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Kernel Integral Legendre Polynomials and Approximations

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Abstract

This paper is concerned with deriving a new system of orthogonal polynomials whose inflection points coincide with their interior roots, primitives of Legendre polynomials,they appear as solutions of linear differential equation. We study orthogonality, and extremal properties and minimization and Fourier development involving of integral Legendre polynomials. There are some important properties and certain identities and extremal properties involving both associated integral Legendre polynomials. We have used mathematical induction to establish the relation between them. Also, we present some results for these orthogonal polynomials by using some properties of Jacobi polynomials. General expressions are found for the kernels polynomials associated to integral Legendre polynomials. These kernel polynomials can be used to describe the approximation of continuous functions by integral Legendre polynomials. They can be used for the representation of the n-th partial sum of the Fourier series expansion of integral Legendre polynomials in the form of an integral. We conclude the paper with some results on finite Fourier series expansion by using polynomials integrals of the kernels polynomials.

Keywords: Legendre polynomials, Integral Legendre polynomials, Kernel polynomials, Christoffel-Darboux formula, Fourier series expansion, Best approximation problems.
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1. Mathematical basis

Associated orthogonal polynomials are a family of polynomials derived from a given set of orthogonal Legendre polynomials by integrating polynomials or shifting the indices in the recurrence coefficients. Specifically, these associated polynomials are also orthogonal, but with respect to a possibly different measure. Orthogonal polynomials are of considerable importance properties in many branches of science and engineering since they represent an indispensable analytical tool for solving various extremal and minimization and approximation problems and Fourier developments [2, 3, 4, 5, 6, 7, 9, 12, 13].

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In this paper we study a system of orthogonal polynomials $\{Q_n\}_{n=2,3,\cdots}$with respect to the measure $\rho(x) = \dfrac{1}{1-x^2}$, $x \in [-1, 1]$, that we call integral of Legendre polynomials, with the first and last roots at $x = \pm 1$, given by

$$Q_n(x) = \left(1 - x^2\right) q_{n-2}(x) \ , \quad n = 2, 3, \cdots .$$

where $q_n(x)$, $n = 0, 1, 2, \cdots$ is a polynomial of degree n.Using these orthogonal polynomials we study a system of orthogonal polynomials $\{q_n\}_{n=0,1,2,\cdots}$ with respect to the measure $w(x) = 1 - x^2$, $x \in [-1, 1]$ associated to Legendre polynomials.The paper will be structured as follows : in section 1, we present some useful terminology as well as some necessary definitions regarding integral Legendre polynomials. In section 2, we present some properties of orthogonal polynomials $\{q_n(x)\}_n$, second, we give some necessary definitions and basic properties of two Fourier polynomials approximability for the polynomials $\{q_n(x)\}_{n=0,1,2,\cdots}$ and for $\{Q_n\}_{n=2,3,\cdots}$ best solutions for two associated extremal problems in respect to orthogonal polynomials $\{Q_n\}_{n=2,3,\cdots}$ and $\{q_n\}_{n=0,1,2,\cdots}$.

These orthogonal polynomials can be used to describe the approximation of continuous functions by integral Legendre polynomials and orthogonal polynomials $\{q_n\}_{n=0,1,2,\cdots.}$ by finite Fourier series and how to compute efficiently such approximations. We show a connection between these orthogonal polynomials and two special cases of Jacobi polynomials, we derive structures relations between Q_n and $P_n^{(\alpha,\beta)}(x)$, for $\alpha = \beta = -1$ we derive also some structures relations between q_n and $P_n^{(\alpha,\beta)}(x)$, for $\alpha = \beta = 1$.All of them may be transferred into formulas for integral Legendre polynomials, we give some necessary definitions and basic extremal and approximations properties of the q-kernel orthogonal polynomials,where $K_n(x, y)$ is the Christoffel-Darboux formula for the polynomials $\{q_n(x)\}$, $n = 0, 1, 2, \cdots$

$$K_n(x, y) = \sum_{k=0}^{n} \frac{q_k(x) q_k(y)}{\|q_k\|_w^2}$$

$\{K_n(x, \lambda)\}_{n=0,1,2,\cdots}$ are orthogonal polynomials on the interval $[-1, 1]$, with respect to the weight function $x \longmapsto (x - \lambda) w(x)$,where $-1 \leq \lambda \leq 1$.

We conclude the paper with some results concerning polynomials integrals of the q-kernels polynomials. In addition, some comparisons with some other methods are made.

Let $\{L_n\}, n = 0, 1, 2, \cdots$ be the Legendre polynomials. They satisfies the orthogonality relation ,

$$\int_{-1}^{1} L_n(x) x^k dx = 0 \quad , \qquad k = 0, 1, 2, \cdots, n-1,$$

and the differential equation

$$\left(\left(1 - x^2\right) L_n'(x)\right)' + n(n+1) L_n(x) = 0 \tag{1.1}$$

Explicit formula for $L_n(x)$ is given as follows, [1, 8, 14, 16, 17]

$$L_n(x) = \frac{1}{(2n)!} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (2n-2k)!}{k!\,(n-k)!\,(n-2k)!} x^{n-2k} \quad , \qquad n \geq 2, \tag{1.2}$$

they are normalized so that $L_n(1) = 1$ for all n.

An immediate consequence of (1.1) is the following second expansion of Q_n , we have

$$Q_n(x) = -\int_{x}^{1} L_{n-1}(t)\, dt \quad , \qquad n = 2, 3, \cdots .$$

We shall consider some and different class of Jacobi polynomials denoted by $P_n^{(\alpha,\beta)}(x)$, orthogonal with respect to the Jacobi weight function $w^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$ over $I = [-1,1]$ [1, 5, 8, 11, 14, 16, 17], namely

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\, P_m^{(\alpha,\beta)}(x)\, w^{\alpha,\beta}(x)\, dx = \delta_{mn} \left\| P_n^{(\alpha,\beta)}(x) \right\|^2 \quad , \tag{1.3}$$

where $\delta_{m,n}$ is the Kronecker symbol, where

$$\left\| P_n^{(\alpha,\beta)}(x) \right\|^2 = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{n!\,\Gamma(n+\alpha+\beta+1)}, \tag{1.4}$$

$P_n^{(\alpha,\beta)}$ are normalized according to

$$P_n^{(\alpha,\beta)}(1) = C_n^{n+\alpha} \quad , \quad n = 0,1,2,3,\cdots \tag{1.5}$$

where $C_n^{n+\alpha}$ is a binomial coefficient. The weight function $w^{\alpha,\beta}$ belongs to $L^1([-1,1])$, if and only if $\alpha,\, \beta > -1$ (to be assumed throughout this section). i-e

$$\int_{-1}^{1} w^{\alpha,\beta}(x)\, dx < \infty$$

The coefficient of term x^n in $P_n^{(\alpha,\beta)}(x)$ is given by, [16] :

$$k_n^{\alpha,\beta} = \frac{\Gamma(2n+\alpha+\beta+1)}{2^n n!\,\Gamma(n+\alpha+\beta+1)} \tag{1.6}$$

We shall need the following important derivative relation of Jacobi polynomials, [16, 17] :

$$\frac{\partial}{\partial x} P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n+\alpha+\beta+1)\, P_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{1.7}$$

and

$$(1+x)\frac{\partial}{\partial x} P_n^{(\alpha,\beta)}(x) = (n+\beta)\, P_n^{(\alpha+1,\beta-1)}(x) - \beta P_n^{(\alpha,\beta)}(x).$$

Applying this formula recursively yields, for $k = 0,1,\cdots,n$.

$$\frac{\partial^k}{\partial x^k} P_n^{(\alpha,\beta)}(x) = \frac{1}{2^k} \frac{\Gamma(n+k+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)} P_{n-k}^{(\alpha+k,\beta+k)}(x) \tag{1.8}$$

The Jacobi polynomials are the eigenfunctions of a singular Sturm-Liouville operator, [8, 9, 14, 16, 17] defined by

$$\left(\left(x^2-1\right) \frac{\partial^2}{\partial x^2} + (\alpha - \beta + (\alpha+\beta+2)\,x) \frac{\partial}{\partial x} \right) P_n^{(\alpha,\beta)}(x) = \lambda_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(x) \tag{1.9}$$

where

$$\lambda_n^{(\alpha,\beta)} = n(n+\alpha+\beta+1) \tag{1.10}$$

If

$$K_n(x,y) = \sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(y)}{\left\| P_k^{(\alpha,\beta)} \right\|^2} \tag{1.11}$$

The sequence $(K_n(x,y))_{n=0}^{\infty}$ satisfies the Christoffel-Darboux formula, [5, 7, 9, 11, 14, 16, 17]

$$K_n(x,y) = \frac{k_n^{\alpha,\beta}}{k_{n+1}^{\alpha,\beta}} \frac{1}{\left\| P_n^{(\alpha,\beta)} \right\|^2} \frac{P_{n+1}^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y) - P_{n+1}^{(\alpha,\beta)}(y) P_n^{(\alpha,\beta)}(x)}{x-y}, \quad x \neq y$$

Moreover $\{K_n(x,\lambda)\}_{n=0,1,2,\cdots}$ are orthogonal with respect to $(x-\lambda)\, w^{\alpha,\beta}(x)$, where $-1 \le \lambda \le 1$.

A remarkable property of $\{K_n(x,y)\}_{n=0,1,2,\cdots}$ is stated in the following property. K_n has the reproducing kernel property [1, 5, 7, 9, 11, 14, 16, 17] :

$$f(x) = \int_{-1}^{1} K_n(x,t)\, f(t)\, w^{\alpha,\beta}(t)\, dt$$

for any continuous function f on the interval $[-1,1]$.

1.1. Approximability of Jacobi polynomials

Theorem 1.1. *Denote by Π_n the set of all algebraic polynomials of degree $\le n$, and*

$$L^2\left([-1,1], w^{\alpha,\beta}\right) = \left\{ f \ : \int_{-1}^{1} (f(x))^2\, w^{\alpha,\beta}(x)\, dx < \infty \right\}$$

For any $f \in L^2\left([-1,1], w^{\alpha,\beta}\right)$ and $n \in mathbbN$, there exists a unique $S_n^* \in \Pi_n$, such that

$$\|f - S_n^*\|^2 = Inf\left\{ \|f - S_n\|^2 \ , \ S_n \in \Pi_n \right\} \tag{1.12}$$

where

$$S_n^*(x) = \sum_{k=0}^{n} c_k P_k^{(\alpha,\beta)}(x) \tag{1.13}$$

with

$$c_k = \frac{\int_{-1}^{1} f(t)\, P_k^{(\alpha,\beta)}(t)\, w^{\alpha,\beta}(t)\, dt}{\left\| P_k^{(\alpha,\beta)} \right\|^2} \tag{1.14}$$

Proof. In the Jacobi case,[1, 5, 9, 11, 14, 16, 17, 10]. In particular, we denote the best approximation polynomial S_n^* by $pr(f)$, which is the $L^2_{w^{\alpha,\beta}}$-orthogonal projection of f , and is characterized by the projection theorem

$$\|f - S_n^*\| = Inf\left\{ \|f - S_n\|^2 \ , \ S_n \in \Pi_n \right\} \tag{1.15}$$

Equivalently, the $L^2_{w^{\alpha,\beta}}$-orthogonal projection can be defined by :

$$\langle f - S_n^* \ , \ \varphi \rangle_{w^{\alpha,\beta}} = 0 \quad , \quad \forall \varphi \in \Pi_n$$

i.e.,

$$\int_{-1}^{1} (f(t) - S_n^*(t))\, \varphi(t)\, w^{\alpha,\beta}(t)\, dt = 0 \quad , \quad \forall \varphi \in \Pi_n$$

So S_n^* is the first $n+1$-term truncation of the Fourier series:

$$f(x) = \sum_{k=0}^{\infty} c_k P_k^{(\alpha,\beta)}(x)$$

□

Theorem 1.2. *[16]The Jacobi polynomial* $P_n^{(\alpha+1,\beta)}$ *is a linear combination of* $P_k^{(\alpha,\beta)}$, $k = 0, 1, 2, \cdots, n$.

$$P_n^{(\alpha+1,\beta)}(x) = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+2)} \sum_{k=0}^{n} (2k+\alpha+\beta+1) \frac{\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\beta+1)} P_k^{(\alpha,\beta)}(x). \tag{1.16}$$

Proof. In the Jacobi case, the kernel polynomial at $x = 1$, see (1.11), takes the form

$$K_n(x,1) = \sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(1)}{\left\| P_k^{(\alpha,\beta)} \right\|^2}$$

The Christoffel-Darboux formula, (1.11)

$$K_n(x,1) = \frac{k_n^{\alpha,\beta}}{k_{n+1}^{\alpha,\beta}} \frac{1}{\left\| P_n^{(\alpha,\beta)} \right\|^2} \frac{P_{n+1}^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(1) - P_{n+1}^{(\alpha,\beta)}(1) P_n^{(\alpha,\beta)}(x)}{x-1}$$

By lemma [16, 17], $\{K_n(x,1)\}_n$ are orthogonal with respect to $w^{\alpha+1,\beta}(x)$. By the uniqueness of orthogonal polynomials, [17] $K_n(x,1)$ must be proportional to $P_n^{(\alpha+1,\beta)}(x)$ i-e.

$$K_n(x,1) = \sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(1)}{\left\| P_k^{(\alpha,\beta)} \right\|^2} = a_n^{\alpha,\beta} P_n^{(\alpha+1,\beta)}(x) \tag{1.17}$$

The proportionality constant $a_n^{\alpha,\beta}$ is determined by comparing the leading coefficients of both sides of (1.17) and working out the constants, namely,

$$\frac{k_n^{\alpha,\beta}}{\left\| P_n^{(\alpha,\beta)} \right\|^2} P_n^{(\alpha,\beta)}(1) = a_n^{\alpha+1,\beta} k_n^{\alpha++1,\beta}$$

we find that

$$a_n^{\alpha+1,\beta} = \frac{k_n^{\alpha,\beta}}{k_n^{\alpha++1,\beta}} \left\| P_n^{(\alpha,\beta)} \right\|^{-2} P_n^{(\alpha,\beta)}(1)$$

Using properties (1.4),(1.5) and(1.6), yields

$$a_n^{\alpha,\beta} = 2^{-\alpha-\beta-1} \frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)}$$

Inserting this constant into (1.17),we obtain (1.16) directly from (1.5) and(1.4). This ends the proof.

Remark 1.3. Thanks to the property

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\alpha,\beta)}(x)$$

it follows from (1.16) that

$$P_n^{(\alpha,\beta+1)}(x) = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+2)} \sum_{k=0}^{n} (-1)^k (2k+\alpha+\beta+1) \frac{\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\alpha+1)} P_k^{(\alpha,\beta)}(x) \tag{1.18}$$

□

There are many important properties for Legendre and Jacobi polynomials,[1, 17]. All of them may be transferred into formulas for Integral of Legendre polynomials.

2. Some families of orthogonal polynomials associated with Legendre polynomials

The Legendre polynomial,$L_n(x)$ satisfies,[1, 5, 7, 9, 11, 14, 16, 17] the orthogonality relation:

$$\int_{-1}^{1} L_n(x)\, x^k dx = 0, \quad k = 0, 1, 2, \cdots, n-1.$$

and

$$\|L_n\|^2 = \int_{-1}^{1} L_n^2(x)\, dx = \frac{2}{2n+1}, \quad n = 0, 1, 2, \cdots \tag{2.1}$$

The Legendre polynomial L_n satisfies the differential equation,[1, 5, 7, 9, 11, 15, 14, 16, 17] :

$$\left(\left(1-x^2\right) L_n'(x)\right)' = -n(n+1) L_n(x) \tag{2.2}$$

The three-term recurrence relation for the Legendre polynomials reads,[1, 5, 7, 11, 15, 14, 16, 17]

$$L_0(x) = 1, \ L_1(x) = x,$$

and

$$(n+1) L_{n+1}(x) = (2n+1) x L_n(x) - n L_{n-1}(x) \quad n = 1, 2, 3, \cdots$$

We also derive that [1, 17]

$$L_n(\pm 1) = (\pm 1)^n \tag{2.3}$$

in the same way :

$$L_n'(\pm 1) = \frac{1}{2} (\pm 1)^{n-1} n(n+1) \tag{2.4}$$

2.1. Integral of Legendre polynomials

We restricted our attention to a polynomial $Q_n(x)$ with the first and last roots at $x = \pm 1$,given by

$$Q_n(x) = \left(x^2 - 1\right) q_{n-2}(x), \quad n \geq 2, \tag{2.5}$$

and satisfies the differential equation:

$$\left(1 - x^2\right) Q_n''(x) + n(n-1) Q_n(x) = 0$$

Straightforwardly,

$$Q_n'(x) = L_{n-1}(x) \quad \text{and} \quad Q_n''(x) = L_{n-1}'(x) \quad , \quad n \geq 2, \tag{2.6}$$

and

$$Q_n(\pm 1) = 0 \quad , \quad n \geq 2 \tag{2.7}$$

also we have,

$$Q_n'(1) = 1 \text{ , and } \quad Q_n'(-1) = (-1)^{n-1} \quad , \quad n \geq 2. \tag{2.8}$$

Theorem 2.1. *If*

$$Q_n(x) = -\int_{x}^{1} L_{n-1}(t)\, dt \quad , \ -1 \leq x \leq 1 \tag{2.9}$$

Integral Legendre polynomials are particular solutions of the differential equation

$$\left(1 - x^2\right) Q_n''(x) + n(n-1) Q_n(x) = 0, \quad n \geq 2 \tag{2.10}$$

therefore,

$$Q_n(x) = \frac{2}{n-1} P_n^{(-1,-1)}(x) \quad , \quad -1 \leq x \leq 1 \tag{2.11}$$

The functions $Q_n(x)$ and $Q_m(x)$ $(n \neq m)$ are orthogonal with respect to the weight function $\rho(x) = \frac{1}{1-x^2}$. Then

$$\int_{-1}^{1} \frac{Q_n(x) Q_m(x)}{1-x^2} dx = 0 \quad , \quad (n \neq m), \quad n, m = 1, 2, \cdots \tag{2.12}$$

and

$$\|Q_n\|_\rho^2 = \int_{-1}^{1} \frac{Q_n^2(x)}{1-x^2} dx = \frac{2}{n(n-1)(2n-1)} \quad , \quad n = 2, 3, \cdots \tag{2.13}$$

Since $w(x)$ is not continuous on $[-1,1]$, but because $Q_n(-1) = 0$ $Q_n(+1) = 0$, all integrals (2.12),(2.13) *are proper.*

Proof. Thanks to,(2.9),(2.2)

$$\left(1-x^2\right) L'_{n-1}(x) = -n(n-1) \int_{x}^{1} L_{n-1}(x) \, dx \tag{2.14}$$

Inserting (2.6) into (2.14) ,we obtain (2.10).Using explicit formula for $L_{n-1}(x)$ in (1.2) and (2.9) we obtain

$$\frac{(-1)^{\frac{n-2}{2}} (n-3)!!}{n!!} = Q_n(0) \tag{2.15}$$

Hence, comparing this equality with (1.9) and (1.10) ,we have for $\alpha = \beta = -1$,

$$(1-x^2)y'' + n(n-1)y = 0$$

The bounded solution of this equation is the Jacobi polynomial $P_n^{(-1,-1)}(x)$.hence, comparing this equality with (6) we have, $P_n^{(-1,-1)}(x) = \lambda_n Q_n(x)$.The Rodrigues' formula for the Jacobi polynomials is stated below,[16], page 72 ,

$$\left(1-x^2\right)^\alpha P_n^{(\alpha,\alpha)}(x) = \frac{(-1)^n}{2^n n!} \left[\left(1-x^2\right)^{n+\alpha}\right]^{(n)} \quad , \quad n \geq 1.$$

If $\alpha = \beta = -1$, it becomes

$$\left(1-x^2\right)^{-1} P_n^{(-1,-1)}(x) = \frac{(-1)^n}{2^n n!} \left[\left(1-x^2\right)^{n-1}\right]^{(n)}$$

Wishing to find $P_n^{(-1,-1)}(0)$, we determine the coefficient of x^n in the binomial $\left(x^2-1\right)^{n-1}$, then evaluate

$$P_n^{(-1,-1)}(0) = \frac{(-1)^{\frac{n-2}{2}} (n-1)!!}{2n!!}$$

The proportionality constant λ_n is determined by comparing the leading coefficients of both sides of Q_n and $P_n^{(-1,-1)}$ working out the constants, namely,

$$P_n^{(-1,-1)}(0) = \frac{n-1}{2} Q_n(0)$$

Hence,

$$Q_n(x) = \frac{2}{n-1} P_n^{(-1,-1)}(x)$$

This ends the proof of (2.11).The Jacobi polynomials $\left\{P_n^{(-1,-1)}\right\}_n$ are orthogonal with respect to the weight function $w(x) = \dfrac{1}{1-x^2}$,and

$$\left\| P_n^{(-1,-1)}(x) \right\|^2 = \frac{n-1}{2n(2n-1)}$$

Thus proprietes,(2.12),and (2.13) are established.Which confirms the desired result. □

Using integral Legendre polynomials ,we study a system of orthogonal polynomials $\{q_n\}_{n=0,1,2,\cdots}$with respect to the measure $w(x) = 1-x^2$, $x \in [-1,1]$ associated to Legendre polynomials. We present some useful terminology as well as some necessary definitions regarding these associated Legendre polynomials, we give some necessary definitions and basic properties of Fourier polynomials approximability for the polynomials $\{q_n(x)\}_{n=0,1,2,\cdots}$. General expressions are found for the q-kernel orthogonal polynomials associated to Legendre polynomials. These kernel polynomials can be used to describe the approximation of continuous functions by $\{q_n(x)\}_{n=0,1,2,\cdots}$. They can be used for the representation of the n-th partial sum of the Fourier series expansion of integral Legendre polynomials in the form of integrals.

2.2. $\{q_n\}_{n=0,1,2,\cdots}$ *orthogonal polynomial*

In fact :

$$Q_{n+2}(x) = \left(1-x^2\right) q_n(x), \quad -1 \le x \le 1, \quad n = 1,2,\cdots$$

Using (2.10) we obtain,

$$Q''_n(x) = -n(n-1)\, q_{n-2}(x) \tag{2.16}$$

Proposition 2.2. *The polynomial $q_n(x)$ of degree n satisfies the differential equation*

$$\left(1-x^2\right) q''_n(x) - 4x q'_n(x) + n(n+3)\, q_n(x) = 0, \quad n \ge 0 \tag{2.17}$$

therefore

$$q_n(x) = -\frac{1}{2(n+1)} P_n^{(1,1)}(x), \quad n = 0,1,2,3,\cdots \tag{2.18}$$

The functions $q_n(x)$ and $q_m(x)$ $(n \neq m)$ are orthogonal with respect to the weight function $w(x) = 1-x^2$. Then

$$\int_{-1}^{1} q_n(x)\, q_m(x) \left(1-x^2\right) dx = 0 \qquad ,(n \neq m) \qquad n,m = 0,1,2,\cdots \tag{2.19}$$

and

$$\|q_n(x)\|_w^2 = \int_{-1}^{1} q_n^2(x) \left(1-x^2\right) dx = \frac{2}{(n+2)(n+1)(2n+3)}, \qquad n = 0,1,2,\cdots \tag{2.20}$$

Proof. Differentiating two times the resulting equality (2.5) we derive,

$$Q''_{n+2}(x) = 2q_n(x\ ) + 4x q'_n(x\ ) + \left(x^2-1\right) q''_n(x\ )$$

If we substitute the formula (2.5) into the equation (2.10), we determine that the polynomial $q_n(x)$ of degree n satisfies the linear differential equation :

$$\left(1-x^2\right) q''_n(x) - 4x q'_n(x) = -n(n+3)\, q_n(x)$$

Hence, comparing this q-linear differential equation with (1.9)and (1.10) , we have for $\alpha = \beta = 1$,

$$(1 - x^2)y'' - 4xy' + n\,(n+3)\,y = 0$$

The bounded solution of this equation is the Jacobi polynomial $P_n^{(1,1)}(x)$.hence, comparing this equality with (2.17) we have, $P_n^{(1,1)}(x) = \mu_n Q_n(x)$.The proportionality constant μ_n is determined by comparing the the two specials values $P_n^{(1,1)}(1)$ and $q_n(1)$ .The normalization of Jacobi polynomials is(1.5) ,$P_n^{(1,1)}(1) = n+1$. Since $q_n(1) = -\frac{1}{2}$, we have:

$$P_n^{(1,1)}(x) = -2\,(n+1)\,q_n(x)$$

Naturally, the Jacobi polynomials $\left\{P_n^{(1,1)}\right\}_n$ are orthogonal with respect to the weight function $w(x) = 1 - x^2$, over $I = [-1, 1]$, according with (1.3),(1.4) we obtain,

$$\left\|P_n^{(1,1)}(x)\right\|_w^2 = \frac{8\,(n+1)}{(2n+3)\,(n+2)}$$

According with (2.18), we have

$$\|q_n(x)\|_w^2 = \frac{1}{4\,(n+1)^2}\left\|P_n^{(1,1)}(x)\right\|_w^2 = \frac{2}{(n+1)\,(n+2)\,(2n+3)}$$

Thus proprieties, (2.19),and (2.20) are established.which confirms the desired result.This ends the proof. □

We state below two useful formulas , we can express P_n^* in terms of p_n^* ,the two Fourier polynomials bests solutions of above extremal problems, are approximability of $\{q_n\}_{n.}$orthogonal polynomials,and integral Legendre polynomials $\{Q_n\}_n$, where the expansion of Fourier coefficients are known as the connection coefficients between them .

Theorem 2.3. *Denote by Π_n the set of all algebraic polynomials of degree $\leq n$. For any $f \in L^2([-1, 1], w(x))$ and $f(1) = f(-1) = 0$.Let σ_n, w_n, γ_n, denote the extremal constants satisfies,*

$$\sigma_n = \|f - p_n^*\|_w^2 = Inf\left\{\|f - p_n\|_w^2\,,\quad p_n \in \Pi_n\right\} \tag{2.21}$$

and

$$\omega_n = \|f - P_n^*\|_\rho^2 = Inf\left\{\|f - P_n\|_\rho^2\,,\quad P_n \in \Pi_n\right\} \tag{2.22}$$

and

$$\gamma_n = \left\|f' - T_n^*\right\|^2 = Inf\left\{\left\|f' - T_n\right\|^2, T_n \in \Pi_n\right\} \tag{2.23}$$

there exists a unique three approximatings Fourier polynomials $p_n^*, P_n^*, T_n^*(x) \in \Pi_n$, such that :

$$p_n^*(x) = \sum_{k=0}^{n} c_k q_k(x) \tag{2.24}$$

and

$$P_n^*(x) = \sum_{k=2}^{n} C_k Q_k(x) \tag{2.25}$$

and

$$T_n^*(x) = \sum_{k=2}^{n} A_k L_k(x) \tag{2.26}$$

We have,

$$c_k = \frac{(k+1)(k+2)(2k+3)}{2} \int_{-1}^{1} f(t)\, Q_{k+2}(t)\, dt, \quad k = 0, 1, 2, \cdots \tag{2.27}$$

and

$$C_k = \frac{k(k-1)(2k-1)}{2} \int_{-1}^{1} f(t)\, q_{k-2}(t)\, dt \quad k = 2, 3, 4, \cdots \tag{2.28}$$

therefore,

$$A_k = \frac{k(k+1)(2k+1)}{2} \int_{-1}^{1} f(t)\, q_{k-1}(t)\, dt \quad k = 1, 2, 3, \cdots \tag{2.29}$$

and

$$\sigma_n = \|f\|_w^2 + 2\sum_{k=0}^{n} \frac{c_k^2}{(k+1)(k+2)(2k+3)} \tag{2.30}$$

also

$$\omega_n = \|f\|_\rho^2 + 2\sum_{k=2}^{n} \frac{C_k^2}{k(k-1)(2k-1)} \tag{2.31}$$

and

$$\gamma_n = \left\|f'\right\|^2 + 2\sum_{k=2}^{n+1} \frac{C_k^2}{2k+1} \tag{2.32}$$

Proof. Since

$$c_k = \frac{\int_{-1}^{1} f(t)\, q_k(t)\, w(t)\, dt}{\|q_k\|^2}$$

Then,

$$c_k = \frac{(k+1)(k+2)(2k+3)}{2} \int_{-1}^{1} f(t)\, q_k(t)\, w(t)\, dt$$

□

Using (2.5), we obtain:

$$c_k = \frac{(k+1)(k+2)(2k+3)}{2} \int_{-1}^{1} f(t)\, Q_{k+2}(t)\, dt$$

Since

$$C_k = \frac{\int_{-1}^{1} f(t)\, \frac{Q_k(t)}{1-t^2} dt}{\|Q_k\|^2}$$

which, together with (2.13), leads to

$$C_k = \frac{k(k-1)(2k-1)}{2} \int_{-1}^{1} f(t)\, q_{k-2}(t)\, dt$$

Using (2.13),(2.22)and (2.20), (2.21), we obtain :

$$\sigma_n = \|f\|_w^2 + 2\sum_{k=0}^{n} \frac{c_k^2}{(k+1)(k+2)(2k+3)}$$

On the other hand,

$$\omega_n = \|f\|_\rho^2 + 2\sum_{k=2}^{n} \frac{C_k^2}{k(k-1)(2k-1)}$$

Since,

$$T_n^*(x) = \sum_{k=2}^{n} A_k L_k(x)$$

we have

$$A_k = \frac{2k+1}{2}\int_{-1}^{1} f'(t)\, L_k(t)\, dt$$

Then we integrate by parts this series termwise from x to 1 , we obtain (use $L_k(1) = 1$, $k = 0, 1, 2...$) ,

$$A_k = \frac{2k+1}{2}\left(f(1)\, L_k(1) - \int_{-1}^{1} f(t)\, L_k'(t)\, dt\right)$$

i-e,

$$A_k = \frac{2k+1}{2}\left(f(1) - \int_{-1}^{1} f(t)\, Q_{k+1}''(t)\, dt\right)$$

According to the property (2.6) (2.10), we obtain

$$A_k = \frac{2k+1}{2}\left(f(1) - k(k+1)\int_{-1}^{1} f(t)\, \frac{Q_{k+1}(t)}{1-t^2} dt\right)$$

Then

$$A_k = \frac{k(k+1)(2k+1)}{2}\int_{-1}^{1} f(t)\, \frac{Q_{k+1}(t)}{1-t^2} dt = C_{k+1}$$

i-e,

$$A_k = \frac{k(k+1)(2k+1)}{2}\int_{-1}^{1} f(t)\, q_{k-1}(t)\, dt$$

Because

$$\gamma_n = \left\|f'\right\|^2 + 2\sum_{k=1}^{n} \frac{A_k^2}{2k+1}$$

According to the property (2.1),(2.23), we obtain

$$\gamma_n = \left\|f'\right\|^2 + 2\sum_{k=2}^{n+1} \frac{C_k^2}{2k+1}$$

where :

$$\left\|f'\right\|^2=\int_{-1}^{1}\left(f'(t)\right)^2dt,$$

Which confirms the desired result.This ends the proof.

Corollary 2.4. *Fourier polynomials $p_n^*, P_n^*, T_n^*(x)$,verifies the the following estimations*

$$|p_n^*(x)|\le\sum_{k=0}^{n}(k+1)(k+2)(2k+3)\frac{(2k+1)!!}{(2k+2)!!}\int_{-1}^{1}|f(t)|\,dt \tag{2.33}$$

and

$$|P_n^*(x)|\le\sum_{k=2}^{n}k(k-1)(2k-1)\frac{(2k-3)!!}{(2k)!!}\int_{-1}^{1}|f(t)|\,dt \tag{2.34}$$

Also we have

$$|T_n^*(x)|\le\frac{1}{4}\sum_{k=2}^{n}k(k+1)(2k+1)\int_{-1}^{1}|f(t)|\,dt \tag{2.35}$$

However, if $\int_{-1}^{1}|f(t)|\,dt=1$ then we have

$$|p_n^*(x)|\le\sum_{k=0}^{n}(k+1)(k+2)(2k+3)\frac{(2k+1)!!}{(2k+2)!!} \tag{2.36}$$

and

$$|P_n^*(x)|\le\sum_{k=2}^{n}k(k-1)(2k-1)\frac{(2k-3)!!}{(2k)!!} \tag{2.37}$$

Also we have

$$|T_n^*(x)|\le\frac{1}{4}\sum_{k=2}^{n}k(k+1)(2k+1) \tag{2.38}$$

In the same way

$$P_n^{*\prime}(x)=\sum_{k=1}^{n-1}C_{k+1}L_k(x) \tag{2.39}$$

and

$$P_n^{*\prime\prime}(x)=-\sum_{k=0}^{n-2}(k+1)(k+2)C_{k+2}q_k(x) \tag{2.40}$$

Therefore

$$\left\|P_n^{*\prime}(x)\right\|^2=\sum_{k=1}^{n-1}\frac{2C_{k+1}^2}{2k+1} \tag{2.41}$$

and

$$\left\|P_n^{*\prime\prime}(x)\right\|_w^2=2\sum_{k=0}^{n-2}\frac{(k+1)(k+2)}{2k+3}C_{k+2}^2 \tag{2.42}$$

Proof. Using the representation of the functions Q_k, q_k, L_k , we can show that for $-1 \leq x \leq 1$ the following estimations hold:

$$|Q_k(x)| \leq 4\frac{(2k-3)!!}{(2k)!!}, \quad k = 2, 3, \cdots$$

and

$$|q_k(x)| \leq \frac{1}{2}, \quad k = 0, 1, 2, \cdots$$

and

$$|L_k(x)| \leq 1, \quad k = 0, 1, 2, \cdots .$$

According to (2.27),(2.28),(2.29) we obtain the following estimations

$$|p_n^*(x)| \leq \sum_{k=0}^{n} (k+1)(k+2)(2k+3)\frac{(2k+1)!!}{(2k+2)!!}\int_{-1}^{1} |f(t)|\, dt$$

and

$$|P_n^*(x)| \leq \sum_{k=2}^{n} k(k-1)(2k-1)\frac{(2k-3)!!}{(2k)!!}\int_{-1}^{1} |f(t)|\, dt$$

and

$$|T_n^*(x)| \leq \frac{1}{4}\sum_{k=2}^{n} k(k+1)(2k+1)\int_{-1}^{1} |f(t)|\, dt$$

Let us differentiate the two sides of equation (2.25) , we get

$$P_n^{*\prime}(x) = \sum_{k=1}^{n-1} C_{k+1}L_k(x)$$

and differentiating the two sides of (2.25) two times ,with according to (2.16) we get

$$P_n^{*\prime\prime}(x) = -\sum_{k=0}^{n-2} (k+1)(k+2)\, C_{k+2}q_k(x)$$

Which confirms the desired results.This ends the proof. □

2.3. q-Kernel polynomials and extremum properties

The n-th q-kernel is given in [14, 16, 17, 10] by

$$K_n(x,y) = \sum_{k=0}^{n} \frac{q_k(x)q_k(y)}{\|q_k\|_w^2} \tag{2.43}$$

The sequence $(K_n(x,y))_{n=0}^{\infty}$ satisfies the Christoffel-Darboux formula [14, 16, 17, 10]

$$K_n(x,y) = \frac{k_n}{k_{n+1}}\frac{1}{\|q_n\|_w^2}\frac{q_{n+1}(x)q_n(y) - q_{n+1}(y)q_n(x)}{x-y}, \qquad x \neq y \tag{2.44}$$

where k_n is the coefficient of x^n in $q_n(x)$. Since,

$$k_n = -\frac{\Gamma(2n+3)}{2^{n+1}(n+1)!\Gamma(n+3)}$$

Hence, we have

$$\frac{k_n}{k_{n+1}} = \frac{n+3}{2n+3} \tag{2.45}$$

For $x = y$ one has

$$K_n\left(x,x\right) = \frac{k_n}{k_{n+1}} \frac{1}{\left\|q_n\right\|_w^2} \left(q'_{n+1}(x)q_n(x) - q_{n+1}(x)q'_n(x)\right) \ .$$

The sequence of kernel polynomials $\left(K_n\left(x,\lambda\right)\right)_{n=0}^{\infty}$, $\lambda \in [-1,1]$ is fixed real constant (see in [14, 16, 17, 10]), are orthogonal polynomials with respect to the measure $\left(x-\lambda\right)w\left(x\right), \, x \in [-1,1]$.

If

$$L_n\left(x,y\right) = \sum_{k=2}^{n} \frac{Q_k(x)Q_k(t)}{\left\|Q_k\right\|^2} \ , \quad n = 1,2,\cdots ,$$

and

$$M_n\left(x,y\right) = \sum_{k=0}^{n} \frac{L_k(x)L_k(t)}{\left\|L_k\right\|^2} \ , \quad n = 1,2,\cdots .$$

For the representation of the three approximatings Fourier polynomials p_n^*, P_n^*, $T_n^*\left(x\right)$ in the form of integrals,because :

$$c_k = \frac{(k+1)(k+2)(2k+3)}{2} \int_{-1}^{1} f\left(t\right) q_k\left(t\right) w\left(t\right) dt, \quad k = 0,1,2,\cdots ,$$

hence

$$p_n^*\left(x\right) = \int_{-1}^{1} f\left(t\right) \sum_{k=0}^{n} \frac{(k+1)(k+2)(2k+3)}{2} q_k(x) q_k\left(t\right) w\left(t\right) dt$$

i-e

$$p_n^*\left(x\right) = \int_{-1}^{1} f\left(t\right) K_n\left(x,t\right) w(t) dt$$

and

$$P_n^*\left(x\right) = \int_{-1}^{1} f\left(t\right) L_n\left(x,t\right) \frac{dt}{1-t^2}$$

also

$$T_n^*\left(x\right) = \int_{-1}^{1} f\left(t\right) M_n\left(x,t\right) dt.$$

Theorem 2.5. *1) $\left\{K_n\left(x,\lambda\right)\right\}_{n=0,1,2,\cdots}$ are orthogonal polynomials on the interval $[-1,1]$, with respect to the weight function $x \longmapsto \left(x-\lambda\right) w\left(x\right)$, where $-1 \leq \lambda \leq 1$.*

2) Denote by Π_n the set of all algebraic polynomials of degree $\leq n$. Let λ be an arbitrary rel constant, $G\left(x\right)$ is an arbitrary real polynomial Π_n normalised by the condition

$$\int_{-1}^{1} \left(G\left(x\right)\right)^2 w\left(x\right) dx = 1 \tag{2.46}$$

The maximum of $\left(G\left(\lambda\right)\right)^2$is given by the kernel plynomials $\left\{K_n\left(x,\lambda\right)\right\}_{n=1,2,\cdots}$ as follows

$$G\left(x\right) = \varepsilon \frac{K_n\left(x,\lambda\right)}{\sqrt{K_n\left(\lambda,\lambda\right)}}, \quad |\varepsilon| = 1, \tag{2.47}$$

The maximum itself is $K_n\left(\lambda,\lambda\right)$.

Proof. Now we employ the Christoffel-Darboux formula(see in [17, 16, 10, 14]), then the polynomials $R_0(x), R_1(x), R_3(x), \cdots, R_n(x), \cdots$ where

$$R_n(x) = C_n \frac{q_{n+1}(x)q_n(\lambda) - q_{n+1}(\lambda)q_n(x)}{x-\lambda}, \quad x \neq \lambda \quad (n = 0, 1, 2, \cdots)$$

are orthogonal on the same segment $[-1, 1]$ in respect of the weight function $x \longmapsto (x-\lambda) w(x)$, because

$$K_n(x, y) = \frac{k_n}{k_{n+1}} \frac{1}{\|q_n\|_w^2} \frac{q_{n+1}(x)q_n(\lambda) - q_{n+1}(\lambda)q_n(x)}{x-\lambda}, \quad x \neq \lambda$$

hence, $C_n = \frac{k_n}{k_{n+1}} \frac{1}{\|q_n\|_w^2}$. It follows that $\{K_n(x, \lambda)\}_{n=0,1,2,\cdots}$ are orthogonal polynomials on the interval $[-1, 1]$, with respect to the weight function $x \longmapsto (x-\lambda) w(x)$,where $-1 \leq \lambda \leq 1$.

If we write, $G(x) = G_0 q_0(x) + G_1 q_1(x) + \cdots + G_n q_n(x)$, condition (2.46) becomes

$$G_0^2 + G_1^2 + \cdots + G_n^2 = 1 \tag{2.48}$$

By Cauchy's inequality it follows that,

$$(G(\lambda))^2 \leq \sum_{k=0}^{n} G_k^2 \sum_{k=0}^{n} q_k^2(\lambda) = K_n(\lambda, \lambda)$$

The later bound is attained for $G_k = \mu q_k(\lambda), k = 0, 1, 2, \cdots, n.$ where μ is determined according to the condition,(2.48)

$$\mu^2 \sum_{k=0}^{n} q_k^2(\lambda) = 1$$

Thus we get

$$\mu = \frac{\pm 1}{\sqrt{K_n(\lambda, \lambda)}}$$

Which confirms the desired results. This evidently completes the proof of Theorem. □

Proposition 2.6. *If* $(x_\nu)_{\nu=0,1,2,\cdots,n}$ *denote the zeros of* $q_n(x)$*, then we have*

$$\int_{-1}^{1} (q_n(x))^2 \frac{w(x)}{(x-x_\nu)^2} dx = \frac{q_n'^2(x_\nu)}{K_n(x_\nu, x_\nu)}$$

Proof. The sequence $(K_n(x, y))_{n=0}^{\infty}$ satisfies the Christoffel-Darboux formula(see in [14, 16, 17, 10])

$$K_n(x, x_\nu) = -\frac{k_n}{k_{n+1}} \frac{1}{\|q_n\|_w^2} \frac{q_{n+1}(x_\nu)q_n(x)}{x-x_\nu}$$

Hence

$$\frac{q_n(x)}{q_n'(x_\nu)(x-x_\nu)} = -\frac{\|q_n\|_w^2}{q_n'(x_\nu)q_{n+1}(x_\nu)} \frac{k_{n+1}}{k_n} K_n(x, x_\nu)$$

it follows that,

$$\int_{-1}^{1} \left(\frac{q_n(x)}{q_n'(x_\nu)(x-x_\nu)} \right)^2 w(x) dx$$

$$= \left(\frac{\|q_n\|_w^2}{q_n'(x_\nu) q_{n+1}(x_\nu)} \frac{k_{n+1}}{k_n} \right)^2 \int_{-1}^{1} K_n^2 (x, x_\nu) \, w(x) \, dx$$

A remarkable property of $\{K_n(x,y)\}_n$ is stated in the following property. K_n has the reproducing kernel property (see in [14, 16, 17, 10]) :

$$K_n(x_\nu, x_\nu) = \int_{-1}^{1} K_n(x, x_\nu) K_n(x, x_\nu) \, w(x) \, dx$$

then

$$\int_{-1}^{1} \left(\frac{q_n(x)}{q_n'(x_\nu)(x - x_\nu)} \right)^2 w(x) \, dx = \left(\frac{\|q_n\|_w^2}{q_n'(x_\nu) q_{n+1}(x_\nu)} \frac{k_{n+1}}{k_n} \right)^2 K_n(x_\nu, x_\nu) .$$

Since

$$K_n(x_\nu, x_\nu) = -\frac{k_n}{k_{n+1}} \frac{1}{\|q_n\|_w^2} q_{n+1}(x_\nu) q_n'(x_\nu)$$

then

$$K_n^{-1}(x_\nu, x_\nu) = -\frac{k_{n+1}}{k_n} \frac{\|q_n\|_w^2}{q_{n+1}(x_\nu) q_n'(x_\nu)}$$

it follows that, for $\nu = 1, 2, \cdots, n$

$$\int_{-1}^{1} \left(\frac{q_n(x)}{q_n'(x_\nu)(x - x_\nu)} \right)^2 w(x) \, dx = \frac{1}{K_n(x_\nu, x_\nu)}.$$

□

Remark 2.7. Let us denote λ_ν, $\nu = 1, 2, 3, \cdots, n$ these constants are named Christoffel numbers. We can construct the Lagrange interpolation polynomial $L(x)$ od degree $n - 1$ which coincide with $\rho(x)$ at the points, λ_ν, $\nu = 1, 2, \cdots, n$, that is

$$L(x) = \sum_{\nu=1}^{n} \rho(x_\nu) \frac{q_n(x)}{q_n'(x_\nu)(x - x_\nu)}.$$

Because

$$\int_{-1}^{1} \rho(x) \, w(x) \, dx = \lambda_1 \rho(x_1) + \lambda_2 \rho(x_2) + \cdots + \lambda_n \rho(x_n),$$

whenever $\rho(x)$ is Π_{2n-1},then

$$\lambda_\nu = \int_{-1}^{1} \frac{q_n(x)}{q_n'(x_\nu)(x - x_\nu)} w(x) \, dx, \quad \nu = 1, 2, 3...n.$$

The Christoffel numbers, are defined as follows

$$\lambda_\nu = \int_{-1}^{1} \left(\frac{q_n(x)}{q_n'(x_\nu)(x - x_\nu)} \right)^2 w(x) \, dx, \quad \nu = 1, 2, \cdots, n.$$

i-e

$$\lambda_\nu = -\frac{k_{n+1}}{k_n}\frac{\|q_n\|_w^2}{q_{n+1}(x_\nu)q_n'(x_\nu)} = -\frac{k_n}{k_{n-1}}\frac{\|q_n\|_w^2}{q_{n-1}(x_\nu)q_n'(x_\nu)} = \frac{1}{K_n\left(x_\nu, x_\nu\right)}, \quad \nu = 1, 2, \cdots, n.$$

Consequently

$$\int_{-1}^{1} w\left(x\right) dx = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

Finally, if $(x_\nu)_{\nu=1,2,\cdots,n}$ denote the zeros of $q_n(x)$, then we have

$$\frac{1}{K_n\left(x_1, x_1\right)} + \frac{1}{K_n\left(x_2, x_2\right)} + \cdots + \frac{1}{K_n\left(x_n, x_n\right)} = \int_{-1}^{1} w\left(x\right) dx$$

Remark 2.8. The following decomposition into partial fraction decomposition holds

$$\frac{q_n(x)}{q_{n+1}(x)} = \sum_{\nu=0}^{n}\frac{a_{\nu,n}}{x - x_\nu}$$

where $\{x_\nu\}_{\nu=0,1,2,\cdots,n}$denote the zeros of $q_{n+1}(x)$.

For we have

$$a_{\nu,n} = \frac{q_n(x_\nu)}{q_{n+1}'(x_\nu)} \quad , \quad \nu = 0, 1, 2, \cdots, n.$$

Then

$$a_{\nu,n} = \frac{q_n(x_\nu)q_{n+1}'(x_\nu) - q_{n+1}(x_\nu)q_n'(x_\nu)}{\left[q_{n+1}'(x_\nu)\right]^2} = \frac{k_{n+1}}{k_n}\left\|q_n\right\|_w^2 K_n\left(x_\nu, x_\nu\right) > 0$$

because

$$\frac{k_{n+1}}{k_n} = \frac{2n+3}{n+3}$$

and

$$\left\|q_n\right\|_w^2 = \frac{2}{\left(n+1\right)\left(n+2\right)\left(2n+3\right)}$$

it follows that,

$$a_{\nu,n} = \frac{2}{\left(n+1\right)\left(n+2\right)\left(n+3\right)} K_n\left(x_\nu, x_\nu\right)$$

becomes

$$\frac{q_n(x)}{q_{n+1}(x)} = \frac{2}{\left(n+1\right)\left(n+2\right)\left(n+3\right)}\sum_{\nu=0}^{n}\frac{K_n\left(x_\nu, x_\nu\right)}{x - x_\nu}$$

Differentiating the two sides

$$\frac{q_n(x)}{q_{n+1}(x)}\left(\frac{q_n'(x)}{q_n(x)} - \frac{q_{n+1}'(x)}{q_{n+1}(x)}\right) = \frac{-2}{\left(n+1\right)\left(n+2\right)\left(n+3\right)}\sum_{\nu=0}^{n}\frac{K_n\left(x_\nu, x_\nu\right)}{\left(x - x_\nu\right)^2}$$

Thus we get

$$\frac{\sum\limits_{\nu=0}^{n}\dfrac{K_n\left(x_\nu, x_\nu\right)}{\left(x - x_\nu\right)^2}}{\sum\limits_{\nu=0}^{n}\dfrac{K_n\left(x_\nu, x_\nu\right)}{x - x_\nu}} = \frac{q_{n+1}'(x)}{q_{n+1}(x)} - \frac{q_n'(x)}{q_n(x)}$$

3. Conclusion

In this article,we demonstrate certain identities involving both the integral Legendre polynomials and some associated Legendre polynomials. This study brings to light some significant results and defines the relationship between these polynomials and Jacobi polynomials. We have used mathematical induction to establish the relation between the integral Legendre polynomials,and two special cases of Jacobi polynomials,we derive structures relations between Q_n and $P_n^{(\alpha,\beta)}(x)$ for $\alpha = \beta = -1$, we derive also some structures relations between q_n and $P_n^{(\alpha,\beta)}(x)$ for $\alpha = \beta = 1$.

We also present some results for Christoffel-Darboux formula,particularly by using the Christoffel-Darboux formula, we prove some results that connect the q-kernel polynomials. In addition, we look at the practical application of q-kernel polynomials in approximation theory.

It is worth mentioning here that the above-achieved results and analysis are fruitful. Some of their presumed uses are given below:

- The integral Legendre polynomials and their kernel polynomials are fruitful in approximation theory.
- These orthogonal polynomials are fruitful in applied to find the minimum value and the minimizing function for various definite integrals and solving extremal problems.
- These results strengthen the knowledge of the kernel polynomials associated to the integral legendre polynomials.
- They are also beneficial in studying problems connected to solve extremal problem and to describe the approximation of continuous functions by kernel polynomials of integral legendre polynomials
- They help study finite linear combinations and finites summations sequences and calculating general summations.
- These orthogonal polynomials are fruitful in applied to find the interpolation problem,we illustrates that one can use Gaussian quadratures for various definite integrals and solving extremal problems.
- These polynomials can be used to solve differential equations, whether they are linear or non-linear and to acquire numerical answers to differential equations,whether linear or nonlinear.
- The connections between the integral Legendre polynomials, and Jacobi polynomials are highly helpful in obtaining the identities related to them.

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References

[1] M. Abramowitz and I. A. Stegun (Eds.), Handbook of Mathematical Functions, 10th Edition, Dover, New York, 1972.
[2] F. Alberto Grunbaum, Random walks and orthogonal polynomials: some challenges, Probability, Geometry and Integrable Systems MSRI Publications, 2007, Volume 55.
[3] R. Askey, Orthogonal Polynomials and Special Functions, Regional Conf. Ser.Appl. Math. 21, SIAM, Philadelphia, 1975.
[4] S. Belmehdi, S. Lewanowicz and A. Ronveaux, *Linearization of the product of orthogonal polynomials of a discrete variable*, Applicationes Mathematicae, **24(4)** (1997) 445–455.
[5] R. Belinsky, *Integrals of Legendre polynomials and solution of some partial differential equations*, Journal of Applied Analysis, **6(2)** (2000) 259–282. 1, 1, 1, 1.1, 1.1, 2, 2, 2
[6] J. P. Boyd, Chebyshev and Fourier spectral methods, (Dover, 2nd Edn., Mineola, 2001). 1

[7] T. S. Chihara, An introduction to orthogonal polynomials, Gordon and Breach, New York, 1978. 1
[8] K. Dilcher and K. B. Stolarsky, *Resultants and Discriminants of Chebyshev and related polynomials*, Transactions of the Amer. Math. Soc. **357** (2004), 965-981. 1
[9] M. Foupouagnigni, On difference and differential equations for modifications of classical orthogonal polynomials. Habilitation Thesis, Department of Mathematics and Computer Science. University of Kassel, Kassel, January 2006. 1, 1, 1, 1.1, 2, 2, 2
[10] O. Farrell and B.Ross, Solved Problems, The Macmollan Company, New York, 1963. 1
[11] J. C. Mason, D. C. Handscomb, Chebyshev polynomials, (Chapman& Hall, London, 2003) 1, 1, 2, 2, 2
[12] A. Rehouma and H. Jafari, Construction of the Shifted Modified Gegenbauer Polynomials and Approximation, Advances in Mathematical Physics, Article ID 8278473
[13] A. Rehouma, Integral of Legendre polynomials and its properties, Mathematics and Computational Sciences, 6(3) (2025) 61-76 1, 1, 1
[14] A. Rehouma and M. J. Atia, Study of an example of Markov chains involving Chebyshev polynomials. Integral transforms and special functions, **34(2)** (2023) 180-195. 1, 1, 1, 1.1, 2, 2
[15] H. Pijeira Cabrera and Y.José Bello Cruz and W.Urbina, On Polar Legendre polynomials, Rocky Mountain Journal of Mathematics, RMJ (2008), **38**, 1-10. 1.1, 2.3, 2.3, 2.3, 2.3, 2.3
[16] J. Shen, T. Tang, L.-L. Wang, Spectral methods: algorithms, analysis and applications, (Springer,2011). 1, 1, 1.1, 2, 2, 2
[17] G. Szego, Orthogonal Polynomials, American Mathematical Society , New York, 1939. 1
1