Available online at http://jprm.sms.edu.pk/
Journal of Prime Research in Mathematics, 22(1) (2026), 93–103

Topological Neighborhood Induced Ideal Convergence of Sequences of Sets

Prasenjit Bal[a,*]

[a] *Department of Mathematics, The ICFAI University Tripura, Kamalghat, INDIA-799210.*

Abstract

This paper investigates the concept of ideal convergence of sequences of sets through neighborhoods in topological context. The operators Δ_x^- and Δ_x^+ are employed as fundamental tools for the establishment of ${}^n\mathcal{I}$-limit inferior and ${}^n\mathcal{I}$-limit superior. For sequence of sets, 'sandwich' theorem like description is presented under ${}^n\mathcal{I}$-convergence. Several inclusion properties of ${}^n\mathcal{I} - lim\ inf$, ${}^n\mathcal{I} - lim\ sup$ and related notions are derived. It has been shown that in an $\mathcal{I}$ compact space, sequence of closed sets having $\mathcal{I}$-intersection property always possess ${}^n\mathcal{I} - lim\ inf$. The scope and limitations of the theory are further tracked down through counterexamples.

Keywords: Ideal convergence, Sequence of sets, Neighborhood convergence.
2010 Mathematics Subject Classification: 54D20, 40A35

1. Introduction

The concept of convergence has always been in the center of interest for analysis and topology. Although classical convergence is a powerful tool, it frequently proves its restrictiveness in situations where exceptional indices are permitted for small deviations from a limit should be ignored. In order to fix this, the set theoretic concept of ideal provides a natural framework for the generalization of classical convergence. Formally, a family $\mathcal{I} \subseteq \mathcal{P}(X)$ is called an ideal on an non empty set X if

(i) If $Y \in \mathcal{I}$ and $Z \subseteq Y$, then $Z \in \mathcal{I}$ (hereditary property).

(ii) If $Y, Z \in \mathcal{I}$, then $Y \cup Z \in \mathcal{I}$ (closed under finite unions).

Unsurprisingly, $\mathcal{I}$ represents the collection of insignificant or negligible sets. This makes it a suitable tool for the analysis of non converging sequences and non-covering topological properties [1, 15]. Some very common ideals of the set of natural numbers found in literature are

(i) $\mathcal{I}_{fin} = \{A \subseteq \mathbb{N} : A \text{ is finite }\}$,

(ii) $\mathcal{I}_\delta = \{A \subseteq \mathbb{N} : \delta(A) = \lim_{n\to\infty} \frac{1}{n}|\{k \leq n : k \in A\}| = 0\}$, $\delta(A)$ being the natural density (also known as asymptotic density [7, 8]) of $A \subseteq \mathbb{N}$.

*Corresponding author
Email address: balprasenjit177@gmail.com (Prasenjit Bal)

Received : 03 November 2025*; Accepted:* 12 February 2026*; Published Online:* 15 February 2026

(iii)$\mathcal{I}_{2\mathbb{N}} = \{A : A \subseteq 2\mathbb{N}\}$ etc.

Kostyrko et. al. [10] introduced the notion of ideal convergence, marking the opening of a new line of generalizations of convergence. They defined a sequence $\{x_n : n \in \mathbb{N}\}$ in a topological space X to be $\mathcal{I}$-convergent to a point $x \in X$ if for every neighborhood U of x, the set $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$.

This definition generalizes ordinary convergence and statistical convergence, i.e., ideal convergence stands for ordinary convergence if $\mathcal{I} = \mathcal{I}_{fin}$ and statistical convergence if $\mathcal{I} = \mathcal{I}_\delta$[5, 13]. The main strength of ideal convergence is its flexibility. Different choices of ideals lead to different convergence behaviors, providing a unique setting for convergence notions.

As documented by Zoretti [18] the concept of the limit of sequences of sets was originally introduced by Painlevé in 1902. Through the influential work of Kuratowski [11], particularly in his seminal book Topologie, this notion gained wider recognition which led to its frequent reference as the Kuratowski limit. Wijsman [16, 17] made separate approaches to study convergence of sequences of convex sets known as Wijsman convergence. As per the definition of Kuratowski, any decreasing sequence of subsets $\{Y_n : n \in \mathbb{N}\}$ has a limit, which is the intersection of their closures. If $Y_n \subseteq Y_m$, when $n \geq m$, then $\lim Y_k = \bigcap_{k\geq 1} Y_k$. So in the real line space with the usual topology, the limit of the sequence $\{Y_k = \{y_k\} : k \in \mathbb{N}\}$ of singletons will always be empty until it's a constant sequence. As per the definition of Wijsman, in a metric space (X, ρ), for non-empty closed subsets Y, $Y_k \subseteq X$, we say that the sequence $\{Y_k : k \in \mathbb{N}\}$ is convergent to Y if $\lim_{k\to\infty} d_k(x) = d(x)$ for each $x \in X$, where $d, d_k : X \to \mathbb{R}^+$are defined as $d(x) = d(x, Y) = \inf_{y\in Y} \rho(x, y)$ and $d_k(x) = d(x, Y_k) = \inf_{y\in Y} \rho(x, y)$.

Despite having an early origin, set convergence has emerged as a powerful analytical tool, particularly in the context of approximations pertaining optimization, systems of equations, and other mathematical structures in last few years. With the introduction of generalized convergence (statistical convergence and ideal convergence), works of Kuratowski and Wijsman achieved new heights and also the limitations of these convergence criterion are revealed. Kuratowski convergence is applicable only for the decreasing sequences of sets and Wijsman convergence is applicable on sequences of closed sets. In 2012, Nuray et. al.[14], started the investigation on statistical convergence of sequence of sets but that too is restricted up to the sequence of closed sets. In 2015, Inan et. al. [9] made an outstanding approach towards ideal convergence of sequence of sets. They defined two operators $N_x^-(Y_k)$ and $N_x^+(Y_k)$ on the sequence of subsets $\{Y_k : k \in \mathbb{N}\}$ in a topological space (X, τ) as

$$N_x^-(Y_k) = \{k \in \mathbb{N} : x \notin Y_k\} \text{ and } N_x^+(Y_k) = \{k \in \mathbb{N} : x \in Y_k\}.$$

They defined ideal limit inferior and ideal limit supremum as

$$\overset{\mathcal{I}-\lim inf}{k \to \infty} Y_k = \{x \in X : N_x^-(Y_k) \in \mathcal{I}\} \text{ and } \overset{\mathcal{I}-\lim sup}{k \to \infty} Y_k = \{x \in X : N_x^+(Y_k) \notin \mathcal{I}\}.$$

If $\overset{\mathcal{I}-\lim inf}{k \to \infty} Y_k = \overset{\mathcal{I}-\lim sup}{k \to \infty} Y_k$, then the common set is defined as the ideal limit of the sequence $\{Y_k : n \in \mathbb{N}\}$. Although all efforts has been given so as to define a convergence criteria applicable to the sequence of sets irrespective of the nature of the sets, the approach totally misses the importance of the neighborhoods of the limit which again makes it applicable only for sequence of closed sets. In this paper, we propose a general form of ideal convergence for sequence of sets applicable to all sequences of sets irrespective of the nature of the sets.

2. Preliminaries

The operators Δ_a^- and Δ_a^+, their properties and some other associated notions that are necessary for this study are stated in this section.

Definition 2.1. [9] Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of sets. The limit inferior and limit superior of the sequence of sets is

$$\lim \sup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} A_k \text{ and } \lim \inf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k\geq n} A_k$$

respectively. If $\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = A$, then $\{A_n\}_{n\in\mathbb{N}}$ is called convergent to the common set A in ordinary sense and denoted by $\lim_{n\to\infty} A_n = A$.

The operators N_x^- and N_x^+ has been modified to Δ_x^- and Δ_x^+ respectively by Bal et. al. [2] to have more general and more analylical flexibility.

Definition 2.2. [2] Let $\{A_k : k \in \mathbb{N}\}$ be a sequence of subsets of X in a topological space (X, τ). Then,

$$\Delta_a^-(A_k) = \{k \in \mathbb{N} : U \cap A_k = \emptyset \text{ for atleast one neighborhood } U \text{ of } a\}$$
$$\text{and } \Delta_a^+(A_k) = \{k \in \mathbb{N} : U \cap A_k \neq \emptyset \text{ for every neighborhood } U \text{ of } a\}.$$

The use of neighborhoods of the limiting elements makes these operators more useful for the study of convergence criteria of sequence of sets. The properties of these operators are extensively studied by Bal et. al. [2]. Some of these properties that are used in our study are stated below.

Proposition 2.3. *[2] Let $\{A_k : k \in \mathbb{N}\}$ be a sequence of subsets of X in a topological space (X, τ). Then,*

(1) $\Delta_a^-(A_k) \cap \Delta_a^+(A_k) = \emptyset$ *for each* $a \in X$.

(2) $\Delta_a^-(A_k) \cup \Delta_a^+(A_k) = \mathbb{N}$ *for each* $a \in X$.

Proposition 2.4. *[2] Let $\{A_k : k \in \mathbb{N}\}$ and $\{B_k : k \in \mathbb{N}\}$ be two sequences of subsets of X in a topological space (X, τ). Then,*

(1) $\Delta_a^-(A_k \cap B_k) \supseteq \Delta_a^-(A_k) \cup \Delta_a^-(B_k)$

(2) $\Delta_a^+(A_k \cap B_k) \subseteq \Delta_a^+(A_k) \cap \Delta_a^+(B_k)$

(3) $\Delta_a^-(A_k \cup B_k) = \Delta_a^-(A_k) \cap \Delta_a^-(B_k)$

(4) $\Delta_a^+(A_k \cup B_k) = \Delta_a^+(A_k) \cup \Delta_a^+(B_k)$

Proposition 2.5. *[2] Let us consider $\{A_n^{(i)} : n \in \mathbb{N}\}_{i=1}^p$ be sequences of subsets of X in a topological space (X, τ). Then,*

(1) $\Delta_a^-(\prod_{i=1}^p A_n^{(i)}) = \bigcup_{i=1}^p \Delta_{a_i}^-(A_n^{(i)})$, *where* $a = (a_1, a_2, a_3, \ldots, a_p)$.

(2) $\Delta_a^+(\prod_{i=1}^p A_n^{(i)}) = \bigcap_{i=1}^p \Delta_{a_i}^+(A_n^{(i)})$, *where* $a = (a_1, a_2, a_3, \ldots, a_p)$.

Definition 2.6. [12] A family $\mathcal{F} \subseteq \mathcal{P}(X)$ is called a filter on a non empty set X if following properties hold
(i) If $Y \in \mathcal{F}$ and $Z \supseteq Y$, then $Z \in \mathcal{F}$.
(ii) If $Y, Z \in \mathcal{F}$, then $Y \cap Z \in \mathcal{F}$.

If $\mathcal{I}$ is an ideal on X then the family $\{X \setminus I : I \in \mathcal{I}\}$ of subsets of X forms a filter on X denoted by $\mathcal{F}(\mathcal{I})$.

Definition 2.7. [3] Consider an ideal $\mathcal{I}$ on the set $\mathbb{N}$ of natural numbers. A space X is called an $\mathcal{I}$-compact space if for every countable open cover $\mathcal{U} = \{A_n : n \in \mathbb{N}\}$ of X, there exists a sub-cover $\mathcal{V} = \{A_{n_k} : k \in \mathbb{N}\}$ such that $\{n_k : A_{n_k} \in \mathcal{V}\} \in \mathcal{I}$.

Definition 2.8. [3] Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. A family $\mathcal{A} = \{\Omega_n : n \in \mathbb{N}\}$ is categorized to posses $\mathcal{I}$-intersection property if $\mathcal{A} \neq \emptyset$ and $\bigcap_{n\in S} \Omega_n \neq \emptyset$ for all $S \in \mathcal{I}$.

Theorem 2.9. *[3] In an $\mathcal{I}$-compact space, every family of closed sets having $\mathcal{I}$-intersection property have nonempty intersection.*

Through out the paper a space X means a topological space X with the associate topology τ. $\mathcal{I}$ stands for any arbitrary ideal $\mathcal{I}$ defined on the set $\mathbb{N}$ of natural numbers. $\mathcal{I}$ is not restricted up to admissible ideal and no separation axioms has been assumed in this paper otherwise stated. For general symbols and notions we follow [6].

3. Main Results

In this section, the concept of neighborhood induced ideal limit has been established. Its relation with the associated concepts of convergence are investigated and their set theoretic properties are revealed.

Definition 3.1. A set Y_{inf} is the neighborhood induced ideal limit inferior (in short ${}^{n}\mathcal{I}-\lim inf$) of a sequence $\{Y_n : n \in \mathbb{N}\}$ of subsets in a space X if

$$Y_{inf} = \overset{{}^{n}\mathcal{I}-\lim inf}{n \to \infty} Y_n = \{x \in X : \Delta_x^-(Y_n) \in \mathcal{I}\}.$$

Similarly, a set Y^{sup} is the neighborhood induced ideal limit superior (in short ${}^{n}\mathcal{I}-\lim sup$) of a sequence $\{Y_n : n \in \mathbb{N}\}$ of subsets in a space X if

$$Y^{sup} = \overset{{}^{n}\mathcal{I}-\lim sup}{n \to \infty} Y_n = \{x \in X : \Delta_x^+(Y_n) \in \mathcal{F}(\mathcal{I}) \cup (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I})\}.$$

If $\overset{{}^{n}\mathcal{I}-\lim inf}{n \to \infty} Y_n = \overset{{}^{n}\mathcal{I}-\lim sup}{n \to \infty} Y_n = Y$, then the common set Y is called the neighborhood induced ideal limit (in short ${}^{n}\mathcal{I}-\lim$) of the sequence $\{Y_n : n \in \mathbb{N}\}$. We express it as

$$\overset{{}^{n}\mathcal{I}-\lim}{n \to \infty} Y_n = Y.$$

If $\overset{{}^{n}\mathcal{I}-\lim inf}{n \to \infty} Y_n \neq \overset{{}^{n}\mathcal{I}-\lim sup}{n \to \infty} Y_n$ or $\overset{{}^{n}\mathcal{I}-\lim inf}{n \to \infty} Y_n = \overset{{}^{n}\mathcal{I}-\lim sup}{n \to \infty} Y_n = \emptyset$, then we say that $\overset{{}^{n}\mathcal{I}-\lim}{n \to \infty} Y_n$ does not exists. It is evident to explain that there might be an $I \subseteq \mathbb{N}$ that belongs to both $\mathcal{I}$ and $\mathcal{F}(\mathcal{I})$. We want to consider that I for both cases i.e., while computing the Y_{inf} as well as while computing Y_{sup}. So we have taken $\Delta_x^+(Y_n) \in \mathcal{F}(\mathcal{I}) \cup (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I})$ rather than taking $\Delta_x^+(Y_n) \in \mathcal{F}(\mathcal{I})$ or $\Delta_x^+(Y_n) \notin \mathcal{I}$. More over if we take only $\Delta_x^+(Y_n) \in \mathcal{F}(\mathcal{I})$, the concepts of Y_{inf} and Y_{sup} does not make any difference.

Example 3.2. Consider the sequence,

$$Y_n = \begin{cases} [0, \frac{1}{n}) & \text{if } n = k^2 \text{ for some } k \in \mathbb{N}, \\ (1 - \frac{1}{n}, 1] & \text{otherwise.} \end{cases}$$

Here, $\Delta_0^-(Y_n) = \{2, 3, 5, \dots\} \notin \mathcal{I}_\delta$; $\Delta_1^-(Y_n) = \{4, 9, 16, \dots\} \in \mathcal{I}_\delta$. For every, $a \in (-\infty, 0) \cup (1, \infty)$, $\Delta_a^-(Y_n) = \mathbb{N} \notin \mathcal{I}_\delta$. For every, $a \in (0, 1)$, $\Delta_a^-(Y_n) = \mathbb{N} \setminus A$(where A is a finite set) $\notin \mathcal{I}_\delta$.

$$\text{Therefore, } \overset{{}^{n}\mathcal{I}_\delta-\lim inf}{n \to \infty} Y_n = \{1\}.$$

On the other side, $\Delta_0^+(Y_n) = \{1, 4, 9, 16, \dots\} \notin (\mathcal{P}(\mathbb{N}) \setminus I_\delta) \cup \mathcal{F}(\mathcal{I}_\delta)$; $\Delta_1^+(Y_n) = \{1, 2, 3, 5, 6, 7, 8, 10, \dots\} \in (\mathcal{P}(\mathbb{N}) \setminus I_\delta) \cup \mathcal{F}(\mathcal{I}_\delta)$. For every, $a \in (-\infty, 0) \cup (1, \infty)$, $\Delta_a^+(Y_n) = \emptyset \notin (\mathcal{P}(\mathbb{N}) \setminus I_\delta) \cup \mathcal{F}(\mathcal{I}_\delta)$. For every, $a \in (0, 1)$, $\Delta_a^+(Y_n) = A$ (where A is a finite set) $\notin (\mathcal{P}(\mathbb{N}) \setminus I_\delta) \cup \mathcal{F}(\mathcal{I}_\delta)$.

$$\text{Therefore, } \overset{{}^{n}\mathcal{I}_\delta-\lim sup}{n \to \infty} Y_n = \{1\}.$$

$$\text{Thus, } \overset{{}^{n}\mathcal{I}_\delta-\lim}{n \to \infty} Y_n = \{1\}.$$

Lemma 3.3. *If $I \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}) \cup \mathcal{F}(\mathcal{I})$, for some ideal $\mathcal{I}$ and $J \supseteq I$, then $J \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}) \cup \mathcal{F}(\mathcal{I})$.*

Proof. If $I \in \mathcal{F}(\mathcal{I})$, then $J \in \mathcal{F}(\mathcal{I})$, by virtue of filter. Let, $I \in \mathcal{P}(\mathbb{N}) \setminus \mathcal{I}$ and $J \notin \mathcal{P}(\mathbb{N}) \setminus \mathcal{I}$. So $I \notin \mathcal{I}$ and $J \in \mathcal{I}$, which is a contradiction. So, $J \in \mathcal{P}(\mathbb{N}) \setminus \mathcal{I}$. □

Proposition 3.4. *For every sequence $\{Y_n : n \in \mathbb{N}\}$ of subsets in a space X,*

$$\overset{{}^{n}\mathcal{I}-\lim inf}{n \to \infty} Y_n \subseteq \overset{{}^{n}\mathcal{I}-\lim sup}{n \to \infty} Y_n.$$

Proof. Let $y \in \overset{n\mathcal{I}-\lim inf}{n \to \infty} Y_n = \{x \in X : \Delta_x^-(Y_n) \in \mathcal{I}\}$. Therefore, $\Delta_y^-(Y_n) \in \mathcal{I}$, i.e., $\{n \in \mathbb{N} : U \cap Y_n = \emptyset$ for at least one neighborhood U of $y\} \in \mathcal{I}$. So, $\{n \in \mathbb{N} : U \cap Y_n \neq \emptyset$ for every neighborhood U of $y\} \in \mathcal{F}(\mathcal{I})$, i.e., $\Delta_y^+(Y_n) \in \mathcal{F}(\mathcal{I})$. Thus, $y \in \overset{n\mathcal{I}-\lim sup}{n \to \infty} Y_n$. Hence, $\overset{n\mathcal{I}-\lim inf}{n \to \infty} Y_n \subseteq \overset{n\mathcal{I}-\lim sup}{n \to \infty} Y_n$. □

Example 3.5. Let us consider the ideal $\mathcal{I} = \mathcal{P}(\mathbb{N}\setminus 2\mathbb{N})$ defined on $\mathbb{N}$, then $\mathcal{F}(\mathcal{I}) = \{(2\mathbb{N} \cup M) : M \subseteq \mathbb{N}\setminus 2\mathbb{N}\}$. Consider the space $\mathbb{R}$ with usual topology and the sequence $\{Y_n : n \in \mathbb{N}\}$ such that

$$Y_n = \begin{cases} (-1, -1 + \frac{1}{n}) \cup (1 - \frac{1}{n}, 1) & \text{if } n \text{ is an odd prime,} \\ (-\frac{1}{n}, \frac{1}{n}) & \text{if } n \text{ is even,} \\ \{0\} & \text{otherwise.} \end{cases}$$

Here, $\Delta_{-1}^-(Y_n) = \{n \in \mathbb{N} : U \cap Y_n = \emptyset$ for at least one neighborhood U of $-1\} = 2\mathbb{N} \cup \{1, 9, 15, 21, ...\} \notin \mathcal{I}$; $\Delta_1^-(Y_n) = 2\mathbb{N} \cup \{1, 9, 15, 21, ...\} \notin \mathcal{I}$; $\Delta_0^-(Y_n) = 2\mathbb{N} \cup \{3, 5, 7, 11, ...\} \in \mathcal{I}$.
For $a \in (-\infty, -1) \cup (1, \infty)$, $\Delta_a^-(Y_n) = \mathbb{N} \notin \mathcal{I}$. For $a \in (-1, 0) \cup (0, 1)$, $\Delta_a^-(Y_n)$ contains infinitely many even numbers. So, $\Delta_a^-(Y_n) \notin \mathcal{I}$.

$$\text{Therefore, } \overset{n\mathcal{I}-\lim inf}{n \to \infty} Y_n = \{0\}.$$

On the other side, $\Delta_{-1}^+(Y_n) = \{3, 5, 7, 11, 13, \dots\} \notin \mathcal{F}(\mathcal{I}) \cup (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I})$; $\Delta_1^+(Y_n) \notin \mathcal{F}(\mathcal{I}) \cup (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I})$ and $\Delta_0^+(Y_n) = 2\mathbb{N} \cup \{1, 9, 15, 21, \dots\} \in \mathcal{F}(\mathcal{I}) \cup (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I})$.
For $a \in (-\infty, -1) \cup (1, \infty) \cup (-\frac{2}{3}, -\frac{1}{2}) \cup (\frac{1}{2}, \frac{2}{3})$, $\Delta_a^+(Y_n) = \emptyset \notin \mathcal{F}(\mathcal{I}) \cup (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I})$. For $a \in (-1, -\frac{2}{3}] \cup [\frac{2}{3}, 1)$, $\Delta_a^+(Y_n) = \{3, 5, 7, 11, \dots\} \notin \mathcal{F}(\mathcal{I}) \cup (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I})$. For $a \in [-\frac{1}{2}, 0) \cup (0, \frac{1}{2}]$, $\Delta_a^+(Y_n) \subseteq 2\mathbb{N} \in \mathcal{F}(\mathcal{I}) \cup (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I})$.

$$\text{Therefore, } \overset{n\mathcal{I}-\lim sup}{n \to \infty} Y_n = \left[-\frac{1}{2}, \frac{1}{2} \right].$$

$$\text{Hence, } \overset{n\mathcal{I}-\lim inf}{n \to \infty} Y_n \subseteq \overset{n\mathcal{I}-\lim sup}{n \to \infty} Y_n.$$

Proposition 3.6. *For every sequence $\{Y_n : n \in \mathbb{N}\}$, in a topological space (X, τ),*

$$(i)\ \overset{\lim inf}{n \to \infty} Y_n \subseteq \overset{n\mathcal{I}_{fin}-\lim inf}{n \to \infty} Y_n.$$

$$(ii)\ \overset{\lim sup}{n \to \infty} Y_n \subseteq \overset{n\mathcal{I}_{fin}-\lim sup}{n \to \infty} Y_n.$$

Proof.

$$(i) \text{ Let } y \in \overset{\lim inf}{n \to \infty} Y_n = \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} Y_k.$$

So, $y \in \bigcap_{k \geq n_0} Y_k$ for at least one $n_0 \in \mathbb{N}$, i.e., $y \in Y_k$ for all $k \geq n_0$, for at least one $n_0 \in \mathbb{N}$. Therefore, $\{n \in \mathbb{N} : y \notin Y_n\} \subseteq \{1, 2, 3, \dots, n_0\}$, i.e., $\{n \in \mathbb{N} : y \notin Y_n\} \in \mathcal{I}_{fin}$. Now $U \cap Y_n = \emptyset$ for at least one neighborhood U of y implies that $y \notin Y_n$. So, $\Delta_y^-(Y_n) = \{n \in \mathbb{N} : U \cap Y_n = \emptyset$ for at least one neighborhood U of $y\} \subseteq \{n \in \mathbb{N} : y \notin Y_n\}$. Therefore, $\Delta_y^-(Y_n) \in \mathcal{I}_{fin}$. Thus $y \in \overset{n\mathcal{I}_{fin}-\lim inf}{n \to \infty} Y_n$. Hence, $\overset{\lim inf}{n \to \infty} Y_n \subseteq \overset{n\mathcal{I}_{fin}-\lim inf}{n \to \infty} Y_n$.

$$(ii) \text{ Let } y \in \overset{\lim sup}{n \to \infty} Y_n = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} Y_k.$$

Thus, for all $n \in \mathbb{N}$, $y \in \bigcup_{k \geq n} Y_k$. However large n is there is a $k \geq n$ such that $y \in Y_k$. So $y \in Y_k$ for infinitely many $n \in \mathbb{N}$. Thus, $\{n \in \mathbb{N} : y \in Y_n\} \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}_{fin}) \cup \mathcal{F}(\mathcal{I}_{fin})$. If $y \in Y_n$ then $U \cap Y_n \neq \emptyset$ for every neighborhood U of y. Thus, $\{n \in \mathbb{N} : y \in Y_n\} \subseteq \{n \in \mathbb{N} : U \cap Y_n \neq \emptyset$ for every neighborhood U of $y\} \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}_{fin}) \cup \mathcal{F}(\mathcal{I}_{fin})$, i.e., $\Delta_y^+(Y_n) \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}_{fin}) \cup \mathcal{F}(\mathcal{I}_{fin})$. Therefore, $y \in \overset{n\mathcal{I}_{fin}-\lim sup}{n \to \infty} Y_n$. Hence, $\overset{\lim sup}{n \to \infty} Y_n \subseteq \overset{n\mathcal{I}_{fin}-\lim sup}{n \to \infty} Y_n$. □

Corollary 3.7. *For every sequence $\{Y_n : n \in \mathbb{N}\}$ of closed sets, in a topological space (X, τ),*

$$(i)\ \overset{\lim inf}{n \to \infty}\ Y_n = \overset{{}^n\mathcal{I}_{fin}-\lim inf}{n \to \infty}\ Y_n = \overset{{}^n\mathcal{I}_{fin}-\lim sup}{n \to \infty}\ Y_n = \overset{\lim sup}{n \to \infty}\ Y_n.$$

Example 3.8. There exists a sequence of subsets which is ${}^n\mathcal{I}$-convergent but not convergent at all.
Let $X = \mathbb{N}$, $\tau = \{\emptyset, \{2\}\} \cup \{\{1, 2, 3, \dots n : n \in \mathbb{N}\}\}$ and $\mathcal{I}_{2\mathbb{N}} = \mathcal{P}(2\mathbb{N})$. Then (X, τ) is a topological space and $\mathcal{I}_{2\mathbb{N}}$ is an ideal on the set of natural numbers. Consider the sequence

$$Y_n = \begin{cases} \{1\}, \text{if } n \text{ is an even number,} \\ \{2\}, \text{otherwise.} \end{cases}$$

Now $\Delta_1^-(Y_n) = \{1, 3, 5, \dots\} \notin \mathcal{I}_{2\mathbb{N}}$; $\Delta_2^-(Y_n) = \{2, 4, 6, \dots\} \in \mathcal{I}_{2\mathbb{N}}$ and for other $a \in \mathbb{N}$, $\Delta_a^-(Y_n) = \emptyset \in \mathcal{I}_{2\mathbb{N}}$. Therefore, $\overset{{}^n\mathcal{I}_{2\mathbb{N}}-\lim inf}{n \to \infty}\ Y_n = \mathbb{N} \setminus \{1\}$.
Also, $\Delta_1^+(Y_n) = \{2, 4, 6, \dots\} \notin (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}_{2\mathbb{N}}) \cup \mathcal{F}(\mathcal{I}_{2\mathbb{N}})$; $\Delta_2^+(Y_n) = \{1, 3, 5, \dots\} \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}_{2\mathbb{N}}) \cup \mathcal{F}(\mathcal{I}_{2\mathbb{N}})$ and for other $a \in \mathbb{N}$, $\Delta_a^+(Y_n) = \mathbb{N} \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}_{2\mathbb{N}}) \cup \mathcal{F}(\mathcal{I}_{2\mathbb{N}})$. Therefore, $\overset{{}^n\mathcal{I}_{2\mathbb{N}}-\lim sup}{n \to \infty}\ Y_n = \mathbb{N} \setminus \{1\}$.

$$\text{Thus, } \overset{{}^n\mathcal{I}_{2\mathbb{N}}-\lim}{n \to \infty}\ Y_n = \mathbb{N} \setminus \{1\}$$

On the other side,

$$\bigcap_{n=1}^{\infty} \bigcup_{k \geq n} Y_n = \{1, 2\} \text{ and } \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} Y_n = \emptyset.$$

Thus $\overset{\lim}{n \to \infty}\ Y_n$ does not exists.

Sandwich theorem is a very important concept in analysis to verify the stability of any convergence criteria. So we try to give a sandwich theorem like characterization for the ${}^n\mathcal{I}$ convergence for the sequence of sets.

Theorem 3.9. *In a space X, $\{\Phi_n : n \in \mathbb{N}\}$, let $\{\Psi_n : n \in \mathbb{N}\}$ and $\{\Omega_n : n \in \mathbb{N}\}$ be three sequence of subsets such that $\overset{{}^n\mathcal{I}-\lim}{n \to \infty}\ \Phi_n = \overset{{}^n\mathcal{I}-\lim}{n \to \infty}\ \Omega_n = \Gamma$. If $\Phi_n \subseteq \Psi_n \subseteq \Omega_n$ for each $n \in \mathbb{N}$, then $\overset{{}^n\mathcal{I}-\lim}{n \to \infty}\ \Psi_n = \Gamma$.*

Proof. Let $\eta \in \Gamma$ be arbitrary. Now, $\overset{{}^n\mathcal{I}-\lim inf}{n \to \infty}\ \Phi_n = \Gamma$ as $\overset{{}^n\mathcal{I}-\lim}{n \to \infty}\ \Phi_n = \Gamma$. So, $\Delta_\eta^-(\Phi_n) \in \mathcal{I}$.
Since $\Phi_n \subseteq \Psi_n$ for each $n \in \mathbb{N}$, $\{n \in \mathbb{N} : U \cap \Psi_n = \emptyset$ for at least one neighborhood U of $\eta\ \} \subseteq \{n \in \mathbb{N} : U \cap \Phi_n = \emptyset$ for at least one neighborhood U of $\eta\ \}$. i.e., $\Delta_\eta^-(\Psi_n) \subseteq \Delta_\eta^-(\Phi_n) \in \mathcal{I}$. Therefore, $\Delta_\eta^-(\Psi_n) \in \mathcal{I}$. So, $\eta \in \overset{{}^n\mathcal{I}-\lim inf}{n \to \infty}\ \Psi_n$.

$$\text{Thus, } \Gamma \subseteq \overset{{}^n\mathcal{I}-\lim inf}{n \to \infty}\ \Psi_n. \tag{3.1}$$

Again let $\xi \in \overset{{}^n\mathcal{I}-\lim inf}{n \to \infty}\ \Psi_n$. So, $\Delta_\xi^-(\Psi_n) \in \mathcal{I}$.
Since $\Psi_n \subseteq \Omega_n$ for each $n \in \mathbb{N}$, $\{n \in \mathbb{N} : U \cap \Omega_n = \emptyset$ for at least one neighborhood U of $\xi\ \} \subseteq \{n \in \mathbb{N} : U \cap \Psi_n = \emptyset$ for at least one neighborhood U of $\xi\ \}$. i.e., $\Delta_\xi^-(\Omega_n) \subseteq \Delta_\xi^-(\Psi_n) \in \mathcal{I}$. This implies that $\Delta_\xi^-(\Omega_n) \in \mathcal{I}$. So, $\xi \in \overset{{}^n\mathcal{I}-\lim inf}{n \to \infty}\ \Omega_n$. But $\overset{{}^n\mathcal{I}-\lim}{n \to \infty}\ \Omega_n = \Gamma$. i.e., $\overset{{}^n\mathcal{I}-\lim inf}{n \to \infty}\ \Omega_n = \overset{{}^n\mathcal{I}-\lim sup}{n \to \infty}\ \Omega_n = \Gamma$. So. $\xi \in \Gamma$.

$$\text{Thus, } \overset{{}^n\mathcal{I}-\lim inf}{n \to \infty}\ \Psi_n \subseteq \Gamma. \tag{3.2}$$

So, from equation 3.1 and equation 3.2,

$$\overset{{}^n\mathcal{I}-\lim inf}{n \to \infty}\ \Psi_n = \Gamma. \tag{3.3}$$

Suppose, $\gamma \in \Gamma$ be arbitrary. $\overset{{}^n\mathcal{I}-\lim sup}{n \to \infty}\ \Phi_n = \Gamma$ as $\overset{{}^n\mathcal{I}-\lim}{n \to \infty}\ \Phi_n = \Gamma$. So, $\Delta_\gamma^+(\Phi_n) \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}) \cup \mathcal{F}(\mathcal{I})$.
Since, $\Phi_n \subseteq \Psi_n$ for each $n \in \mathbb{N}$,

$\{n \in \mathbb{N} : U \cap \Phi_n \neq \emptyset$ for every neighborhood U of $\gamma\} \subseteq \{n \in \mathbb{N} : U \cap \Psi_n \neq \emptyset$ for every neighborhood U of $\gamma\}$. i.e., $\Delta^+_\gamma(\Phi_n) \subseteq \Delta^+_\gamma(\Psi_n)$. Therefore, $\Delta^+_\gamma(\Psi_n) \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}) \cup \mathcal{F}(\mathcal{I})$ (By Lemma 3.3) . So, $\gamma \in \overset{n\mathcal{I}-\lim sup}{n \to \infty} \Psi_n$.

$$\text{Thus, } \Gamma \subseteq \overset{n\mathcal{I}-\lim sup}{n \to \infty} \Psi_n. \tag{3.4}$$

Now let $\zeta \in \overset{n\mathcal{I}-\lim sup}{n \to \infty} \Psi_n$. So, $\Delta^+_\zeta(\Psi_n) \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}) \cup \mathcal{F}(\mathcal{I})$.
Since, $\Psi_n \subseteq \Gamma_n$, for each $n \in \mathbb{N}$,
$\{n \in \mathbb{N} : U \cap \Psi_n \neq \emptyset$ for every neighborhood U of $\zeta\} \subseteq \{n \in \mathbb{N} : U \cap \Omega_n \neq \emptyset$ for every neighborhood U of $\zeta\}$. i.e., $\Delta^+_\zeta(\Psi_n) \subseteq \Delta^+_\zeta(\Omega_n)$. Therefore, $\Delta^+_\zeta(\Omega_n) \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}) \cup \mathcal{F}(\mathcal{I})$ (By Lemma 3.3) . So, $\zeta \in \overset{n\mathcal{I}-\lim sup}{n \to \infty} \Omega_n$. But, $\overset{n\mathcal{I}-\lim}{n \to \infty} \Omega_n = \Gamma$. So, $\zeta \in \Gamma$.

$$\text{Thus, } \overset{n\mathcal{I}-\lim sup}{n \to \infty} \Psi_n \subseteq \Gamma. \tag{3.5}$$

So, from equation 3.4 and equation 3.5,

$$\overset{n\mathcal{I}-\lim sup}{n \to \infty} \Psi_n = \Gamma. \tag{3.6}$$

Thus, from equation 3.3 and equation 3.6,

$$\overset{n\mathcal{I}-\lim}{n \to \infty} \Psi_n = \Gamma.$$

□

Proposition 3.10. *Let $\{\Phi_n : n \in \mathbb{N}\}$ and $\{\Psi_n : n \in \mathbb{N}\}$ be two sequences of subsets. Then*

$$(i)\ \overset{n\mathcal{I}-\lim inf}{n \to \infty} \Phi_n \cup \overset{n\mathcal{I}-\lim inf}{n \to \infty} \Psi_n \subseteq \overset{n\mathcal{I}-\lim inf}{n \to \infty} (\Phi_n \cup \Psi_n).$$

$$(ii)\ \overset{n\mathcal{I}-\lim sup}{n \to \infty} \Phi_n \cup \overset{n\mathcal{I}-\lim sup}{n \to \infty} \Psi_n \subseteq \overset{n\mathcal{I}-\lim sup}{n \to \infty} (\Phi_n \cup \Psi_n).$$

Proof. (i) Let $\beta \in \overset{n\mathcal{I}-\lim inf}{n \to \infty} \Phi_n \cup \overset{n\mathcal{I}-\lim inf}{n \to \infty} \Psi_n$. So, $\beta \in \overset{n\mathcal{I}-\lim inf}{n \to \infty} \Phi_n$ or $\beta \in \overset{n\mathcal{I}-\lim inf}{n \to \infty} \Psi_n$. Therefore, $\Delta^-_\beta(\Phi_n) \in \mathcal{I}$ or $\Delta^-_\beta(\Psi_n) \in \mathcal{I}$. i.e., $\Delta^-_\beta(\Phi_n) \cap \Delta^-_\beta(\Psi_n) \in \mathcal{I}$. So, $\Delta^-_\beta(\Phi_n \cup \Psi_n) \in \mathcal{I}$ (By Property 2.4). Thus, $\beta \in \overset{n\mathcal{I}-\lim inf}{n \to \infty} (\Phi_n \cup \Psi_n)$. Hence, $\overset{n\mathcal{I}-\lim inf}{n \to \infty} \Phi_n \cup \overset{n\mathcal{I}-\lim inf}{n \to \infty} \Psi_n \subseteq \overset{n\mathcal{I}-\lim inf}{n \to \infty} (\Phi_n \cup \Psi_n)$.
(ii) Let $\beta \in \overset{n\mathcal{I}-\lim sup}{n \to \infty} \Phi_n \cup \overset{n\mathcal{I}-\lim sup}{n \to \infty} \Psi_n$. So, $\beta \in \overset{n\mathcal{I}-\lim sup}{n \to \infty} \Phi_n$ or $\beta \in \overset{n\mathcal{I}-\lim sup}{n \to \infty} \Psi_n$. Therefore, $\Delta^+_\beta(\Phi_n) \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}) \cup \mathcal{F}(\mathcal{I})$ or $\Delta^+_\beta(\Psi_n) \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}) \cup \mathcal{F}(\mathcal{I})$. i.e., $\Delta^+_\beta(\Phi_n) \cup \Delta^+_\beta(\Psi_n) \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}) \cup \mathcal{F}(\mathcal{I})$ (By Lemma 3.3). So, $\Delta^+_\beta(\Phi_n \cup \Psi_n) \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}) \cup \mathcal{F}(\mathcal{I})$ (By Proposition 2.4). Thus, $\beta \in \overset{n\mathcal{I}-\lim sup}{n \to \infty} (\Phi_n \cup \Psi_n)$. Hence, $\overset{n\mathcal{I}-\lim sup}{n \to \infty} \Phi_n \cup \overset{n\mathcal{I}-\lim sup}{n \to \infty} \Psi_n \subseteq \overset{n\mathcal{I}-\lim sup}{n \to \infty} (\Phi_n \cup \Psi_n)$. □

Corollary 3.11. *Let $\{\Phi_n : n \in \mathbb{N}\}$ and $\{\Psi_n : n \in \mathbb{N}\}$ be two sequences of subsets. If $\overset{n\mathcal{I}-\lim}{n \to \infty} \Phi_n$, $\overset{n\mathcal{I}-\lim}{n \to \infty} \Psi_n$ and $\overset{n\mathcal{I}-\lim}{n \to \infty} (\Phi_n \cup \Psi_n)$ exists then*

$$\overset{n\mathcal{I}-\lim}{n \to \infty} \Phi_n \cup \overset{n\mathcal{I}-\lim}{n \to \infty} \Psi_n \subseteq \overset{n\mathcal{I}-\lim}{n \to \infty} (\Phi_n \cup \Psi_n).$$

Example 3.12. Consider the set $X = (-1, 1)$ and the topology τ induced by the usual topology. Consider the sequences $\{\Phi_n : n \in \mathbb{N}\}$ and $\{\Psi_n : n \in \mathbb{N}\}$, where

$$\Phi_n = \begin{cases} (-1, 0], & \text{if } n \in 2\mathbb{N}, \\ [0, 1), & \text{otherwise.} \end{cases} \qquad \Psi_n = \begin{cases} [0, 1), & \text{if } n \in 2\mathbb{N}, \\ (-1, 0], & \text{otherwise.} \end{cases}$$

Now, $\Delta^-_0(\Phi_n) = \emptyset \in \mathcal{I}_\delta$; for every $a \in (-1, 0)$, $\Delta^-_a(\Phi_n) = \{1, 3, 5, \dots\} \notin \mathcal{I}_\delta$ and for every $a \in (0, 1)$, $\Delta^-_a(\Phi_n) = \{2, 4, 6, \dots\} \notin \mathcal{I}_\delta$. Therefore, $\overset{n\mathcal{I}_\delta-\lim inf}{n \to \infty} \Phi_n = \{0\}$.

Also, $\Delta_0^-(\Psi_n) = \emptyset \in \mathcal{I}_\delta$; for every $a \in (-1,0)$, $\Delta_a^-(\Psi_n) = \{2,4,6,\dots\} \notin \mathcal{I}_\delta$ and for every $a \in (0,1)$, $\Delta_a^-(\Psi_n) = \{1,3,5,\dots\} \notin \mathcal{I}_\delta$. Therefore, $\overset{n\mathcal{I}_\delta-\lim inf}{n\to\infty}\ \Psi_n = \{0\}$.

$$\text{So, } \overset{n\mathcal{I}_\delta-\lim inf}{n\to\infty}\ \Phi_n \cup \overset{n\mathcal{I}_\delta-\lim inf}{n\to\infty}\ \Psi_n = \{0\}.$$

On the other side, $\Phi_n \cup \Psi_n = X,$ for every $n \in \mathbb{N}$ and for every $x \in X$, $\Delta_x^-(\Phi_n \cup \Psi_n) = \emptyset \in \mathcal{I}_\delta$. Therefore, $\overset{n\mathcal{I}_\delta-\lim inf}{n\to\infty}\ (\Phi_n \cup \Psi_n) = X$.

$$\text{Thus, } \overset{n\mathcal{I}_\delta-\lim inf}{n\to\infty}\ \Phi_n \cup \overset{n\mathcal{I}_\delta-\lim inf}{n\to\infty}\ \Psi_n \subsetneq \overset{n\mathcal{I}_\delta-\lim inf}{n\to\infty}\ (\Phi_n \cup \Psi_n).$$

Example 3.13. Consider the set $X = (-1,1)$ and the topology τ induced by the usual topology. Consider the sequences $\{\Phi_n : n \in \mathbb{N}\}$ and $\{\Psi_n : n \in \mathbb{N}\}$, where

$$\Phi_n = \begin{cases} (-\frac{1}{n}, 0], \text{if } n = k^2, k \in \mathbb{N}, \\ [0, \frac{1}{n}), \text{otherwise.} \end{cases} \qquad \Psi_n = \begin{cases} (-1, \frac{1}{n}] \cup (0,1), \text{ if } n = k^2, k \in \mathbb{N}, \\ (-1,0) \cup [\frac{1}{n}, 1), \text{otherwise.} \end{cases}$$

Now, $\Delta_0^+(\Phi_n) = \mathbb{N} \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}_\delta) \cup \mathcal{F}(\mathcal{I}_\delta)$. For every $a \in (-1,0)$, $\Delta_a^+(\Phi_n) = F$, where $F \subseteq \{1,4,9,16,\dots\}$ is finite. So, $\Delta_a^+(\Phi_n) \notin (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}_\delta) \cup \mathcal{F}(\mathcal{I}_\delta)$. For every $a \in (0,1)$, $\Delta_a^+(\Phi_n) = E$, where $E \subseteq \{2,3,5,6,\dots\}$ is finite. So, $\Delta_a^+(\Phi_n) \notin (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}_\delta) \cup \mathcal{F}(\mathcal{I}_\delta)$. Therefore, $\overset{n\mathcal{I}_\delta-\lim sup}{n\to\infty}\ \Phi_n = \{0\}$.
Also, $\Delta_0^+(\Psi_n) = \mathbb{N} \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}_\delta) \cup \mathcal{F}(\mathcal{I}_\delta)$; for every $a \in (-1,0)$, $\Delta_a^+(\Psi_n) = \{2,3,5,6,\dots\} \cup G$ where $G \subseteq \{1,4,9,16,\dots\}$ is finite. So, $\Delta_a^+(\Psi_n) \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}_\delta) \cup \mathcal{F}(\mathcal{I}_\delta)$. For every $a \in (0,1)$, $\Delta_a^+(\Psi_n) = \{1,4,9,16,\dots\} \cup H$ where $H \subseteq \{2,3,5,6,\dots\}\}$ is finite. So, $\Delta_a^+(\Psi_n) \notin (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}_\delta) \cup \mathcal{F}(\mathcal{I}_\delta)$. Therefore, $\overset{n\mathcal{I}_\delta-\lim sup}{n\to\infty}\ \Psi_n = (-1,0]$.

$$\text{So, } \overset{n\mathcal{I}_\delta-\lim sup}{n\to\infty}\ \Phi_n \cup \overset{n\mathcal{I}_\delta-\lim sup}{n\to\infty}\ \Psi_n = (-1,0].$$

On the other side, $\Phi_n \cup \Psi_n = X,$ for every $n \in \mathbb{N}$ and for every $x \in X$, $\Delta_x^+(\Phi_n \cup \Psi_n) = \mathbb{N} \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}_\delta) \cup \mathcal{F}(\mathcal{I}_\delta)$. Therefore, $\overset{n\mathcal{I}_\delta-\lim sup}{n\to\infty}\ (\Phi_n \cup \Psi_n) = X$.

$$\text{Thus, } \overset{n\mathcal{I}_\delta-\lim sup}{n\to\infty}\ \Phi_n \cup \overset{n\mathcal{I}_\delta-\lim sup}{n\to\infty}\ \Psi_n \subsetneq \overset{n\mathcal{I}_\delta-\lim sup}{n\to\infty}\ (\Phi_n \cup \Psi_n).$$

Proposition 3.14. *For any sequence* $\{Y_n : n \in \mathbb{N}\}$,

$$(i) \quad \big(\overset{n\mathcal{I}-\lim inf}{n\to\infty} Y_n\big) \cap \big(\bigcup_{n\in\Delta_a^-(Y_n)} \overline{Y_n}\big) = \emptyset \text{ for all } a \in \overset{n\mathcal{I}-\lim inf}{n\to\infty} Y_n.$$

$$(ii) \quad \overset{n\mathcal{I}-\lim sup}{n\to\infty} Y_n \subseteq \bigcap_{n\in\Delta_a^+(Y_n)} \overline{Y_n} \text{ for all } a \in \overset{n\mathcal{I}-\lim sup}{n\to\infty} Y_n.$$

Proof. (i) Let $a \in \overset{n\mathcal{I}-\lim inf}{n\to\infty} Y_n$ be arbitrary. So, $\Delta_a^-(Y_n) \in \mathcal{I}$. Thus, for all $n \in \Delta_a^-(Y_n)$, we can find a neighborhood U of a such that $U \cap Y_n = \emptyset$. So a can not be a limit point of Y_n if $n \in \Delta_a^-(Y_n)$. Therefore, $a \notin \overline{Y_n}$ for all $n \in \Delta_a^-(Y_n)$.

$$\text{So, } a \in \bigcap_{n\in\Delta_a^-(Y_n)} X \setminus \overline{Y_n}, \text{ Therefore, } a \in X \setminus \Big(\bigcup_{n\in\Delta_a^-(Y_n)} \overline{Y_n}\Big).$$

$$\text{Therefore, } \overset{n\mathcal{I}-\lim inf}{n\to\infty} Y_n \subseteq X \setminus \Big(\bigcup_{n\in\Delta_a^-(Y_n)} \overline{Y_n}\Big) \text{ for all } a \in \overset{n\mathcal{I}-\lim inf}{n\to\infty} Y_n.$$

$$\text{Hence, } \overset{n\mathcal{I}-\lim inf}{n\to\infty} Y_n \cap \Big(\bigcup_{n\in\Delta_a^-(Y_n)} \overline{Y_n}\Big) = \emptyset \text{ for all } a \in \overset{n\mathcal{I}-\lim inf}{n\to\infty} Y_n.$$

(ii) Let $a \in \overset{n\mathcal{I}-\lim sup}{n \to \infty} Y_n$ be arbitrary. So, $\Delta_a^+(Y_n) \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}) \cup \mathcal{F}(\mathcal{I})$. Thus, for all $n \in \Delta_a^+(Y_n)$, every neighborhood U of a is such that $U \cap Y_n \neq \emptyset$. So a is a limit point of Y_n if $n \in \Delta_a^+(Y_n)$. Therefore, $a \in \overline{Y_n}$ for all $n \in \Delta_a^+(Y_n)$.

$$\text{So, } a \in \bigcap_{n \in \Delta_a^+(Y_n)} \overline{Y_n}.$$

$$\text{Hence, } \overset{n\mathcal{I}-\lim sup}{n \to \infty} Y_n \subseteq \bigcap_{n \in \Delta_a^+(Y_n)} \overline{Y_n} \text{ for all } a \in \overset{n\mathcal{I}-\lim sup}{n \to \infty} Y_n.$$

□

Proposition 3.15. *In a topological space X, if $\{\Omega_n^{(i)} : n \in \mathbb{N}\}_{i=1}^p$ are sequences of subsets of X, then*

$$(i) \quad \overset{n\mathcal{I}-\lim inf}{n \to \infty}\Big(\prod_{i=1}^p \Omega_n^{(i)}\Big) = \prod_{i=1}^p \overset{n\mathcal{I}-\lim inf}{n \to \infty} \Omega_n^{(i)}.$$

$$(ii) \quad \overset{n\mathcal{I}-\lim sup}{n \to \infty}\Big(\prod_{i=1}^p \Omega_n^{(i)}\Big) \subseteq \prod_{i=1}^p \overset{n\mathcal{I}-\lim sup}{n \to \infty} \Omega_n^{(i)}.$$

Proof. (i) Let $z = (z_1, z_2, \dots, z_p) \in \overset{n\mathcal{I}-\lim inf}{n \to \infty}\Big(\prod_{i=1}^p \Omega_n^{(i)}\Big)$. So $\Delta_z^-\Big(\prod_{i=1}^p \Omega_n^{(i)}\Big) \in \mathcal{I}$. By Proposition 2.5, $\bigcup_{i=1}^p \Delta_{z_i}^-\Big(\Omega_n^{(i)}\Big) \in \mathcal{I}$. Therefore $\Delta_{z_i}^-\Big(\Omega_n^{(i)}\Big) \in \mathcal{I}$ for each $i = 1, 2, 3, \dots, p$. Thus, $z_i \in \overset{n\mathcal{I}-\lim inf}{n \to \infty} \Omega_n^{(i)}$ for each $i = 1, 2, 3, \dots, p$. So, $z = (z_1, z_2, \dots, z_p) \in \prod_{i=1}^p \overset{n\mathcal{I}-\lim inf}{n \to \infty} \Omega_n^{(i)}$. Therefore, $\overset{n\mathcal{I}-\lim inf}{n \to \infty}\Big(\prod_{i=1}^p \Omega_n^{(i)}\Big) \subseteq \prod_{i=1}^p \overset{n\mathcal{I}-\lim inf}{n \to \infty} \Omega_n^{(i)}$.

Conversely, let $z = (z_1, z_2, \dots, z_p) \in \prod_{i=1}^p \overset{n\mathcal{I}-\lim inf}{n \to \infty} \Omega_n^{(i)}$. So, $z_i \in \overset{n\mathcal{I}-\lim inf}{n \to \infty} \Omega_n^{(i)}$ for each $i = 1, 2, 3, \dots, p$. i.e., $\Delta_{z_i}^-\Big(\Omega_n^{(i)}\Big) \in \mathcal{I}$, for each $i = 1, 2, 3, \dots, p$. Therefore, $\bigcup_{i=1}^p \Delta_{z_i}^-\Big(\Omega_n^{(i)}\Big) \in \mathcal{I}$. Now, by Proposition 2.5, $\Delta_z^-\Big(\prod_{i=1}^p \Omega_n^{(i)}\Big) \in \mathcal{I}$. So, $z \in \overset{n\mathcal{I}-\lim inf}{n \to \infty}\Big(\prod_{i=1}^p \Omega_n^{(i)}\Big)$. Therefore, $\prod_{i=1}^p \overset{n\mathcal{I}-\lim inf}{n \to \infty} \Omega_n^{(i)} \subseteq \overset{n\mathcal{I}-\lim inf}{n \to \infty}\Big(\prod_{i=1}^p \Omega_n^{(i)}\Big)$.

$$\text{Hence, } \overset{n\mathcal{I}-\lim inf}{n \to \infty}\Big(\prod_{i=1}^p \Omega_n^{(i)}\Big) = \prod_{i=1}^p \overset{n\mathcal{I}-\lim inf}{n \to \infty} \Omega_n^{(i)}.$$

(ii) Let $z = (z_1, z_2, \dots, z_p) \in \overset{n\mathcal{I}-\lim sup}{n \to \infty}\Big(\prod_{i=1}^p \Omega_n^{(i)}\Big)$. So $\Delta_z^+\Big(\prod_{i=1}^p \Omega_n^{(i)}\Big) \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}) \cup \mathcal{F}(\mathcal{I})$. By Proposition 2.5, $\bigcap_{i=1}^p \Delta_{z_i}^+\Big(\Omega_n^{(i)}\Big) \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}) \cup \mathcal{F}(\mathcal{I})$. Therefore, $\Delta_{z_i}^+\Big(\Omega_n^{(i)}\Big) \in (\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}) \cup \mathcal{F}(\mathcal{I})$ for each $i = 1, 2, 3, \dots, p$. Thus, $z_i \in \overset{n\mathcal{I}-\lim sup}{n \to \infty} \Omega_n^{(i)}$ for each $i = 1, 2, 3, \dots, p$. So, $z = (z_1, z_2, \dots, z_p) \in \prod_{i=1}^p \overset{n\mathcal{I}-\lim sup}{n \to \infty} \Omega_n^{(i)}$.

$$\text{Therefore, } \overset{n\mathcal{I}-\lim sup}{n \to \infty}\Big(\prod_{i=1}^p \Omega_n^{(i)}\Big) \subseteq \prod_{i=1}^p \overset{n\mathcal{I}-\lim sup}{n \to \infty} \Omega_n^{(i)}.$$

□

Example 3.16. Let $X = \{1, 2\}$, then $\tau = \{\emptyset, \{1\}, X\}$ is a topology on X.
$\mathcal{B} = \tau \times \tau = \{\emptyset, \{(1,1)\}, \{(1,1),(1,2)\}, \{(1,1),(2,1)\}, X \times X\}$ is the base for the product space defined on $X \times X = \{(1,1),(1,2),(2,1),(2,2)\}$. Consider the sequences

$$\Phi_n = \begin{cases} \{1\}, \text{if } n \text{ is even,} \\ \{2\}, \text{otherwise.} \end{cases} \quad \text{and} \quad \Psi_n = \begin{cases} \{2\}, \text{if } n \text{ is even,} \\ \{1\}, \text{otherwise.} \end{cases}$$

Now, $\Delta_1^+(\Phi_n) = 2\mathbb{N} \notin \mathcal{I}_\delta$; $\Delta_2^+(\Phi_n) = \mathbb{N} \notin \mathcal{I}_\delta$; $\Delta_1^+(\Psi_n) = \mathbb{N} \setminus 2\mathbb{N} \notin \mathcal{I}_\delta$ and $\Delta_2^+(\Psi_n) = \mathbb{N} \notin \mathcal{I}_\delta$. So, $\overset{^n\mathcal{I}-\lim sup}{n \to \infty}(\Phi_n) = \{1, 2\}$ and $\overset{^n\mathcal{I}-\lim sup}{n \to \infty}(\Psi_n) = \{1, 2\}$. Therefore, $\overset{^n\mathcal{I}-\lim sup}{n \to \infty}(\Phi_n) \times \overset{^n\mathcal{I}-\lim sup}{n \to \infty}(\Psi_n) = \{(1, 1),(1, 2),(2, 1),(2, 2)\}$.

$$\text{But, } \Phi_n \times \Psi_n = \begin{cases} \{(1, 2)\} \text{if } n \text{ is even,} \\ \{(2, 1)\} \text{otherwise.} \end{cases}$$

$\Delta^+_{(1,1)}(\Phi_n \times \Psi_n) = \emptyset \in \mathcal{I}_\delta$; $\Delta^+_{(1,2)}(\Phi_n \times \Psi_n) = 2\mathbb{N} \notin \mathcal{I}_\delta$; $\Delta^+_{(2,1)}(\Phi_n \times \Psi_n) = \mathbb{N} \setminus 2\mathbb{N} \notin \mathcal{I}_\delta$ and $\Delta^+_{(2,2)}(\Phi_n \times \Psi_n) = \mathbb{N} \notin \mathcal{I}_\delta$. So, $\overset{^n\mathcal{I}-\lim sup}{n \to \infty}(\Phi_n \times \Psi_n) = \{(1, 2),(2, 1),(2, 2)\}$.

$$\text{Thus, } \overset{^n\mathcal{I}-\lim sup}{n \to \infty}\Big(\Phi_n \times \Psi_n\Big) \subseteq \overset{^n\mathcal{I}-\lim sup}{n \to \infty} \Phi_n \times \overset{^n\mathcal{I}-\lim sup}{n \to \infty} \Psi_n.$$

Theorem 3.17. *In an $\mathcal{I}$-compact space, every sequence $\{\Omega_n : n \in \mathbb{N}\}$ of closed sets having I-intersection property possess a $^n\mathcal{I} - \lim inf$.*

Proof. Let X be an $\mathcal{I}$-compact space and the sequence $\{\Omega_n : n \in \mathbb{N}\}$ of closed sets possess I-intersection property.

Let $^n\mathcal{I} - \lim inf$ of $\{\Omega_n : n \in \mathbb{N}\}$ does not exists. So, $\overset{^n\mathcal{I}-\lim inf}{n \to \infty}\Omega_n = \emptyset$. Therefore, for every $x \in X$, $\Delta^-_x(\Omega_n) \notin \mathcal{I}$. Let $a \in X$ is arbitrary.

$$\text{So, } \Delta^-_a(\Omega_n) = \{k \in \mathbb{N} : U \cap \Omega_k = \emptyset \text{ for at least one neighborhood } U \text{ of } a\} \notin \mathcal{I}.$$

$$\text{i.e., } \Delta^-_a(\Omega_n) = \{k \in \mathbb{N} : a \notin \overline{\Omega_k}\} \notin \mathcal{I}.$$

But Ω_k is closed for all $k \in \mathbb{N}$ and $\emptyset \in \mathcal{I}$. Therefore, $\Delta^-_a(\Omega_n) = \{k \in \mathbb{N} : a \notin \Omega_k\} \neq \emptyset$.

$$\text{Thus, } a \notin \bigcap_{\Delta^-_a(\Omega_n)} \Omega_n \supseteq \bigcap_{n\in\mathbb{N}} \Omega_n.$$

$$\bigcap_{n\in\mathbb{N}} \Omega_n = \emptyset, \text{ which is a contradiction to } \mathcal{I}\text{-intersection theorem (Theorem 2.9).}$$

Therefore, $\overset{^n\mathcal{I}-\lim inf}{n \to \infty}\Omega_n \neq \emptyset$. Hence, $^n\mathcal{I} - \lim inf$ of $\{\Omega_n : n \in \mathbb{N}\}$ exists. □

4. Conclusion

This study established an ideal convergence criteria for the sequence of sets that is applicable for any sequence of sets irrespective of the nature of the sets. Sandwich theorem holds for this $^n\mathcal{I}$ convergence criteria. The $^n\mathcal{I} - \lim inf$ of finite product of sequence of sets is equal to finite product of $^n\mathcal{I} - \lim inf$ of sequence of sets. But this property does not hold for $^n\mathcal{I} - \lim sup$. Moreover, in an $\mathcal{I}$-compact space, sequence of closed sets having $\mathcal{I}$ intersection property maintains a nonempty $^n\mathcal{I} - \lim inf$. The concept can further be used for the analysis of bounded sequence of sets. In light of [1] and [4],this study explores a new avenue for the application of ideal convergence in the field of selection principles.

Acknowledgement

We are thankful to the reviewers for their careful comments and suggestions that have immensely improved the presentation of the results. The second au thor (Prasenjit Bal) acknowledges the support of the National Board for Higher Mathematics (NBHM), Department of Atomic Energy (DAE), Government of India, under project file no. 020211/233/2025/NBHM(R.P.)/R&D II/10465.

References

[1] P. Bal. On the class of i-γ-open cover and i-st-γ-open cover. *Hacettepe Journal of Mathematics and Statistics*, 52(3):630–639, 2023. 1, 4

[2] P. Bal and T. Datta. Statistical convergence for sequences of sets via topological neighborhoods. 2025(communicated). 2, 2.2, 2, 2.3, 2.4, 2.5

[3] P. Bal, D. Rakshit, and S. Sarkar. Countable compactness modulo an ideal of natural numbers. *Ural Mathematical Journal*, 9(2 (17)):28–35, 2023. 2.7, 2.8, 2.9

[4] P. Das, L. D. R. Kočinac, and D. Chandra. Some remarks on open covers and selection principles using ideals. *Topol. Appl.*, 202:183–193, 2016. 4

[5] G. Di Maio and L. D. Kočinac. Statistical convergence in topology. *Topology and its Applications*, 156(1):28–45, 2008. 1

[6] R. Engelking. General topology, sigma series in pure mathematics, vol. 6, helder-mann, berlin. *Revised and completed edition*, 1989. 2

[7] H. Fast. Sur la convergence statistique. 2(3-4):241–244, 1951. 1

[8] J. A. Fridy. On statistical convergence. *Analysis*, 5(4):301–314, 1985. 1

[9] B. Inan and M. Küçükaslan. Ideal convergence of sequence of sets. *Contem. Anal. Appl. Math*, 3(2):184–212, 2015. 1, 2.1

[10] P. Kostyrko, W. Wilczyński, and T. Šalát. I-convergence. *Real Analysis Exchange*, 26(2):669–686, 2000. 1

[11] K. Kuratowski. *Topology: Volume II.* Academic Press, 1966. 1

[12] B. K. Lahiri and P. Das. I and I^*-convergence in topological spaces. *Mathematica Bohemica*, 130(2):153–160, 2005. 2.6

[13] M. Mursaleen. Statistical convergence of sequences of fuzzy numbers. *Mathematica Slovaca*, 50(1):111–115, 2000. 1

[14] F. Nuray and B. Rhoades. Statistical convergence of sequences of sets. *Fasciculi Mathematici*, 49(2):87–99, 2012. 1

[15] S. Sarkar, P. Bal, and M. Datta. On star rothberger spaces modulo an ideal. *Applied General Topology*, 25(2):407–414, 2024. 1

[16] R. A. Wijsman. Convergence of sequences of convex sets, cones and functions. *Bull. Amer. Math. Soc.*, 70:186–188, 1964. 1

[17] R. A. Wijsman. Convergence of sequences of convex sets, cones and functions. ii. *Transactions of the American Mathematical Society*, 123(1):32–45, 1966. 1

[18] L. Zoretti. *Leçons sur le prolongement analytique professés au Collège de France*, volume 14. Gauthier-Villars, 1911. 1