



Singular Perturbation for Beam Equation Involving Conformable Derivative: Exact and Asymptotic Analysis

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Abstract

This work deals with an analysis of a singularly perturbed nonlinear beam equation that involves Khalil conformable derivatives. Our study is presented in two parts. We begin first by establishing the mathematical framework to derive the proposed perturbed equation and performing non-dimensionalization of the problem. For the case $\varepsilon_\alpha = 0$, we obtain exact traveling wave solutions of the form $S(\xi) = A_0 + A_1 \tanh(\mu\xi)$. The second part addresses the physically relevant case $\varepsilon_\alpha \neq 0$ using matched asymptotic expansions. We show that the solution displays multi-scale behavior, characterized by outer regions that satisfy the reduced equation and inner boundary layers of thickness $O(\varepsilon_\alpha)$. The composite solution connects these regions and preserves the wave structure. Our analysis derives analytical expressions and establishes validity conditions, showing that the Khalil derivative introduces refined scaling.

Keywords: Khalil conformable derivative, Singular perturbation, Traveling waves, Matched asymptotic expansion, Nonlinear beam equation, Boudary layer.

2010 MSC: 35C07, 35Q74, 34E15, 74K10.

1. Introduction

The study of beam structures has long been an important topic in structural mechanics and engineering, with applications including railway tracks, pavement systems, and micro-electromechanical systems [22, 30, 39, 42]. The classical Winkler foundation model has been considered and provides the theoretical basis for many engineering design procedures [17, 21]. However, several investigations have revealed that many foundation materials show significant nonlinear behavior, particularly under large deformations [6, 20, 27, 31].

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The incorporation of nonlinear foundation models has led to richer mathematical structures in beam problems, enabling the prediction of complex phenomena such as bifurcations and localized deformation patterns [41]. The problem studied in this work, characterized by a quadratic nonlinearity in the displacement field, captures essential features of soil-structure interaction where stiffness varies with deformation magnitude [43].

Recently, the application of fractional calculus to structural mechanics has opened new avenues to study materials with complex microstructures [32, 36, 38]. Although the present work focuses on structural mechanics, relevant references from other areas of fractional calculus—unrelated to this specific field—are also incorporated to enrich the theoretical framework [8, 9, 10, 19, 35]. Several recent contributions have advanced this field; the authors in [37] investigated fractional differential equations with applications, in [28, 33], the authors explored advanced methods in fractional calculus, and the authors of [4] contributed to the theory of fractional boundary value problems. These works complement the growing literature on fractional modeling in mechanics, see also [11, 15].

This work substitutes classical spatial derivatives in the beam equation with Khalil conformable derivatives [1, 24] to account for scale-dependent mechanical behavior. Unlike Caputo or Riemann-Liouville operators, which embed memory effects through nonlocal kernels, the Khalil derivative acts locally and modifies deformation gradients via the factor $x^{1-\alpha}$. This factor captures how effective stiffness varies with position in functionally graded or micro-structured beams, preserves product and chain rules for analytical tractability. The resulting framework models size effects without the computational complexity of nonlocality in integral terms. For detailed comparisons among fractional derivatives, see [2, 3, 5, 34].

As a local operator defined by $D_x^\alpha g(x) = x^{1-\alpha} g'(x)$, the Khalil derivative retains the product, quotient, and chain rules—greatly facilitating analytical methods like tanh expansions and matched asymptotics. Its limitation is the inability to capture long-range memory, as the value at any point depends only on an infinitesimal neighborhood. Conversely, Caputo and Riemann-Liouville derivatives (nonlocal, kernel $(x-t)^{-\alpha}$) describe hereditary phenomena but are computationally heavier and violate the classical Leibniz rule. Hence, Caputo differential problems are preferred when viscoelastic memory dominates, whereas the Khalil derivative excels for scale-dependent local stiffness variations, such as in functionally graded beams. Recent comparative studies [2, 3, 5] reinforce this distinction. We adopt the Khalil derivative precisely because its locality enables a clean singular perturbation analysis and introduces fractional scaling.

We emphasize that we work with the sequential derivative $D_x^\alpha D_x^\alpha D_x^\alpha D_x^\alpha$, which is not equivalent to a single Khalil derivative of order 4α . Our analysis focuses on the regime where $4\alpha \in (3, 4]$, ensuring that the operator captures the fourth-order bending behavior characteristic of beam equations. This sequential interpretation is essential for the problem.

Taking into account that many fractional derivatives capture non-local phenomena and long-range interactions in heterogeneous materials [13, 26], the Khalil conformable derivative [24] offers a different approach as a local operator with several advantageous properties. It satisfies classical derivative rules and maintains physical study through its connection to standard derivatives via a simple scaling relationship.

The application of conformable derivatives to beam problems has gained attention due to their ability to model scale-dependent effects in micro- and nano-structures and several researchers have investigated the behavior of beams on elastic foundations using this approach [1, 7, 12, 14, 16]. However, the combination of nonlinear foundation behavior with conformable derivative operators in beam governing equations remains relatively unexplored, in the context of traveling wave solutions and singular perturbation analysis.

In the beam context, the spatical fractional order α ($0 < \alpha \leq 1$) quantifies the degree of spatial heterogeneity or scale-dependent stiffness in the beam. When $\alpha = 1$, the material is homogeneous and classical beam theory is recovered. For $\alpha < 1$, the effective bending stiffness becomes scale-sensitive: the term $H^{4\alpha}$ in the non-dimensional parameter $\varepsilon_\alpha^2 = EI/(k_0 H^{4\alpha})$ shows that as α decreases, the foundation appears relatively stiffer for a given beam length H , or equivalently, the bending resistance is weaker. This models microstructured or functionally graded beams where material properties vary with the scale of deformation. The sequential Khalil derivative $D_x^{4\alpha}$ captures a form of progressive local scaling: each derivative introduces a factor $x^{1-\alpha}$, reflecting how the strain gradient is modulated by position in a heterogeneous medium. Thus

α serves as a tunable exponent that bridges classical continuum mechanics ($\alpha = 1$) and fractal-like structural responses ($\alpha < 1$).

This work aims to bridge this gap by investigating the nonlinear dynamics of beams on nonlinear elastic foundations using the Khalil conformable derivative framework in the context of singular perturbations [29]. We develop a mathematical formulation, perform non-dimensionalization, and analyze traveling wave solutions through both exact methods for simplified cases and singular perturbation techniques for the full problem. The results provide new insights into the relationship between nonlinear foundation behavior, fractional derivative order, and wave propagation characteristics in structural systems.

As a related contribution, the authors [25] studied the governing equation for the beam deflection $w(x, t)$ at position x and time t which is given by:

$$EI \frac{\partial^4 w}{\partial x^4} - T \frac{\partial^2 w}{\partial x^2} + k(w) w + \rho A \frac{\partial^2 w}{\partial t^2} = p,$$

where EI denotes the bending stiffness of the beam, T represents an applied tension, and $k(w)$ describes the nonlinear stiffness distribution of the elastic foundation.

Compared to some existing beam models [23, 25, 40], our approach using the Khalil derivative offers the advantage of preserving the rules of classical calculus while introducing fractional scaling. This yields analytical solutions that are easier to study, as demonstrated by the traveling wave solutions obtained in Section 5. However, unlike Caputo-based models, our approach does not capture memory effects.

Using the replacement of classical spatial derivatives with Khalil conformable derivatives of order $0 < \alpha \leq 1$, we study the following problem.

$$EI D_x^\alpha D_x^\alpha D_x^\alpha D_x^\alpha f(x, t) - T D_x^\alpha D_x^\alpha f(x, t) + k(f) f(x, t) + \rho A \frac{\partial^2 f}{\partial t^2}(x, t) = p, \quad (1.1)$$

where $k(f) := k_0 \left(1 - a \frac{f}{H} + b \left(\frac{f}{H} \right)^2 \right)$, k_0 is the reference linear stiffness and a, b, H are parameters.

Note that $D_x^\alpha D_x^\alpha D_x^\alpha D_x^\alpha$ denotes the sequential application of the Khalil derivative of order α four times. This is not equivalent to $D_x^{4\alpha}$, a single Khalil derivative of order 4α . For $\alpha \in (0.75, 1]$, the effective order 4α lies in the interval $(3, 4]$, capturing the fourth-order bending behavior. The case where $4\alpha < 3$ is not relevant for beam equations and is excluded from our analysis.

The remainder of this paper is structured as follows. In Section 2, we recall the necessary preliminaries on the Khalil conformable derivative and its scaling properties. Section 3 deals with the non-dimensionalization of our proposed fractional-order beam equation, by identifying the parameter ε_α . Section 4 presents a traveling-wave reduction, transforming the governing problem into a fourth-order nonlinear ODE. The analytical results are presented in two main parts: Section 5 analyzes the unperturbed equation ($\varepsilon_\alpha = 0$), employing the tanh method to derive exact traveling wave solutions, and Section 6 discusses a singular perturbation analysis for the full fourth-order problem using matched asymptotic expansions to resolve boundary layer effects. In the same section some illustrative numerical simulations are discussed. Finally, a conclusion follows.

2. Preliminaries on Khalil conformable derivative

For a differentiable function $g(x)$ with $x > 0$, the Khalil conformable derivative of order $\alpha \in (0, 1]$ is defined by

$$D_x^\alpha g(x) = \lim_{\varepsilon \rightarrow 0} \frac{g(x + \varepsilon x^{1-\alpha}) - g(x)}{\varepsilon} = x^{1-\alpha} \frac{dg}{dx}(x). \quad (2.1)$$

For sequential applications, we define $D_x^\alpha D_x^\alpha g = D_x^\alpha (D_x^\alpha g)$, and similarly for higher orders. This sequential operator is not equivalent to a single Khalil derivative of order 2α , unless specific conditions are met. For higher orders where $n\alpha > 1$ with $n \in \mathbb{N}$, we interpret $D_x^{n\alpha}$ as the sequential application D_x^α repeated

n times. This interpretation ensures dimensional consistency and proper scaling properties throughout our analysis.

In this work, we focus on the regime $4\alpha \in (3, 4]$ to ensure that the sequential operator $D_x^\alpha D_x^\alpha D_x^\alpha D_x^\alpha$ captures fourth-order bending behavior.

Under the change of variable $x = H\bar{x}$, one obtains

$$D_x^\alpha g(x) = H^{-\alpha} D_{\bar{x}}^\alpha \bar{g}(\bar{x}), \tag{2.2}$$

which generalizes the usual scaling law for classical derivatives.

The following properties are crucial in our work. Let

$$x = H\bar{x}, \quad g(x) = \bar{g}(\bar{x}).$$

Then, we have

$$\frac{dg}{dx}(x) = \frac{1}{H} \frac{d\bar{g}}{d\bar{x}}(\bar{x}).$$

Hence

$$D_x^\alpha g(x) = (H\bar{x})^{1-\alpha} \frac{1}{H} \frac{d\bar{g}}{d\bar{x}}(\bar{x}) = H^{-\alpha} \bar{x}^{1-\alpha} \frac{d\bar{g}}{d\bar{x}}(\bar{x}). \tag{2.3}$$

By definition, this gives

$$D_x^\alpha g(x) = H^{-\alpha} D_{\bar{x}}^\alpha \bar{g}(\bar{x}). \tag{2.4}$$

Now, applying D_x^α , we obtain

$$D_x^{2\alpha} g(x) = D_x^\alpha (D_x^\alpha g(x)).$$

From the previous result, it yields that

$$D_x^\alpha g(x) = H^{-\alpha} D_{\bar{x}}^\alpha \bar{g}(\bar{x}).$$

Therefore,

$$D_x^{2\alpha} g(x) = H^{-\alpha} D_x^\alpha (D_{\bar{x}}^\alpha \bar{g}(\bar{x})).$$

Applying the scaling rule, we obtain

$$D_x^\alpha (D_{\bar{x}}^\alpha \bar{g}(\bar{x})) = H^{-\alpha} D_{\bar{x}}^\alpha (D_{\bar{x}}^\alpha \bar{g}(\bar{x})),$$

that is

$$D_x^{2\alpha} g(x) = H^{-2\alpha} D_{\bar{x}}^{2\alpha} \bar{g}(\bar{x}). \tag{2.5}$$

By induction, for the sequential application m times, we can write

$$D_x^{m\alpha} g(x) = H^{-m\alpha} D_{\bar{x}}^{m\alpha} \bar{g}(\bar{x}), \tag{2.6}$$

where $D_x^{m\alpha}$ denotes the sequential application D_x^α repeated m times, not a single Khalil derivative of order $m\alpha$. In the remainder of this paper, we will use the notation $D_x^{4\alpha}$ as a shorthand for the sequential operator $D_x^\alpha D_x^\alpha D_x^\alpha D_x^\alpha$, with the understanding that this represents four successive applications of the Khalil derivative of order α .

3. Non-dimensionalization

To reduce the number of governing parameters, we non-dimensionalize our equation. We set:

$$\bar{f} = \frac{f}{H}, \quad \bar{x} = \frac{x}{H}, \quad \bar{t} = \frac{t}{t_0}, \quad t_0 = \sqrt{\frac{\rho A}{k_0}},$$

and also

$$\bar{p} = \frac{p}{k_0 H}.$$

Using Khalil derivatives, equation (1.1) becomes

$$\varepsilon_\alpha^2 D_{\bar{x}}^{4\alpha} \bar{f} - \tau_\alpha D_{\bar{x}}^{2\alpha} \bar{f} + V(\bar{f}) \bar{f} + \frac{\partial^2 \bar{f}}{\partial \bar{t}^2} = \bar{p}, \tag{3.1}$$

where

$$V(\bar{f}) = a + b\bar{f} + \bar{f}^2,$$

and:

$$\varepsilon_\alpha^2 = \frac{EI}{k_0 H^{4\alpha}}, \quad \tau_\alpha = \frac{T}{k_0 H^{2\alpha}}. \tag{3.2}$$

In equation (3.1), the notation $D_{\bar{x}}^{4\alpha}$ is a shorthand for the sequential operator $D_{\bar{x}}^\alpha D_{\bar{x}}^\alpha D_{\bar{x}}^\alpha D_{\bar{x}}^\alpha$, justified by the scaling property derived in (2.6).

The small parameter in our asymptotic analysis is ε_α (not ε_α^2). Although ε_α^2 appears in the equation, the expansions $S = S_0 + \varepsilon_\alpha S_1 + \varepsilon_\alpha^2 S_2 + \dots$ are performed in powers of ε_α . This choice is standard in singular perturbation theory when the highest derivative is multiplied by a squared small parameter; the boundary layer thickness is $O(\varepsilon_\alpha)$, as will be seen from the stretched coordinate $\zeta = (\xi - \xi_0)/\varepsilon_\alpha$. We emphasise that the perturbation parameter is ε_α ; the fact that it appears as ε_α^2 in the equation does not alter the boundary layer thickness, which remains $O(\varepsilon_\alpha)$.

Remark 3.1. (1): For $\alpha = 1$, the classical nondimensional equation is recovered.

(2): For $\alpha < 1$, the effective bending and tensile terms are weighted by powers of $H^{-\alpha}$, which shows the locality of spatial effects imposed by the Khalil approach.

(3): The parameter ε_α^2 is the singular perturbation parameter that controls bending stiffness.

(4): The nondimensional parameter is

$$\varepsilon_\alpha^2 = \frac{EI}{k_0 H^{4\alpha}}.$$

For the problem to be seen as a singular perturbation, the condition

$$\varepsilon_\alpha^2 \ll 1 \quad \iff \quad EI \ll k_0 H^{4\alpha}$$

must hold.

(4i): For $\alpha = 1$, one recovers the classical condition that corresponds to a beam supported on a relatively stiff foundation.

(4ii): For $0 < \alpha < 1$, the denominator is $H^{4\alpha} < H^4$ (for $H > 1$). Hence, $EI \ll k_0 H^{4\alpha}$ is more restrictive than in the classical case.

(5): For dimensional analysis, we remark that the expression $H^{4\alpha}$ requires careful interpretation. H has units of length, and the sequential Khalil operator $D_{\bar{x}}^\alpha D_{\bar{x}}^\alpha D_{\bar{x}}^\alpha D_{\bar{x}}^\alpha$ has units of $(\text{length})^{-4\alpha}$. This ensures that each term in the nondimensional equation remains dimensionally consistent. For $4\alpha \in (3, 4]$, this corresponds to the range for fourth-order beam equations.

4. Traveling-Wave Reduction for the Problem

We introduce the wave variable:

$$\bar{f}(\bar{x}, \bar{t}) = S(\xi), \quad \xi = \frac{\bar{x}^\alpha}{\alpha} - c\bar{t}. \tag{4.1}$$

This specific form is chosen because it satisfies $D_{\bar{x}}^\alpha \xi = 1$, which eliminates the variable coefficient $\bar{x}^{1-\alpha}$ from the transformed equation. In the same sense, the particular choice $\xi = \bar{x}^\alpha/\alpha$ is crucial: it satisfies $D_{\bar{x}}^\alpha \xi = 1$, which cancels the variable coefficient $\bar{x}^{1-\alpha}$ in all applications of $D_{\bar{x}}^\alpha$. This is a unique technique of the Khalil derivative that enables a constant-coefficient ODE reduction. For a more general transformation, such a simplification would not occur, which highlights an advantage of the Khalil framework.

Then, we have

$$D_{\bar{x}}^{\alpha} \bar{f} = \bar{x}^{1-\alpha} \frac{\partial S}{\partial \bar{x}} = \bar{x}^{1-\alpha} S'(\xi) \frac{\partial \xi}{\partial \bar{x}} = S'(\xi),$$

$$D_{\bar{x}}^{2\alpha} \bar{f} = S''(\xi), \quad D_{\bar{x}}^{4\alpha} \bar{f} = S^{(4)}(\xi).$$

Here, the reduction to ordinary derivatives $S''(\xi)$ and $S^{(4)}(\xi)$ follows from the property $D_{\bar{x}}^{\alpha} \xi = 1$ applied sequentially. Note that the dependence on \bar{x} vanishes completely due to this property.

Also, we can write

$$\frac{\partial \bar{f}}{\partial t} = -cS'(\xi), \quad \frac{\partial^2 \bar{f}}{\partial t^2} = c^2 S''(\xi).$$

Consequently, we obtain the following ODE:

$$\varepsilon_{\alpha}^2 S^{(4)}(\xi) + (c^2 - \tau_{\alpha}) S''(\xi) + S(\xi) [a + bS(\xi) + S^2(\xi)] = \bar{p}. \quad (4.2)$$

Remark 4.1. The dependence on \bar{x} disappears because the traveling-wave variable is specifically chosen as

$$\xi = \frac{\bar{x}^{\alpha}}{\alpha} - c\bar{t}.$$

For the Khalil (conformable) derivative, if F is differentiable then $D_{\bar{x}}^{\alpha} F(\bar{x}) = \bar{x}^{1-\alpha} F'(\bar{x})$. Applying this to \bar{x}^{α}/α gives

$$D_{\bar{x}}^{\alpha} \left(\frac{\bar{x}^{\alpha}}{\alpha} \right) = \bar{x}^{1-\alpha} \cdot \frac{d}{d\bar{x}} \left(\frac{\bar{x}^{\alpha}}{\alpha} \right) = \bar{x}^{1-\alpha} \cdot \bar{x}^{\alpha-1} = 1.$$

Hence $D_{\bar{x}}^{\alpha} \xi = 1$ (the time part contributes zero).

Now, using the chain rule for the Khalil derivative (valid when the inner function has a conformable derivative and the outer function is differentiable),

$$D_{\bar{x}}^{\alpha} S(\xi) = S'(\xi) \cdot D_{\bar{x}}^{\alpha} \xi = S'(\xi).$$

Therefore, no factor $\bar{x}^{1-\alpha}$ appears after differentiation; the variable coefficient that arises from $D_{\bar{x}}^{\alpha}$ acting on a function of \bar{x} is canceled by the choice of ξ . Consequently, in the transformed equation, all spatial derivatives become ordinary derivatives with respect to ξ with constant coefficients, and any explicit \bar{x} in the original PDE must be re-expressed via $\bar{x} = (\alpha(\xi + c\bar{t}))^{1/\alpha}$; but such \bar{x} would reintroduce dependence unless the original equation is properly structured. In practice, the method applies to PDEs where the only spatial derivatives are $D_{\bar{x}}^{\alpha}$ acting on \bar{f} , and the transformation eliminates \bar{x} entirely, yielding a constant-coefficient ODE in ξ .

This property is unique to the Khalil derivative and does not hold for Riemann-Liouville or Caputo fractional derivatives, where a simple chain rule of this form is absent.

5. Result 1: Unperturbed Equation Analysis

We begin our first main result by studying the unperturbed problem related to the case $\varepsilon_{\alpha} = 0$. The tanh method is applied to derive traveling wave solutions for this reduced problem.

5.1. Tanh Method

For the case $\varepsilon_{\alpha} = 0$, the fourth-order derivative term disappears, giving the unperturbed reduced second-order ODE:

$$(c^2 - \tau_{\alpha}) S''(\xi) + S(\xi) [a + bS(\xi) + S^2(\xi)] = \bar{p}.$$

Using the tanh method with its balancing techniques, we propose a traveling wave solution of the form:

$$S(\xi) = A_0 + A_1 Y, \quad Y = \tanh(\mu\xi),$$

where A_0, A_1, μ are constants to be determined.

The derivatives of Y and S are computed as:

$$Y' = \frac{dY}{d\xi} = \mu(1 - Y^2), \quad Y'' = \frac{d^2Y}{d\xi^2} = -2\mu^2Y(1 - Y^2),$$

$$S' = A_1Y' = A_1\mu(1 - Y^2), \quad S'' = A_1Y'' = -2A_1\mu^2Y(1 - Y^2).$$

Expanding the derivative term in the ODE gives:

$$(c^2 - \tau_\alpha)S'' = (c^2 - \tau_\alpha)(-2A_1\mu^2Y + 2A_1\mu^2Y^3) = -2(c^2 - \tau_\alpha)A_1\mu^2Y + 2(c^2 - \tau_\alpha)A_1\mu^2Y^3.$$

The nonlinear term is expanded as:

$$S(a + bS + S^2) = (A_0 + A_1Y)[a + b(A_0 + A_1Y) + (A_0 + A_1Y)^2].$$

Hence:

$$a + b(A_0 + A_1Y) + (A_0 + A_1Y)^2 = (a + bA_0 + A_0^2) + (bA_1 + 2A_0A_1)Y + A_1^2Y^2.$$

Multiplying by $(A_0 + A_1Y)$ and collecting terms, we have:

$$\begin{aligned} S(a + bS + S^2) &= A_0(a + bA_0 + A_0^2) \quad (Y^0 \text{ term}) \\ &\quad + A_1(a + 2bA_0 + 3A_0^2)Y \quad (Y^1 \text{ term}) \\ &\quad + (b + 3A_0)A_1^2Y^2 \quad (Y^2 \text{ term}) \\ &\quad + A_1^3Y^3 \quad (Y^3 \text{ term}). \end{aligned}$$

Substituting the expansions of the derivative and nonlinear terms, we obtain:

$$\begin{aligned} A_0(a + bA_0 + A_0^2) - \bar{p} + \left[-2(c^2 - \tau_\alpha)A_1\mu^2 + A_1(a + 2bA_0 + 3A_0^2) \right] Y \\ + (b + 3A_0)A_1^2Y^2 \\ + \left[2(c^2 - \tau_\alpha)A_1\mu^2 + A_1^3 \right] Y^3 = 0. \end{aligned}$$

Coefficient Equations: Substituting into the reduced ODE yields

$$\begin{cases} Y^0 : & A_0(a + bA_0 + A_0^2) - \bar{p} = 0, \\ Y^1 : & -2(c^2 - \tau_\alpha)A_1\mu^2 + A_1(a + 2bA_0 + 3A_0^2) = 0, \\ Y^2 : & (b + 3A_0)A_1^2 = 0, \\ Y^3 : & 2(c^2 - \tau_\alpha)A_1\mu^2 + A_1^3 = 0. \end{cases}$$

Traveling Wave Solution: For a non-constant solution ($A_1 \neq 0$):

$$A_0 = -\frac{b}{3}.$$

From the Y^0 -term:

$$\bar{p} = -\frac{b}{3} \left(a - \frac{2b^2}{9} \right).$$

From the Y^1 -term:

$$\mu^2 = \frac{a + 2bA_0 + 3A_0^2}{2(c^2 - \tau_\alpha)} = \frac{a - \frac{b^2}{3}}{2(c^2 - \tau_\alpha)}.$$

From the Y^3 -term:

$$A_1^2 = -(a + 2bA_0 + 3A_0^2) = -\left(a - \frac{b^2}{3} \right).$$

Parameter constraints for physically meaningful solutions:

- $a - \frac{b^2}{3} < 0$ (ensures A_1 is real and non-zero).
- $c^2 - \tau_\alpha < 0$ (ensures $\mu > 0$).
- $A_1 \neq 0$ (non-constant solution).
- Additionally, for the beam to be physically realisable, $\varepsilon_\alpha > 0$, $EI > 0$, $k_0 > 0$, $H > 0$, and $\alpha \in (0.75, 1]$ to have $4\alpha \in (3, 4]$ (fourth-order bending behaviour).
- The pressure \bar{p} is then fixed by $\bar{p} = -\frac{b}{3} \left(a - \frac{2b^2}{9} \right)$, which must be compatible with external loading.

For a physical interpretation, the condition $a - b^2/3 < 0$ relates to the nonlinear foundation parameters. For some particular nonlinear properties, a is negative and b is positive, making this condition satisfied. The condition $c^2 - \tau_\alpha < 0$ implies that the wave speed c must satisfy $c^2 < \tau_\alpha$, which means that wave speed is bounded by the parameter τ_α and the order α . This ensures that the traveling wave solution is real and physically meaningful. For beams on elastic foundations, this condition indicates that the wave speed cannot exceed a critical value.

The final solution is given by:

$$S(\xi) = A_0 + A_1 \tanh(\mu\xi), \quad A_0 = -\frac{b}{3}, \quad A_1 = \pm \sqrt{-\left(a - \frac{b^2}{3}\right)}, \quad \mu = \sqrt{\frac{a - b^2/3}{2(c^2 - \tau_\alpha)}}.$$

5.2. Numerical Simulations

In Figure 1, we plot the profiles of the traveling wave

$$S(\xi) = A_0 + A_1 \tanh(\mu\xi),$$

with numerical values computed from each parameter set.

6. Result 2: Singular Perturbation Analysis

Having established the solution for the case $\varepsilon_\alpha = 0$, we now address the situation where bending stiffness, though small, is finite ($\varepsilon_\alpha \neq 0$). This requires singular perturbation methods to handle the multi-scale behavior.

6.1. Matched Asymptotic Expansions Approach

Since we have $\varepsilon_\alpha^2 = \frac{EI}{k_0 H^{4\alpha}}$ in the highest derivative term of our problem, we are in the presence of a singular perturbation problem. This yields a structure where the solution presents different behaviors in distinct regions: slow variation in the outer regions and rapid transitions in thin boundary layers. The method of matched asymptotic expansions [29] will allow us to study the separation and construct our solution.

We note again that the small parameter in our asymptotic analysis is ε_α , which appears squared in equation (4.2). The expansions are performed in powers of ε_α .

We employ this method in five steps, by developing separate expansions in each region and enforcing matching conditions to ensure a smooth transition between them.

Step 1: Outer Expansion Formulation: In regions away from rapid transitions, we assume:

$$S_{\text{outer}}(\xi) = S_0(\xi) + \varepsilon_\alpha S_1(\xi) + \varepsilon_\alpha^2 S_2(\xi) + O(\varepsilon_\alpha^3). \tag{6.1}$$

Substituting into (4.2), we obtain the leading order ($\mathcal{O}(1)$) terms given by:

$$(c^2 - \tau_\alpha)S_0'' + S_0(a + bS_0 + S_0^2) = \bar{p}. \tag{6.2}$$

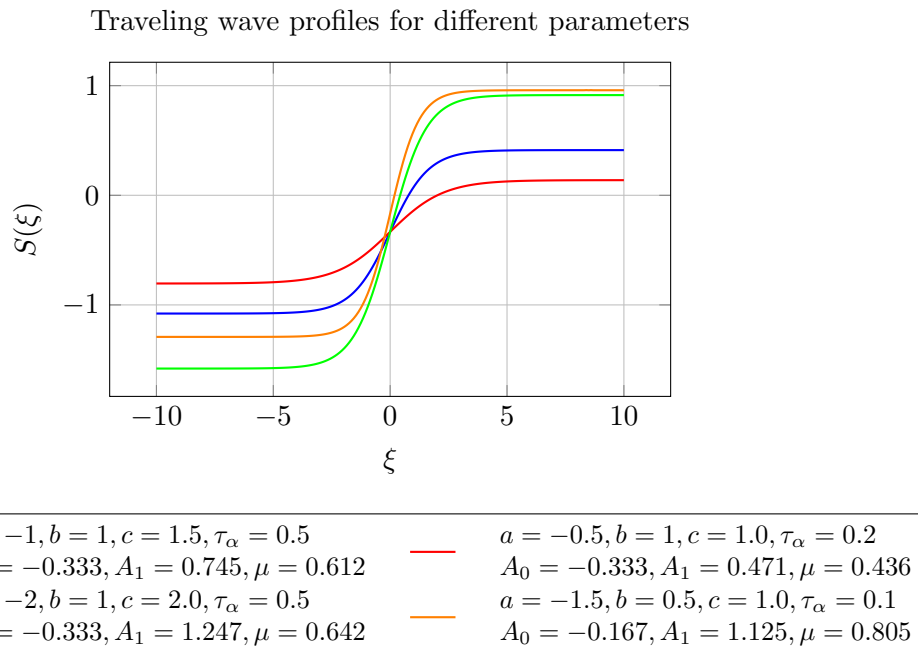


Figure 1: Traveling wave solutions $S(\xi) = A_0 + A_1 \tanh(\mu\xi)$ for the case $\varepsilon_\alpha = 0$. Each curve corresponds to a different parameter set with the values shown in the legend, including the computed A_0 , A_1 , and μ . The x-axis represents the coordinate ξ , and the y-axis shows the deflection $S(\xi)$. The profiles illustrate how varying the parameters a , b , c , and τ_α affects the amplitude (A_1) and width ($1/\mu$) of the solution.

This recovers the reduced (unperturbed) equation.

We also have the first correction ($\mathcal{O}(\varepsilon_\alpha)$) terms:

$$(c^2 - \tau_\alpha)S_1'' + (a + 2bS_0 + 3S_0^2)S_1 = 0. \tag{6.3}$$

This represents the linearization around the leading order solution.

For the second correction ($\mathcal{O}(\varepsilon_\alpha^2)$) terms, we have

$$S_0^{(4)} + (c^2 - \tau_\alpha)S_2'' + (a + 2bS_0 + 3S_0^2)S_2 + (b + 3S_0)S_1^2 = 0. \tag{6.4}$$

Step 2: Leading Order Outer Solution: From our previous analysis, the leading order outer solution is:

$$S_0(\xi) = A_0 + A_1 \tanh(\mu\xi), \tag{6.5}$$

with:

$$\begin{aligned} A_0 &= -\frac{b}{3} \\ A_1 &= \pm \sqrt{-\left(a - \frac{b^2}{3}\right)} \\ \mu &= \sqrt{\frac{a - b^2/3}{2(c^2 - \tau_\alpha)}} \\ \bar{p} &= -\frac{b}{3} \left(a - \frac{2b^2}{9}\right). \end{aligned}$$

It is to note that this outer solution is valid away from boundary layers where the second-order equation (6.2) cannot satisfy all conditions of the full fourth-order problem. The boundary layer regions are necessary to accommodate the constraints imposed by the fourth-order term.

Step 3: Boundary layer analysis: We introduce the stretched coordinate:

$$\zeta = \frac{\xi - \xi_0}{\varepsilon_\alpha}, \tag{6.6}$$

where ξ_0 is the location of the boundary layer.

Thus, the derivative transformations become:

$$\frac{d}{d\xi} = \frac{1}{\varepsilon_\alpha} \frac{d}{d\zeta} \tag{6.7}$$

$$\frac{d^2}{d\xi^2} = \frac{1}{\varepsilon_\alpha^2} \frac{d^2}{d\zeta^2} \tag{6.8}$$

$$\frac{d^4}{d\xi^4} = \frac{1}{\varepsilon_\alpha^4} \frac{d^4}{d\zeta^4}. \tag{6.9}$$

Substituting into (4.2) and multiplying by ε_α^2 , we obtain the following inner equation:

$$\frac{d^4 S}{d\zeta^4} + (c^2 - \tau_\alpha) \frac{d^2 S}{d\zeta^2} + \varepsilon_\alpha^2 (S(\zeta)[a + bS(\zeta) + S^2(\zeta)] - \bar{p}) = 0. \tag{6.10}$$

Note the scaling: the term $(c^2 - \tau_\alpha)S''$ appears without any factor of ε_α , and the nonlinear term is multiplied by ε_α^2 , which is consistent with the transformation and ensures balancing at each order.

Now, in the boundary layer, we assume:

$$S_{\text{inner}}(\zeta) = S_0^I(\zeta) + \varepsilon_\alpha S_1^I(\zeta) + \varepsilon_\alpha^2 S_2^I(\zeta) + O(\varepsilon_\alpha^3). \tag{6.11}$$

From the inner equation at $\mathcal{O}(1)$, we obtain

$$\frac{d^4 S_0^I}{d\zeta^4} + (c^2 - \tau_\alpha) \frac{d^2 S_0^I}{d\zeta^2} = 0, \tag{6.12}$$

and its general solution is given by:

$$S_0^I(\zeta) = C_0 + C_1\zeta + C_2\zeta^2 + C_3\zeta^3. \tag{6.13}$$

For the first correction ($\mathcal{O}(\varepsilon_\alpha)$), we can write

$$\frac{d^4 S_1^I}{d\zeta^4} + (c^2 - \tau_\alpha) \frac{d^2 S_1^I}{d\zeta^2} = 0. \tag{6.14}$$

For the correction ($\mathcal{O}(\varepsilon_\alpha^2)$), we have:

$$\frac{d^4 S_2^I}{d\zeta^4} + (c^2 - \tau_\alpha) \frac{d^2 S_2^I}{d\zeta^2} + S_0^I(a + bS_0^I + (S_0^I)^2) - \bar{p} = 0. \tag{6.15}$$

This provides equations for determining higher-order coefficients.

Step 4: Matching procedure: The matching principle requires:

$$\lim_{\zeta \rightarrow +\infty} S_{\text{inner}}(\zeta) = \lim_{\xi \rightarrow \xi_0^+} S_{\text{outer}}(\xi) \tag{6.16}$$

$$\lim_{\zeta \rightarrow -\infty} S_{\text{inner}}(\zeta) = \lim_{\xi \rightarrow \xi_0^-} S_{\text{outer}}(\xi). \tag{6.17}$$

For our tanh-type outer solution, boundary layer locations occur at:

$$\xi_0 = \frac{1}{\mu} \tanh^{-1} \left(\frac{S_{\text{targ}} - A_0}{A_1} \right), \tag{6.18}$$

where S_{targ} are the asymptotic states $A_0 \pm A_1$.

Matching $S_0^I(\zeta)$ to the outer solution gives:

$$\lim_{\zeta \rightarrow +\infty} S_0^I(\zeta) = A_0 + A_1 \tag{6.19}$$

$$\lim_{\zeta \rightarrow -\infty} S_0^I(\zeta) = A_0 - A_1. \tag{6.20}$$

From the general solution, these limits are finite only if the polynomial terms vanish, which requires specific conditions on the coefficients. The detailed matching analysis shows that $S_0^I(\zeta) = A_0$ at leading order.

Step 5: Composite solution: The composite solution is constructed as:

$$S_{\text{comp}}(\xi) = S_{\text{outer}}(\xi) + S_{\text{inner}}\left(\frac{\xi - \xi_0}{\varepsilon_\alpha}\right) - S_{\text{match}}, \tag{6.21}$$

where S_{match} is the common part determined by matching.

To leading order, we have:

$$S_{\text{comp}}^{(0)}(\xi) = S_0(\xi) + S_0^I\left(\frac{\xi - \xi_0}{\varepsilon_\alpha}\right) - A_0. \tag{6.22}$$

Since matching gives $S_0^I(\zeta) = A_0$, the leading order composite solution simplifies to:

$$S_{\text{comp}}^{(0)}(\xi) = S_0(\xi). \tag{6.23}$$

The boundary layer corrections appear at higher orders. The thickness of the boundary layer is $O(\varepsilon_\alpha)$, and the corrections represent localized deformations near $\xi = \xi_0$ that satisfy the fourth-order equation. These corrections manifest as steepening of the wave front and localized oscillations, as illustrated schematically in Figure 2. The analysis shows that the singular perturbation introduces a fine structure to the traveling wave, which is not captured by the reduced second-order equation.

6.2. Numerical Results Visualization

Physical Interpretation: The parameter $\varepsilon_\alpha^2 = \frac{EI}{k_0 H^{4\alpha}}$ introduces a scaling with α . For $\alpha = 1$, we obtain the standard scaling $\varepsilon^2 = \frac{EI}{k_0 H^4}$. For $\alpha \in (0.75, 1)$, we have a modified one $\varepsilon_\alpha^2 = \frac{EI}{k_0 H^{4\alpha}}$, where $4\alpha \in (3, 4)$ lies in the physical regime for fourth-order beam behavior. This means that for a fixed physical beam (EI constant) and foundation (k_0 constant), the effective parameter depends on the order α . As α decreases towards 0.75, $H^{4\alpha}$ decreases (for $H > 1$), making ε_α^2 larger and the boundary layer effects more pronounced. Figure 2 illustrates this behavior for increasing ε_α values with $\alpha = 0.8$, showing how the wave profile develops localized deformations near the transition regions. These deformations correspond to the boundary layer corrections derived in the asymptotic analysis and represent the influence of the small but finite bending stiffness on the traveling wave structure.

7. Conclusion

This work investigated the nonlinear dynamics of beams on nonlinear elastic foundations using Khalil conformable derivatives. Non-dimensionalization revealed that the perturbation parameter $\varepsilon_\alpha^2 = \frac{EI}{k_0 H^{4\alpha}}$ depends strongly on α : for $\alpha < 1$, the rigid foundation condition becomes more restrictive, which is crucial for micro- and nanostructures where H is small. The physical interpretation of α as a measure of material heterogeneity or scale-dependence was clarified. Exact traveling wave solutions for the unperturbed problem require two conditions: $a - b^2/3 < 0$ (related to foundation nonlinearity) and $c^2 - \tau_\alpha < 0$ (an upper bound on wave speed). Using matched asymptotic expansions, we showed that boundary layers of thickness $O(\varepsilon_\alpha)$ appear near $\xi = \pm \xi_0$, with higher-order corrections steepening the wave front.

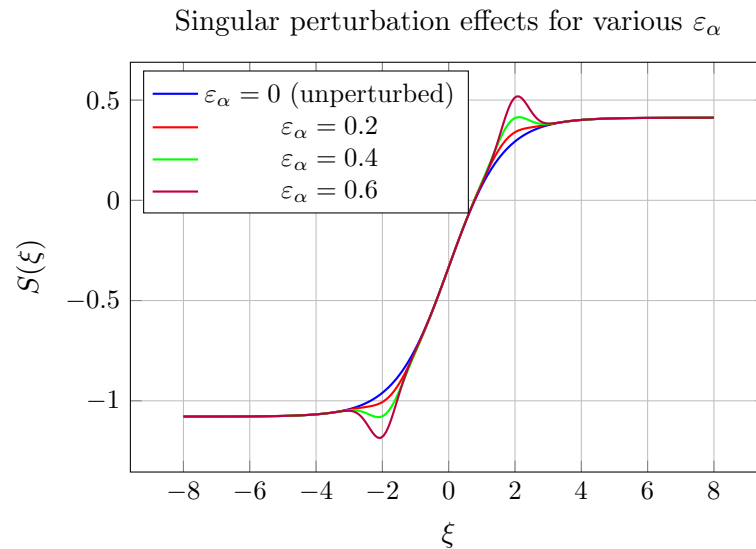


Figure 2: Traveling wave solutions for different values of ε_α , with $a = -1$, $b = 1$, $c = 1.5$, $\tau_\alpha = 0.5$, $\alpha = 0.8$, giving $A_0 = -0.3333$, $A_1 = 0.7454$, $\mu = 0.6124$. The unperturbed solution ($\varepsilon_\alpha = 0$) is the tanh profile. For $\varepsilon_\alpha > 0$, boundary layer corrections appear, creating localized deformations that increase with ε_α . The x-axis represents the coordinate ξ , and the y-axis shows the beam deflection $S(\xi)$. For $\alpha = 0.8$, we have $4\alpha = 3.2 \in (3, 4]$, satisfying the physical regime required for fourth-order beam behavior.

Existing fractional beam models using Caputo or Riemann–Liouville derivatives [18, 23, 25, 40] describe viscoelasticity or long-range interactions via integrodifferential equations requiring numerical discretization. In contrast, our Khalil-based model yields a differential equation that reduces to a constant-coefficient ODE under the traveling-wave transformation, enabling explicit solutions and rigorous matched asymptotics. However, our model does not capture memory effects; it is best suited for materials where the fractional order reflects spatial heterogeneity rather than temporal memory. For beams with significant viscoelastic damping, a Caputo-based model remains preferable. The present work thus complements the existing literature by providing a tractable framework for scale-dependent bending stiffness.

The Khalil approach offers a useful tool for studying scale-dependent effects in structural mechanics, bridging classical nonlinear beam theory with fractional calculus while preserving locality. Future work will address traveling wave stability, existence and uniqueness theory, bifurcation behavior with respect to α , perturbation analysis with multiple small parameters, numerical schemes consistent with conformable derivatives, and forced or damped dynamics.

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