathcal{W}-\mathcal{E},$$

by the given hypothesis.

Remark 3.7. setting $\mathcal{W}=1$ and $\mathcal{E}=\gamma$ in Theorem 3.6, we refine (setting $\lambda=1$) the conclusion established in ([20], Theorem 3.7).

Theorem 3.8. *If*

$$u_{p_1}(1)+u_{p_2}(1)\le 2, \tag{3.6}$$

then

$$\Upsilon(\mathcal{S}^*_{\mathcal{H}}(\mathcal{E},\mathcal{W}))\subset \mathcal{S}^*_{\mathcal{H}}(\mathcal{E},\mathcal{W}).$$

Proof. By applying Lemma 1.3, the proof reduces to showing that $P_2\le \mathcal{W}-\mathcal{E}$. □

$$P_2=\sum_{\daleth=2}^{\infty}[\daleth(1+\mathcal{W})-(1+\mathcal{E})]\,\frac{(-e_1/4)^{\daleth-1}}{(t_1)_{\daleth-1}(\daleth-1)!}|a_\daleth|$$
$$+\sum_{\daleth=1}^{\infty}[\daleth(1+\mathcal{W})+(1+\mathcal{E})]\,\frac{(-e_2/4)^{\daleth-1}}{(t_2)_{\daleth-1}(\daleth-1)!}|b_\daleth|.$$

Using Remark 2.6, we have

$$P_2 \le (\mathcal{W}-\mathcal{E})\left[\sum_{\daleth=2}^{\infty}\frac{(-e_1/4)^{\daleth-1}}{(t_1)_{\daleth-1}(\daleth-1)!}+\sum_{\daleth=1}^{\infty}\frac{(-e_2/4)^{d-1}}{(t_2)_{\daleth-1}(\daleth-1)!}\right]$$
$$=(\mathcal{W}-\mathcal{E})\left[\sum_{\daleth=0}^{\infty}\frac{(-e_1/4)^{\daleth+1}}{(t_1)_{\daleth+1}(\daleth+1)!}+\sum_{\daleth=0}^{\infty}\frac{(-e_2/4)^{\daleth}}{(t_2)_{\daleth}\daleth!}\right]$$
$$=(\mathcal{W}-\mathcal{E})\left[u_{p_1}(1)-1+u_{p_2}(1)\right]$$

by applying the criteria given in (3.6), we obtain $P_2 \le \mathcal{W}-\mathcal{E}$, which completes the proof.

Remark 3.9. Setting $\mathcal{W}=1$ and $\mathcal{E}=\gamma$ in Theorem 3.4, we refine (setting $\lambda=1$) the conclusion established in ([20], Theorem 3.8).

Theorem 3.10. *If*

$$u_{p_1}(1)+u_{p_2}(1)\le 2, \tag{3.7}$$

then

$$\Upsilon(\mathcal{ST}^*_{\mathcal{H}}(\mathcal{E},\mathcal{W}))\subset \mathcal{ST}^*_{\mathcal{H}}(\mathcal{E},\mathcal{W}).$$

Proof. The proof proceeds along the same lines as that of Theorem 3.8; therefore, the details are omitted. □

□

4. Conclusion

By employing the operator Υ defined in (2.4), related to the generalized Bessel function, we establish several inclusion relationships for the harmonic class $\mathcal{ST}^*_{\mathcal{H}}(\mathcal{E},\mathcal{W})$ and other classes of harmonic analytic functions defined in $U=\{\xi\in C:|\xi|<1\}$. Furthermore, we highlighted certain connections between Janowski-type harmonic function classes and generalized Bessel functions. Building on the conclusions of this study. The operator Υ provides a useful framework for establishing further inclusion properties of newly defined subclasses of harmonic analytic functions.

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