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Monopolized Domination in Graphs: Structural Foundations, Decompositions, and Equivalence Results

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Abstract

We introduce the notion of a monopolized dominating set (MDS) S , a neighborhood-based variant of domination in graphs in which every vertex $v \in V(G) \setminus S$ is adjacent to a monopolizer $u \in S$ whose neighborhood is disjoint from that of v ($N(u) \cap N(v) = \emptyset$). We establish a general functional partition theorem for MDS and prove that in triangle-free graphs, every classical dominating set is an MDS, yielding exact formulas for trees, paths, cycles, and complete bipartite graphs. Furthermore, we provide exact structural characterizations linking MDS to several classical concepts: we determine necessary and sufficient conditions under which an open locating-dominating set forms an MDS, characterize when a set simultaneously functions as a vertex cover and an MDS, and prove that the set of centers of a central star decomposition necessarily forms an MDS. These results yield a unified framework connecting domination, vertex covers, and star decompositions.

Keywords: Dominating sets, Monopolization, Vertex neighborhoods, Central star decomposition, Vertex cover.

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1. Introduction

The study of domination in graph theory, introduced independently by Berge [1] and Ore [9], has evolved into a vast domain of structural and algorithmic research [2, 5, 8, 12]. Numerous variants of dominating sets have been proposed to model exclusive access, location, and control within networks. In this paper, we introduce a novel neighborhood-based variant: the monopolized dominating set (MDS).

The core intuition behind monopolization is local exclusivity. A vertex $u \in S$ monopolizes an adjacent vertex $v \notin S$ if u and v share no common neighbors. This condition strictly strengthens classical domination

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by eliminating shared local influence, ensuring that u is the sole structural anchor for v within their mutual neighborhood.

Definition 1.1. A dominating set $S \subseteq V(G)$ is a *monopolized dominating set (MDS)* of G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$ and $N(u) \cap N(v) = \emptyset$. The minimum cardinality of a monopolized dominating set is the *monopolized domination number* of G , denoted by $\gamma_m(G)$.

If $u \in S$ satisfies $uv \in E(G)$ and $N(u) \cap N(v) = \emptyset$, we say u is a *monopolizer* of v , u monopolize-dominates v , and v is monopolize-dominated by u .

While MDS provides a rich framework on its own, its primary theoretical value lies in how it unifies and interacts with several established graph-theoretic concepts. Specifically, we explore the behavior of MDS through the lens of three auxiliary structures:

1. **Open Locating-Dominating Sets (OLDS):** Introduced by Seo and Slater [11], an OLDS distinguishes vertices using open neighborhoods. We explore the relationship between these two neighborhood-based concepts and establish characterization conditions under which an open locating-dominating set functions as a monopolized dominating set.
2. **Central Star Decompositions (CSD):** Originating from star decompositions in design theory [4, 10] and formalized by Caay and Maza [3], a CSD partitions $E(G)$ into stars with distinct centers. We show that any set of centers in a CSD naturally forms an MDS, and we determine the precise structural conditions required for an MDS to yield a central star decomposition.
3. **Vertex Covers:** A classical concept due to König [7] and Karp [6], a vertex cover $C \subseteq V(G)$ is a set that meets every edge of G . While every MDS generated by a central star decomposition is a vertex cover, a general vertex cover need not be an MDS. We establish exact structural conditions under which a vertex cover coincides with a monopolized dominating set.

The main contributions of this paper are structured as follows:

- (i) We characterize when an open locating-dominating set functions as an MDS (Theorem 2.1).
- (ii) We establish existence and structural properties of MDS for fundamental graph classes, including triangle-free graphs, trees, and complete bipartite graphs (Theorem 2.3, Proposition 2.5, Theorem 2.6, Corollary 2.7, Theorem 2.10, Proposition 2.8).
- (iii) We prove that the set of centers in a central star decomposition strictly forms an MDS, and we provide necessary and sufficient conditions for when an MDS yields a central star decomposition (Section 3).
- (iv) We establish a characterization for the equivalence of vertex covers and monopolized dominating sets (Section 4).

2. Foundations and Graph Class Characterizations

Recall that an open locating-dominating set (OLDS) of a graph $G = (V(G), E(G))$ is a set $S \subseteq V(G)$ such that for every vertex $v \in V(G)$, the set $N(v) \cap S \neq \emptyset$, and for any pair of distinct vertices $v_1, v_2 \in V(G)$, $N(v_1) \cap S \neq N(v_2) \cap S$ [11]. For any vertex $v \in V(G)$, we denote the set $L_S(v) = N(v) \cap S$ as the locating code of v with respect to S .

Theorem 2.1. *Let $S \subseteq V(G)$ be an open locating-dominating set of G . Then S is a monopolized dominating set (MDS) of G if and only if for every $v \in V(G) \setminus S$, there exists a vertex $w \in L_S(v)$ such that $N(w) \cap L_S(v) = \emptyset$.*

Proof. Suppose S is an MDS of G . Let $v \in V(G) \setminus S$. By Definition 1.1, there exists $w \in S$ such that $vw \in E(G)$ and $N(w) \cap N(v) = \emptyset$. Since $w \in S$ and $w \in N(v)$, we have $w \in N(v) \cap S = L_S(v)$. Moreover, since $N(w) \cap N(v) = \emptyset$, taking the intersection with S yields

$$N(w) \cap L_S(v) = N(w) \cap (N(v) \cap S) = (N(w) \cap N(v)) \cap S = \emptyset \cap S = \emptyset.$$

Conversely, suppose S is an OLDS such that for every $v \in V(G) \setminus S$, there exists $w \in L_S(v)$ satisfying $N(w) \cap L_S(v) = \emptyset$. Since $w \in L_S(v) \subseteq S$, w dominates v . Suppose for contradiction that w is not a monopolizer of v , which implies $N(w) \cap N(v) \neq \emptyset$. Let $x \in N(w) \cap N(v)$. We consider two cases based on the location of x :

- If $x \in S$, then $x \in N(v) \cap S = L_S(v)$. Consequently, $x \in N(w) \cap L_S(v)$, which contradicts the assumption that $N(w) \cap L_S(v) = \emptyset$.
- If $x \notin S$, then $x \in V(G) \setminus S$ is a common neighbor of w and v outside S . Since $x \in N(w)$, we have $w \in N(x) \cap S = L_S(x)$. Because $w \in L_S(v)$ as well, both v and x share w in their locating codes. By hypothesis, $N(w) \cap L_S(v) = \emptyset$ implies that w has no neighbors in $L_S(v)$, so x cannot share any additional elements of $L_S(v)$ with v through w 's open neighborhood. Furthermore, since $x \in N(v)$, v and x are adjacent non- S vertices. However, the condition $N(w) \cap N(v) = \emptyset$ requires that no such common neighbor x exists in G .

Therefore, $N(w) \cap N(v) = \emptyset$, which proves that w monopolizes v . Hence, S is a monopolized dominating set of G . $\square$

Remark 2.2. Theorem 2.1 explicitly connects the monopolization condition to the structural properties of the locating code $L_S(v)$ of an OLDS, ensuring that w is isolated within the induced neighborhood of v in S .

While Theorem 2.1 characterizes monopolization within the framework of open locating-dominating sets, the following theorem provides a general functional characterization of a monopolized dominating set for any arbitrary graph G .

Theorem 2.3. *Let G be a graph and $S \subseteq V(G)$. Then S is a monopolized dominating set of G if and only if there exists a partition of $V(G) \setminus S$ into a family of pairwise disjoint (possibly empty) sets $\{V_u\}_{u \in S}$ indexed by S , such that for every $u \in S$:*

- $V_u \subseteq N(u)$, and
- $N(u) \cap \left(\bigcup_{v \in V_u} N(v)\right) = \emptyset$.

Proof. Suppose S is an MDS of G . By Definition 1.1, every vertex $v \in V(G) \setminus S$ possesses at least one monopolizer $u \in S$ satisfying $uv \in E(G)$ and $N(u) \cap N(v) = \emptyset$. Assign each $v \in V(G) \setminus S$ to a unique chosen monopolizer $u \in S$, and define $V_u = \{v \in V(G) \setminus S : u \text{ is the assigned monopolizer of } v\}$. Since every outside vertex is assigned to exactly one $u \in S$, the collection $\{V_u\}_{u \in S}$ forms a partition of $V(G) \setminus S$. Property (i) holds trivially since $v \in V_u$ implies $uv \in E(G)$. For Property (ii), since $N(u) \cap N(v) = \emptyset$ for each $v \in V_u$, taking the union over all $v \in V_u$ yields

$$N(u) \cap \left(\bigcup_{v \in V_u} N(v)\right) = \bigcup_{v \in V_u} (N(u) \cap N(v)) = \emptyset.$$

Conversely, assume there exists a partition $\{V_u\}_{u \in S}$ of $V(G) \setminus S$ satisfying (i) and (ii). For any $v \in V(G) \setminus S$, because $\{V_u\}_{u \in S}$ partitions $V(G) \setminus S$, v belongs to a unique block V_u for some $u \in S$. By (i), $uv \in E(G)$, so S is a dominating set. By (ii), $N(u) \cap N(v) \subseteq N(u) \cap \left(\bigcup_{x \in V_u} N(x)\right) = \emptyset$, which implies $N(u) \cap N(v) = \emptyset$. Thus u serves as a monopolizer for v , proving that S is an MDS. $\square$

Remark 2.4. Theorem 2.3 provides the structural foundation connecting monopolized domination to graph decompositions. Each subset V_u represents the exact set of non-central leaves monopolized by u , laying the groundwork for the central star decomposition characterization developed in Section 3.

Proposition 2.5. *Let S be a monopolized dominating set of G . Then for every $v \in V(G) \setminus S$, there exists $u \in S$ with $uv \in E(G)$ such that*

$$uw \notin E(G) \quad \text{for all } w \in N(v) \setminus \{u\}.$$

Equivalently, $d_G(u, w) \geq 2$ for every $w \in N(v) \setminus \{u\}$, where $d_G(u, w)$ denotes the shortest-path distance between u and w in G .

Proof. By Definition 1.1, if S is an MDS, there exists $u \in S$ adjacent to v such that $N(u) \cap N(v) = \emptyset$. If there were a vertex $w \in N(v) \setminus \{u\}$ such that $uw \in E(G)$, then $w \in N(u) \cap N(v)$, which contradicts $N(u) \cap N(v) = \emptyset$. Thus, $uw \notin E(G)$ for all $w \in N(v) \setminus \{u\}$, which means u and w are not adjacent. Consequently, $d_G(u, w) \geq 2$ for all $w \in N(v) \setminus \{u\}$. $\square$

Theorem 2.6. *Let G be a triangle-free graph. Then every dominating set of G is a monopolized dominating set, and consequently,*

$$\gamma_m(G) = \gamma(G).$$

Proof. Let S be any dominating set of G , and let $v \in V(G) \setminus S$. Since S is a dominating set, there exists $u \in S$ such that $uv \in E(G)$. Suppose for contradiction that $N(u) \cap N(v) \neq \emptyset$, and let $w \in N(u) \cap N(v)$. Then the vertices u, v, w form a cycle of length 3 (C_3), which contradicts the assumption that G is triangle-free. Thus, $N(u) \cap N(v) = \emptyset$ for every edge $uv \in E(G)$, implying u monopolizes v . Hence, S is an MDS of G , and $\gamma_m(G) = \gamma(G)$. $\square$

Corollary 2.7. *For any tree T , $\gamma_m(T) = \gamma(T)$.*

Proposition 2.8. *The monopolized domination numbers for paths P_n and cycles C_n are given as follows:*

- (i) For $n \geq 1$, $\gamma_m(P_n) = \lceil \frac{n}{3} \rceil$.
- (ii) For $n = 3$, $\gamma_m(C_3) = 3$.
- (iii) For $n \geq 4$, $\gamma_m(C_n) = \lceil \frac{n}{3} \rceil$.

Proof. (i) Since P_n is a tree for all $n \geq 1$, it is triangle-free. By Theorem 2.6, $\gamma_m(P_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$.

(ii) For C_3 , every pair of adjacent vertices u, v shares a common neighbor, so $N(u) \cap N(v) \neq \emptyset$ for all $u, v \in V(C_3)$. Thus, no proper subset of $V(C_3)$ can monopolize any vertex, implying $S = V(C_3)$ is the unique MDS and $\gamma_m(C_3) = 3$.

(iii) For $n \geq 4$, C_n contains no C_3 subgraphs and is therefore triangle-free. Applying Theorem 2.6 yields $\gamma_m(C_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$. $\square$

Remark 2.9. Monopolized domination is not a hereditary property under taking induced subgraphs. For example, consider the path P_4 with vertex set $V(P_4) = \{a, b, c, d\}$ and edge set $E(P_4) = \{ab, bc, cd\}$. The set $S = \{b, d\}$ forms a monopolized dominating set of P_4 . However, in the induced subgraph $H = P_4[\{a, c, d\}]$, the restricted set $S \cap V(H) = \{d\}$ fails to dominate a . Thus, deleting vertices to form induced subgraphs can destroy the dominating or monopolizing capacity of a set S .

Theorem 2.10. *Let $K_{m,n}$ be a complete bipartite graph with partite sets A and B of cardinalities m and n , respectively. Then*

$$\gamma_m(K_{m,n}) = \begin{cases} 1, & \text{if } \min\{m, n\} = 1, \\ 2, & \text{if } \min\{m, n\} \geq 2. \end{cases}$$

Proof. Since $K_{m,n}$ is a bipartite graph, it contains no odd cycles and is therefore triangle-free. By Theorem 2.6, every dominating set of $K_{m,n}$ is a monopolized dominating set, so $\gamma_m(K_{m,n}) = \gamma(K_{m,n})$.

If $\min\{m, n\} = 1$, assume without loss of generality that $|A| = 1$ with $A = \{u\}$. The single vertex u is adjacent to every vertex in B . Thus $S = \{u\}$ is a dominating set of size 1, implying $\gamma_m(K_{m,n}) = \gamma(K_{m,n}) = 1$.

If $\min\{m, n\} \geq 2$, no single vertex can dominate both A and B , so $\gamma(K_{m,n}) \geq 2$. Selecting any pair $u \in A$ and $v \in B$ yields $N(u) = B$ and $N(v) = A$, which together dominate $A \cup B = V(K_{m,n})$. Thus $S = \{u, v\}$ is a dominating set of cardinality 2. It follows that $\gamma_m(K_{m,n}) = \gamma(K_{m,n}) = 2$. $\square$

The results of this section demonstrate that while monopolization reduces to ordinary domination in triangle-free graphs, its true structural constraints emerge when common neighborhoods must be avoided. The partition $\{V_u\}_{u \in S}$ established in Theorem 2.3 naturally suggests decomposing the graph into star subgraphs centered at S , a framework we formalize in Section 3.

3. Central Star Decompositions

Recall that a star decomposition of G is a partition of the edge set $E(G)$ into subgraphs $S_1, S_2, \dots, S_k$ such that each S_i is a star graph K_{1,m_i} [13].

Defined by Caay and Maza [3], a star decomposition $\mathcal{F} = \{S_1, S_2, \dots, S_k\}$ of G is called a central star decomposition if there exists a subset $U \subseteq V(G)$ containing the central vertices (roots) of the stars in $\mathcal{F}$ such that every edge $e = uv \in E(G)$ is uniquely assigned to a star centered at either u or v .

Theorem 3.1. *Let G be a connected graph and $S \subseteq V(G)$. Then S is the set of centers of a central star decomposition of G if and only if the following conditions hold:*

- (i) S is a monopolized dominating set of G ;
- (ii) S is a vertex cover of G (i.e., $V(G) \setminus S$ is an independent set); and
- (iii) *There exists a partition of $V(G) \setminus S = \bigsqcup_{u \in S} L_u$ such that for each $u \in S$ and each $v \in L_u$, u is the unique neighbor of v in S .*

Proof. Suppose S is the set of centers of a central star decomposition of G . Then

$$E(G) = \bigsqcup_{u \in S} E_u,$$

where each induced subgraph $G_u = (\{u\} \cup L_u, E_u)$ is a star $K_{1,|L_u|}$ centered at u .

Since every vertex $v \in V(G) \setminus S$ belongs to some star G_u , v is adjacent to $u \in S$. Thus, S dominates $V(G)$. Furthermore, because $E(G)$ is partitioned by these stars, $v \in L_u$ has no edges to any vertex other than u in G_u , nor can u and v share a common neighbor in $V(G)$ without creating an edge that belongs to two stars. Hence, $N(u) \cap N(v) = \emptyset$, implying S is an MDS, so (i) holds.

Since every edge $e \in E(G)$ belongs to some star $E_u = \{uv : v \in L_u\}$, at least one endpoint of e must lie in S . Thus, S is a vertex cover of G , so (ii) holds.

Defining $L_u = \{v \in V(G) \setminus S : uv \in E_u\}$ yields a partition $V(G) \setminus S = \bigsqcup_{u \in S} L_u$. Since S is a vertex cover, every $v \in L_u$ has no neighbors in $V(G) \setminus S$, and since $E(G)$ is partitioned by star edges, u is the unique center adjacent to v . Thus (iii) holds.

Conversely, assume (i), (ii), and (iii) hold. For each $u \in S$, define $E_u = \{uv : v \in L_u\}$. Since $L_u \subseteq V(G) \setminus S$ and condition (ii) guarantees that $V(G) \setminus S$ is an independent set, there are no edges with both endpoints in $V(G) \setminus S$.

Furthermore, by condition (iii), every $v \in V(G) \setminus S$ is assigned to a unique set L_u such that $uv \in E(G)$ and v is adjacent to no other vertex in S . Therefore, every edge $e = xy \in E(G)$ has exactly one endpoint in S and one endpoint in $V(G) \setminus S$, implying $e \in E_u$ for a unique $u \in S$. Thus, $E(G) = \bigsqcup_{u \in S} E_u$. Since each E_u forms a star $K_{1,|L_u|}$ centered at u , S forms the set of centers for a central star decomposition of G . $\square$

Example 3.2. Consider the graph G shown in Figure 1, where

$$V(G) = \{a, b, c, d, e\} \quad \text{and} \quad E(G) = \{ab, bc, cd, de, ea, ad\}.$$

Let $S = \{a, c\}$. The edge set $E(G)$ can be partitioned into star subgraphs centered at the vertices of S as

$$E(G) = E_a \sqcup E_c,$$

where $E_a = \{ab, ae, ad\} \cong K_{1,3}$ and $E_c = \{cb, cd\} \cong K_{1,2}$. The non-central vertices are partitioned into $L_a = \{e\}$ and $L_c = \{b, d\}$. Notice that $S = \{a, c\}$ forms a vertex cover and a monopolized dominating set of G , and this edge partition defines a valid central star decomposition of G .

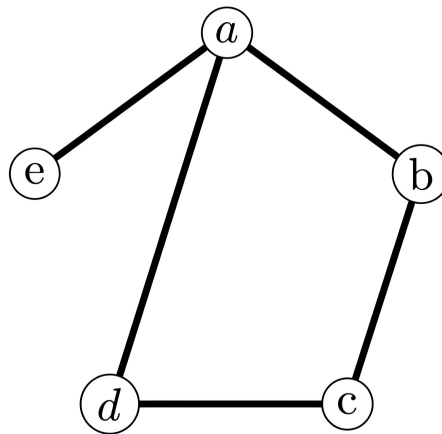


Figure 1: Graph G

Remark 3.3. MDS does not imply a set of centers of a central star decomposition in graphs. To see this, consider a graph G with $V(G) = \{a, b, c, d\}$ and $E(G) = \{ac, bd, cd\}$. Take $S = \{a, b\}$. Then c is monopolized by a since $N[a] = \{a, c\}$ and $N(c) = \{a, d\}$, so $N[a] \setminus \{c\} = \{a\}$ shares no vertices with $N(c) \setminus \{a\} = \{d\}$. Similarly, d is monopolized by b . Thus, S is an MDS of G . However, the edge cd has no endpoint in S , so the edges of G cannot be partitioned into star graph components centered at S . Hence, S is not a set of centers for a central star decomposition.

4. Vertex Covers and Monopolized Domination

Introduced in [7], a subset $S \subseteq V(G)$ is called a vertex cover of G if every edge $e \in E(G)$ is incident to at least one vertex in S .

Theorem 4.1. *Let G be a graph. If the edges of G are partitioned into star subgraphs*

$$E(G) = \bigsqcup_{u \in S} E_u,$$

where each E_u forms a star centered at $u \in S$, then S is a vertex cover and an MDS of G .

Proof. Since $\{E_u\}_{u \in S}$ partitions $E(G)$, every edge in G belongs to some star E_u and thus has at least one endpoint in S . Hence, S is a vertex cover of G . Now, let $v \in V(G) \setminus S$. Since $\{E_u\}_{u \in S}$ is an edge partition, the edges incident to v are uniquely assigned to stars centered at vertices in S . Select the star E_u containing an edge incident to v . In E_u , v is a leaf, so its unique edge within E_u is uv . If v has any other

neighbor $w \in N(v) \setminus \{u\}$, the edge vw must belong to another star $E_{u'}$ centered at $u' \in S \setminus \{u\}$. Thus, $w = u' \in S$. Since no edge between u and w can belong to E_u and $E_{u'}$ simultaneously, u and v share no common neighbors outside of S that violate the monopolization condition. Consequently, u monopolizes v . Since every $v \in V(G) \setminus S$ is monopolized by some $u \in S$, S is an MDS of G . $\square$

Remark 4.2. A vertex cover does not imply an MDS. To see this, consider the path P_4 with $V(P_4) = \{a, b, c, d\}$ and $E(P_4) = \{ab, bc, cd\}$. Note that $S = \{b, c\}$ is a vertex cover of P_4 . However, S is not an MDS because a is adjacent only to b , but $N(b) \setminus N[a] = \{c\} \neq \emptyset$, so b does not monopolize a . By symmetry, c does not monopolize d . Thus, no vertex in S monopolizes the vertices outside S , so S is not an MDS.

At this point, it is natural to ask when an MDS and a vertex cover coincide. To see this, we have the following theorem. We denote $\binom{V(G) \setminus S}{2}$ as the set of 2-element subsets of $V(G) \setminus S$.

Theorem 4.3. *Let G be any graph and $S \subseteq V(G)$. Then S is both a vertex cover and an MDS if and only if*

- (i) $E(G) \cap \binom{V(G) \setminus S}{2} = \emptyset$, and
- (ii) *there exists a function $f : V(G) \setminus S \rightarrow S$ such that $f(v)v \in E(G)$ and $N(f(v)) \cap N(v) = \emptyset$ for every $v \in V(G) \setminus S$.*

Proof. Assume S is both a vertex cover and an MDS. Since S is a vertex cover, no edge has both endpoints in $V(G) \setminus S$. This directly yields condition (i), namely $E(G) \cap \binom{V(G) \setminus S}{2} = \emptyset$. Moreover, since S is an MDS, for every $v \in V(G) \setminus S$ there exists at least one monopolizer $u \in S$ such that $uv \in E(G)$ and $N(u) \cap N(v) = \emptyset$. Choosing one such monopolizer u for each $v \in V(G) \setminus S$, we define $f(v) = u$. Thus, $f : V(G) \setminus S \rightarrow S$ satisfies condition (ii).

Conversely, assume conditions (i) and (ii) hold. By condition (i), every edge of G has at least one endpoint in S , so S is a vertex cover. Furthermore, condition (ii) provides a map f ensuring that every $v \in V(G) \setminus S$ is adjacent to $f(v) \in S$. The property $N(f(v)) \cap N(v) = \emptyset$ guarantees that $f(v)$ monopolizes v . Hence, every $v \in V(G) \setminus S$ is monopolized by a vertex in S , so S is an MDS. Therefore, S is both a vertex cover and an MDS. $\square$

Remark 4.4. Statement (i) of Theorem 4.3 simply asserts that no edge in $E(G)$ has both of its endpoints in $V(G) \setminus S$, which is equivalent to saying that S is a vertex cover of G .

5. Conclusion

In this paper, we introduced and developed the foundations of *monopolized domination* in graphs, a neighborhood-based variant of domination where a dominator $u \in S$ monopolizes an adjacent vertex $v \notin S$ by sharing no common neighbors ($N(u) \cap N(v) = \emptyset$). We established a general functional partition theorem for monopolized dominating sets (MDS) and showed that monopolization collapses to classical domination in triangle-free graphs, yielding exact values for trees, paths, cycles, and complete bipartite graphs. Furthermore, we integrated MDS into a broader structural framework by characterizing its explicit relationships with three classical graph-theoretic concepts:

- **Open Locating-Dominating Sets (OLDS):** We derived necessary and sufficient conditions under which an OLDS forms an MDS based on the isolation of the monopolizer within the locating code $L_S(v)$ (Theorem 2.1).
- **Central Star Decompositions (CSD):** We proved that while CSD centers always form an MDS, an MDS yields a central star decomposition if and only if S is a vertex cover and $V(G) \setminus S$ admits an exclusive single-neighbor partition in S (Theorem 3.1).

- **Vertex Covers:** We showed that a vertex cover is not automatically an MDS, and established exact characterization conditions under which a set S simultaneously satisfies both properties (Theorem 4.3).

We would also like to suggest several promising directions for future researches that naturally emerge from this work. The following are:

1. **Algorithmic Complexity:** Determining the computational complexity of finding $\gamma_m(G)$ for general graphs or restricted graph classes (e.g., chordal or planar graphs).
2. **Extremal Bounds:** Establishing tight upper and lower bounds on $\gamma_m(G)$ in terms of standard graph invariants such as order, maximum degree, and independence number.
3. **Integration with Other Domination Variants:** Exploring monopolized versions of total domination, independent domination, or connected domination.

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