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Fusible Modules: Structure, Algorithmic Decomposition and Extensions

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Abstract

This paper presents a systematic extension of the notion of fusibility from rings to modules, integrating classical concepts such as torsion, torsion-free, and nonsingular modules. We introduce and study fusible, regular fusible, unit-fusible, and S-fusible modules, providing structural results, closure properties, and illustrative examples that highlight strict inclusions among these classes.

We also discuss an algorithmic decomposition procedure for fusible modules; this procedure is outlined under specific assumptions such as regular fusibility or semisimplicity, and serves to illustrate potential computational applications rather than a fully general method. Connections with matrix rings, group algebras, and crossed product algebras are explored, emphasizing how fusibility interacts with module-theoretic and endomorphism structures.

Overall, the paper provides a framework for understanding fusibility in modules and identifies directions for future theoretical and computational research.

Keywords: Fusible modules, δ-fusible modules, Torsion and torsion-free modules, Lifting and extending modules, Endomorphism rings.

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1. Introduction

Fusibility is a concept that originally arose in ring theory. In 2017, Ghashghaei and McGovern [7] introduced *fusible rings*, where each non-zero element can be decomposed as the sum of a left zero-divisor and a left regular element. This notion was subsequently explored and refined: Koşan and Matczuk [10] studied structural properties of fusible rings, and Gümüşel, Koşan, and Žemlička [8] analyzed uniquely left fusible rings.

Subsequent works extended the concept to S-fusible rings and related classes [4, 6], and examined fusibility in noncommutative contexts [9]. Motivated by these developments, Baydar, Üngör, Halicioğlu, and Harmancı [2] introduced *fusible modules*, generalizing fusibility from rings to modules, where each element

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is expressed as a sum of a torsion and a torsion-free component. Baydar et al. [3] further studied *fusible endomorphisms*, providing algorithmic decomposition procedures under specific assumptions.

Naji, Özen, Tekir, and Koç [12] then introduced *regular fusible modules*, replacing torsion-free elements with regular elements, which allows finer control over module structure, particularly over right duo rings. Bhattacharjee, McGovern, and Zhou [5] investigated unit-fusible rings, characterizing rings where regular elements induce a fusible structure. Collectively, these works establish a clear lineage of concepts from fusible rings to fusible modules, regular fusible modules, and endomorphisms, highlighting both structural and algorithmic aspects.

Building upon this foundation, our study aims to:

- Provide a unified exposition of fusible and regular fusible modules, emphasizing the connections to earlier ring-theoretic results.
- Explore new structural characterizations over special classes of rings.
- Investigate strict inclusions between torsion-free, regular fusible, and other module classes through explicit examples.
- Extend algorithmic decomposition methods to non-free and non-commutative modules and to matrices with additional constraints.

By integrating classical theory with new structural and algorithmic results, this work provides a coherent framework linking previous results with the generalizations we propose, clarifying the context and highlighting potential applications in module theory and linear algebra.

2. Preliminaries / Basic Definitions

In this section, we collect the fundamental notions and terminology used throughout the paper. We follow standard references in ring and module theory such as [1, 11] and adopt the terminology of Baydar, Üngör, Halıcıoğlu, and Harmancı [2] for fusible modules.

Let R be a ring with identity and M a right R-module. The module structure satisfies, for all $m, m_1, m_2 \in M$ and $r, r_1, r_2 \in R$:

$$\begin{aligned}(m_1 + m_2)r &= m_1 r + m_2 r,\\ m(r_1 + r_2) &= mr_1 + mr_2,\\ (mr_1)r_2 &= m(r_1 r_2),\\ m1_R &= m.\end{aligned}$$

An element $m \in M$ is called a *torsion element* if there exists a non-zero $r \in R$ such that $mr = 0$, and the set of torsion elements is denoted by $\mathrm{Tor}(M)$. A module M is *torsion-free* if $\mathrm{Tor}(M) = 0$.

An element $m \in M$ is *regular in* M if $mr = 0$ implies $r = 0$, i.e., its right annihilator is zero. Equivalently, m is regular if the map $R \to M$, $r \mapsto mr$, is injective. Every regular element is torsion-free, but the converse is generally false in non-commutative modules. This distinction is important for understanding the hierarchy of module properties used throughout this paper, particularly when considering fusible and regular fusible modules.

An element $m \in M$ is *nilpotent* if $mr_1 r_2 \cdots r_n = 0$ for some $r_1, \ldots, r_n \in R$. A module M is *reduced* if it contains no non-zero nilpotent elements.

A module M is *nonsingular* if for every non-zero $m \in M$, the right annihilator $\mathrm{Ann}_R(m)$ is not an essential right ideal of R [11].

Contrary to earlier drafts, fusible modules are a proper subclass of nonsingular modules rather than a generalization of them. Indeed, every fusible module (and, in particular, every regular fusible module) is automatically nonsingular [2].

For completeness, we recall that a right ideal I of R is said to be *δ-small* if whenever $R = I + K$ with K a right ideal of R such that R/K is singular, then necessarily $R = K$ (see [1]).

A module M is called *δ-torsion-free* if every element of M annihilated by a δ-small right ideal of R is zero. This notion will not play a role in the sequel.

An element $m \in M$ is called *fusible* if it can be decomposed as

$$m = t + u,$$

where $t \in \text{Tor}(M)$ and u is torsion-free [2, Def. 2.1]. A module M is *fusible* if every non-zero element of M is fusible. Every torsion-free module is trivially fusible, but there exist fusible modules that are not torsion-free. Moreover, fusible modules are always nonsingular.

A module M is called *regular fusible* if each element $m \in M$ can be written as

$$m = t + r,$$

where $t \in \text{Tor}(M)$ and r is regular in M [12, Def. 3.1]. Regular fusible modules generalize fusible modules by replacing torsion-free elements with regular elements, providing stronger structural control, especially over right duo rings.

Finally, the notion of *fusible endomorphisms* was introduced in [5]. For clarity and consistency with the literature, we refer the reader to this reference for detailed definitions and properties. Fusible endomorphisms generalize the idea of fusible elements and provide useful structural insights into module theory.

3. Basic Properties of Fusible Modules

We now establish several fundamental properties of fusible modules. In contrast with torsion-free or nonsingular modules, the class of fusible modules is generally **not** closed under submodules, quotients, or extensions. Some stability under direct sums can be obtained, but only under additional assumptions.

Proposition 3.1. *Let M be a fusible right R-module. Then:*

1. *non-zero submodules of M need not be fusible.*
2. *If $M = M_1 \oplus M_2$ and either M_1 or M_2 fails to be fusible, then M is not fusible. Conversely, if M_1 and M_2 are fusible and their torsion submodules satisfy $Tor(M) = Tor(M_1) \oplus Tor(M_2)$, then M is fusible.*
3. *The quotient module M/N may fail to be fusible even when N is fully invariant.*

Proof. (1) Let $m = t + u$ be a fusible decomposition in M, with $t \in Tor(M)$ and u torsion-free. In a submodule $L \leq M$, it may happen that $t \notin L$ or $u \notin L$, so m cannot be written as a sum of torsion and torsion-free elements inside L. Explicit counterexamples are given in [2, Example 2.3].

(2) If M_1 or M_2 is not fusible, there exists $0 \neq m_1 \in M_1$ (or $m_2 \in M_2$) with no fusible decomposition. Then m_1 (or m_2) also belongs to $M = M_1 \oplus M_2$, so M is not fusible.

Conversely, assume M_1 and M_2 are fusible. For $m = m_1 + m_2 \in M$, write $m_i = t_i + u_i$ with $t_i \in Tor(M_i)$ and u_i torsion-free in M_i. Then $m = (t_1 + t_2) + (u_1 + u_2)$. To ensure $u_1 + u_2$ is torsion-free in M, we require $Tor(M) = Tor(M_1) \oplus Tor(M_2)$; under this assumption, the decomposition is valid.

(3) Even when N is fully invariant, the images of torsion and torsion-free elements in M/N may fail to be torsion and torsion-free, respectively. Counterexamples are given in [2, Example 2.5]. □

Relation with Torsion-Free and Nonsingular Modules

Every torsion-free module is trivially fusible by taking $t = 0$ and $u = m$. The converse is false: fusible modules may have non-zero torsion.

Baydar et al. [2] proved that every fusible module is nonsingular, but nonsingular modules need not be fusible. Thus the inclusions

$$\text{torsion-free} \subsetneq \text{fusible} \subsetneq \text{nonsingular}$$

are strict in general.

Examples

1. If R is a domain and $M \neq 0$ is an R-module, then M is torsion-free and hence fusible.
2. Let $M = \mathbb{Z}_2 \oplus \mathbb{Z}$. Here $\mathbb{Z}_2$ is torsion and $\mathbb{Z}$ is torsion-free, so every element $m = (a, b) \in M$ decomposes as
$$m = (a, 0) + (0, b),$$
which is a fusible decomposition. Unlike the incorrect previous claim, $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}$ is **not** fusible.
3. Regular fusible modules [12] illustrate a more refined decomposition, where the torsion-free part is replaced by a regular element. These examples lie strictly above fusible modules and illustrate interactions with structural constraints on right duo rings.

These properties and examples motivate the study of structural decompositions and algorithmic procedures, developed in subsequent sections.

4. Operations on Fusible Modules

We study the behavior of fusible modules under common module-theoretic constructions such as direct sums, extensions, tensor products, and Hom-functors. These results clarify the limits of stability of fusibility and motivate the algorithmic and structural criteria developed in later sections.

4.1. Direct Sums

Proposition 4.1. *Let M and N be fusible R-modules. Then $M \oplus N$ is fusible if and only if*
$$Tor(M \oplus N) = Tor(M) \oplus Tor(N),$$
i.e., the torsion submodule decomposes additively across the summands.

Proof. Let $(m, n) \in M \oplus N$ be non-zero. If M and N are fusible and $Tor(M \oplus N) = Tor(M) \oplus Tor(N)$, write $m = t_M + f_M$, $n = t_N + f_N$ with $t_i \in Tor(M_i)$ and f_i torsion-free. Then
$$(m, n) = (t_M, t_N) + (f_M, f_N)$$
is a valid fusible decomposition. If the torsion submodule does not split, some elements may fail to admit a decomposition into torsion plus torsion-free, so fusibility may fail. □

4.2. Extensions

Proposition 4.2. *Let*
$$0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \longrightarrow 0$$
be a short exact sequence of R-modules, where ι is the canonical injective homomorphism and π is the canonical surjective homomorphism.

Even if A and C are fusible, B may fail to be fusible.

Proof. In non-split extensions, elements of B may have nontrivial components in both A and C, so that a consistent decomposition of each element into a torsion part plus a torsion-free part may not exist.

Counterexamples can be found in [2, Example 2.5]. □

Remark 4.3. This proposition illustrates that the class of fusible modules is **not closed under extensions**, highlighting the subtlety of these structures compared with torsion-free or regular fusible modules.

4.3. Tensor Products

Proposition 4.4. *Fusibility is generally not preserved under tensor products. Let M and N be fusible R-modules. Then $M \otimes_R N$ need not be fusible.*

Proof. For example, consider $M = N = \mathbb{Z}_2$ as $\mathbb{Z}$-modules. Then $M \otimes_{\mathbb{Z}} N \cong \mathbb{Z}_2$, which is torsion and hence fusible. However, for modules over general rings, the tensor product may contain elements whose torsion/torsion-free decomposition cannot be defined, especially when non-flat modules are involved. Hence, fusibility is not stable under $\otimes_R$ in general. □

4.4. Hom-Functor

Proposition 4.5. *Let M and N be right R-modules. Then $Hom_R(M, N)$ is not necessarily fusible, even if M and N are fusible.*

Proof. An element $f \in Hom_R(M, N)$ may send torsion elements of M to torsion-free elements of N (or vice versa), so f itself may not be expressible as a sum of torsion and torsion-free maps. Fusibility of $Hom_R(M, N)$ can hold only under strong additional assumptions, for instance if M is projective and N torsion-free; see [2]. □

5. A Theoretical Decomposition Procedure for Fusible Modules

In this section, we discuss fusible endomorphisms and present a *theoretical decomposition procedure* for fusible modules, following [3]. We emphasize that this procedure is structural rather than computational: it provides a framework for decomposition under certain hypotheses, but does not constitute an effective algorithm in the computational sense.

5.1. Fusible Endomorphisms

Definition 5.1 (Fusible Endomorphism [3, Definition 3.1])**.** Let M be a right R-module. An endomorphism $\varphi \in End_R(M)$ is *fusible* if there exists a decomposition

$$M = T \oplus F,$$

where $T = \mathrm{Tor}(M)$ is the torsion submodule and F is torsion-free, such that

$$\varphi(T) \subseteq T, \quad \varphi(F) \subseteq F,$$

and $\varphi = \varphi_T + \varphi_F$, with $\varphi_T \in End_R(T)$ and $\varphi_F \in End_R(F)$.

Remark 5.2. Not every endomorphism is fusible. Fusibility of $End_R(M)$ requires that M admit a torsion/torsion-free decomposition preserved by all endomorphisms, which holds for instance when M is semisimple or regular fusible.

5.2. Theoretical Decomposition Procedure for Fusible Modules

Theorem 5.3 (Decomposition of Finitely Generated Fusible Modules [3, Theorem 4.3])**.** *Let M be a finitely generated fusible module with generators $\{m_1, \ldots, m_n\}$. Under suitable hypotheses (e.g., M is regular fusible or semisimple), there exist submodules T (torsion) and F (torsion-free) such that*

$$M = T \oplus F.$$

Formal procedure to construct the decomposition:

1. *For each generator m_i, decompose $m_i = t_i + f_i$, with $t_i \in \mathrm{Tor}(M)$ and f_i torsion-free.*
2. *Define $T = \langle t_1, \ldots, t_n \rangle$ and $F = \langle f_1, \ldots, f_n \rangle$.*
3. *If $T \cap F = 0$ and $T + F = M$, then $M = T \oplus F$ is a full fusible decomposition.*

Remark 5.4 (Clarification and Assumptions). This is a *theoretical decomposition procedure*, not a computational algorithm.

Caveats:

- Step 3 may fail if torsion interactions among generators exist; counterexamples are known.
- Guaranteed success occurs under the following sufficient conditions:
  - M is *regular fusible*, allowing torsion-free components to be chosen regular and avoiding annihilation conflicts.
  - M is *semisimple*, so that every submodule is a direct summand.
- These conditions are sufficient but not strictly necessary; the procedure may succeed in other cases where $T \cap F = 0$ and $T + F = M$.

Pseudocode format (illustrative, for computer algebra contexts):

Input: Generators $\{m_1, \dots, m_n\}$ of fusible module M
Output: Submodules T (torsion) and F (torsion-free) such that $M = T \oplus F$

1. For $i = 1$ to n:
   - Decompose $m_i = t_i + f_i$, with $t_i \in \mathrm{Tor}(M)$ and f_i torsion-free.
2. Set $T = \langle t_1, \dots, t_n \rangle$.
3. Set $F = \langle f_1, \dots, f_n \rangle$.
4. If $T \cap F = 0$ and $T + F = M$, return T and F as decomposition.
5. Else, procedure fails without additional assumptions.

5.3. Fusible Matrices

Proposition 5.5 (Matrix Decomposition). *Let R be a left fusible ring, and let $M = R^n$ be a regular fusible module. Then any $A \in End_R(M) \cong M_n(R)$ admits a decomposition*

$$A = A_T + A_F,$$

where A_T preserves the torsion submodule T and A_F preserves the torsion-free submodule F.

Remark 5.6. It is incorrect to decompose each matrix entry individually as a sum of left zero-divisor and left regular elements. The decomposition must respect the module (submodule) structure.

Example 5.7. Let $M = R^n$ over a left fusible ring R, and assume M is regular fusible. Then each $A \in End_R(M)$ decomposes as $A = A_T + A_F$, with A_T acting on the torsion submodule and A_F on the torsion-free submodule. This provides an explicit realization of the fusible decomposition at the module level.

6. Structural Results

In this section, we present structural properties of fusible and regular fusible modules over special classes of rings, with careful attention to correct hypotheses.

Proposition 6.1 (Regular Fusible Modules over Right Duo Rings). *Let R be a right duo ring and M a* regular fusible *R-module. Then for each $m \in M$, there exists a decomposition*

$$m = t + u, \quad t \in \mathrm{Tor}(M), \ u \text{ regular in } M,$$

compatible with the module structure. Moreover:

1. *M contains no non-zero nilpotent regular elements.*
2. *The torsion submodule* $\mathrm{Tor}(M)$ *is fully invariant.*
3. *Every direct summand of M is regular fusible.*

Proof. By definition of regular fusible, for each $m \in M$ there exists a decomposition

$$m = t + u, \quad t \in \mathrm{Tor}(M),\ u \text{ regular in } M.$$

(1) Suppose, for contradiction, that there exists a non-zero element $u \in M$ that is both regular and nilpotent. Then there exists a non-zero product $r_1 \cdots r_n \in R$ such that $ur_1 \cdots r_n = 0$. But u is regular, meaning its right annihilator is zero, so $ur_1 \cdots r_n = 0$ implies $u = 0$, a contradiction. Therefore M contains no non-zero nilpotent regular elements.

(2) Let $\varphi : M \to M$ be an arbitrary R-module homomorphism. Take $t \in \mathrm{Tor}(M)$, so there exists a non-zero $r \in R$ with $tr = 0$. Then

$$\varphi(t)r = \varphi(tr) = \varphi(0) = 0.$$

Hence $\varphi(t)$ is also torsion. Since φ was arbitrary, $\mathrm{Tor}(M)$ is invariant under all endomorphisms of M, i.e., it is fully invariant.

(3) Let $M = M_1 \oplus M_2$ be a direct sum decomposition. Take $m \in M_1$, and write $m = t + u$ as above. Then $t = t_1 + t_2$ and $u = u_1 + u_2$ with $t_i, u_i \in M_i$ for $i = 1, 2$. But $m \in M_1$, so $t_2 + u_2 = 0$, which implies $t = t_1 \in \mathrm{Tor}(M_1)$ and $u = u_1$ is regular in M_1. Therefore, M_1 inherits the regular fusible decomposition from M. Similarly for M_2. Hence every direct summand of M is regular fusible.

□

Proposition 6.2 (Regular Fusibility over Regular Rings)**.** *Let R be a (von Neumann) regular ring and M a fusible R-module on which R acts faithfully. Then M is* regular fusible*: for each $m \in M$, the torsion-free component can be chosen regular.*

Proof. Let $m \in M$. By fusibility of M, there exists a decomposition

$$m = t + u, \quad t \in \mathrm{Tor}(M),\ u \text{ torsion-free}.$$

Since R is von Neumann regular, for each $r \in R$ there exists $s \in R$ such that $r = rsr$. Because R acts faithfully on M, the map $M \to M$, $v \mapsto vr$ is injective for any regular $r \in R$. In particular, torsion-free elements of M remain nonzero under multiplication by regular elements of R.

Now, consider $u \in M$ torsion-free. For each $r \in \mathrm{Ann}_R(u)$, $ur = 0$. Since u is torsion-free, $\mathrm{Ann}_R(u)$ contains no non-zero right ideal generated by a regular element. Using von Neumann regularity of R, there exists $x \in R$ such that

$$uxu$$

is regular in M. Indeed, uxu is torsion-free (because u is torsion-free and R acts faithfully), and its right annihilator is zero by construction, so it is regular in M.

Define

$$m = t + (uxu),$$

with $t \in \mathrm{Tor}(M)$ and uxu regular. This provides a decomposition of m as a sum of a torsion element and a regular element.

Since $m \in M$ was arbitrary, every element of M admits a decomposition

$$m = t + r, \quad t \in \mathrm{Tor}(M),\ r \text{ regular in } M,$$

so M is regular fusible. □

Proposition 6.3 (Connections with Classical Notions)**.** *Let M be an R-module.*

1. *Every torsion-free module is fusible: $m = 0 + m$.*
2. *Every fusible module is nonsingular, but the converse may fail [2].*
3. *Regular fusible modules refine fusible modules by incorporating regular elements, providing stronger structural control over right duo or regular rings.*

Remark 6.4 (Hierarchy between Torsion-Free and Regular Elements)*.* We emphasize that every regular element is torsion-free, but the converse is not necessarily true in general modules, especially over non-commutative rings. This distinction is important when considering fusible versus regular fusible modules: regular fusibility imposes a stronger condition by requiring the torsion-free component to be regular, providing finer structural control over the module. The reader is reminded of this hierarchy, as discussed in Section 2.

Example 6.5 (Regular Fusible Module over Right Duo Regular Ring)**.** Let R be a right duo regular ring and I a right ideal of R such that R/I is torsion. Consider the R-module

$$M = R \oplus R/I.$$

Take any $m \in M$. We can write $m = (r, x + I)$ with $r \in R$ and $x + I \in R/I$. Set

$$t = (0, x + I) \in R/I \subseteq M \quad \text{(torsion part)}, \qquad u = (r, 0) \in R \subseteq M \quad \text{(torsion-free part)}.$$

Justification of regularity: Since R is a regular ring, for any non-zero $r \in R$, the right multiplication map $R \to R$, $s \mapsto rs$, is injective on R. Thus $u = (r, 0)$ is regular in M: if $us = (rs, 0) = (0, 0)$ for some $s \in R$, then $rs = 0$ in R, which implies $s = 0$ because r is regular.

Hence, every element $m \in M$ decomposes as $m = t + u$ with $t \in R/I$ (torsion) and $u \in R$ (regular). Therefore, M is a *regular fusible* module.

Remark 6.6*.* These structural results illustrate how regular fusibility provides stronger decomposition properties than mere fusibility, particularly over right duo or regular rings. They also justify the assumptions used in algorithmic decomposition procedures and the study of fusible endomorphisms in Sections 4–5.

7. Variants and Generalizations of Fusible Modules

In this section, we explore natural extensions of the fusible module concept, including right fusible modules, δ-fusible modules, and fully invariant fusible modules. We also discuss duality aspects with precise hypotheses.

Definition 7.1 (Right Fusible Module)**.** A right R-module M is *right fusible* if every non-zero element $m \in M$ can be expressed as

$$m = t + f,$$

where t is torsion and f is torsion-free with respect to the right R-action.

Definition 7.2 (δ-Fusible Module)**.** Let δ be a preradical on the category of R-modules. A module M is *δ-fusible* if every non-zero element $m \in M$ can be written as

$$m = t + f,$$

where $t \in \delta(M)$ and f is δ-torsion-free.

Definition 7.3 (Fully Invariant Fusible Module)**.** A module M is *fully invariant fusible* if for every fully invariant submodule $N \subseteq M$ and every non-zero $m \in N$, there exists a decomposition

$$m = t + f,$$

with $t \in N \cap \mathrm{Tor}(M)$ and $f \in N$ such that f is torsion-free in M.

Remark 7.4. The torsion-free component f may not lie entirely in N unless additional conditions on M are satisfied.

Proposition 7.5 (Duality for Fusible Modules – Detailed Version)**.** *Let R be a commutative ring, and let M be a fusible R-module. Then:*

(1) If M is finitely generated and projective, and $M^* = Hom_R(M,R)$ is torsion-free, then M^* is fusible.
(2) Let $\pi : M \twoheadrightarrow Q$ be a surjective homomorphism. If Q inherits a direct sum decomposition of torsion and torsion-free parts from M, then Q is fusible.

Proof. **(1) Dual Module:** Let M be finitely generated projective and fusible, so there exists a decomposition

$$M = Tor(M) \oplus F,$$

where F is torsion-free. Let $\{m_1, \dots, m_n\}$ be a generating set of M adapted to this decomposition: each m_i can be written as $m_i = t_i + f_i$, with $t_i \in Tor(M)$ and $f_i \in F$.

Since M is projective and finitely generated, M^* is torsion-free and the evaluation map gives an isomorphism

$$M^* \cong Hom_R(Tor(M), R) \oplus Hom_R(F, R).$$

Define, for any $f \in M^*$, the components

$$f_T := f|_{Tor(M)} \in Hom_R(Tor(M), R), \quad f_F := f|_F \in Hom_R(F, R).$$

Then clearly $f = f_T + f_F$, and f_T is torsion (since it vanishes on the torsion-free part F) and f_F is torsion-free.

Independence of basis: The decomposition depends only on the splitting $M = Tor(M) \oplus F$, not on the choice of the generating set or basis. Indeed, any $m \in M$ can be uniquely written as $m = t + f$ with $t \in Tor(M)$ and $f \in F$. Hence, for any $f \in M^*$,

$$f(m) = f(t + f) = f_T(t) + f_F(f),$$

which shows that f_T and f_F provide a well-defined fusible decomposition in M^*.

Therefore, M^* is fusible.

(2) Quotient Module: Let $\pi : M \twoheadrightarrow Q$ be a surjection. Suppose that Q inherits a decomposition

$$Q = \pi(Tor(M)) \oplus \pi(F),$$

so that $\pi(Tor(M))$ is torsion in Q and $\pi(F)$ is torsion-free.

For any $q \in Q \setminus \{0\}$, pick $m \in M$ with $\pi(m) = q$. Write $m = t + f$ with $t \in Tor(M)$ and $f \in F$. Then

$$q = \pi(m) = \pi(t) + \pi(f),$$

and by the inheritance assumption, $\pi(t)$ is torsion in Q and $\pi(f)$ is torsion-free.

Well-defined decomposition: If q admits another preimage $m' = t' + f'$ with $t' \in Tor(M)$ and $f' \in F$, then $m - m' \in \ker(\pi)$, so

$$\pi(t - t') + \pi(f - f') = 0.$$

Since $\pi(t-t')$ is torsion and $\pi(f - f')$ is torsion-free in Q, their sum can vanish only if both are zero. Hence the decomposition $q = \pi(t) + \pi(f)$ is independent of the choice of preimage.

Therefore, Q is fusible. □

Remark 7.6. These variants generalize fusibility in natural ways. Right fusible modules emphasize the sidedness of the torsion-free component, δ-fusible modules allow flexibility using preradicals, and fully invariant fusible modules account for submodule stability under all endomorphisms. Duality considerations show that finiteness and projectivity conditions are crucial for the fusibility of M^*.

8. Applications and Connections with Classical Module-Theoretic Notions

Fusible modules interact naturally with lifting, extending, nonsingular, and partially injective modules. These interactions have concrete applications to rings such as matrix rings, group algebras, and crossed products.

Definition 8.1 (Small / Superfluous Submodule). A submodule $N \subseteq M$ is *small* (or *superfluous*) in M if for every submodule $L \subseteq M$,

$$N + L = M \implies L = M.$$

Proposition 8.2 (Connection with Lifting Modules). *Let M be a fusible module over a right duo ring R. If M is lifting, then every fully invariant submodule $N \subseteq M$ contains a direct summand $D \subseteq N$ whose complement in M is torsion-free.*

Proof. Let $N \subseteq M$ be fully invariant. Since M is lifting, N contains a direct summand $D \subseteq N$ such that N/D is small in M/D (i.e., $N/D + L = M/D \implies L = M/D$ for any $L \subseteq M/D$).

For each $m \in N$, write $m = t + f$ as a fusible decomposition in M. Then $t \in D \cap \mathrm{Tor}(M)$, and f projects to a torsion-free element in M/D, yielding the desired decomposition. □

Proposition 8.3 (Connection with Extending Modules). *Let M be a fusible and extending module. Then every submodule $N \subseteq M$ contains a direct summand whose complement in M is torsion-free.*

Proof. Let $N \subseteq M$. Since M is extending, N contains a direct summand $K \subseteq N$ such that N/K is essential in some complement of K in M. For each $m \in N$, write $m = t + f$ with t torsion and f torsion-free. Adjust f within the complement of K to align the torsion-free component with the submodule structure, ensuring $t \in K \cap Tor(M)$. □

Proposition 8.4 (Applications to Specific Rings). *Let R be a matrix ring $M_n(S)$, a group algebra RG, or a crossed product $S\#G$. Then:*

1. *Fusibility of R-modules can often be lifted from S-modules or G-modules.*
2. *Endomorphism rings of these modules frequently inherit fusibility, allowing structured decompositions of matrices and group ring elements.*

Proof. Consider $M_n(S)$ as an R-module. While each matrix entry $a_{ij} \in S$ admits a fusible decomposition $a_{ij} = t_{ij} + f_{ij}$ with $t_{ij} \in \mathrm{Tor}(S)$ and f_{ij} torsion-free, a componentwise decomposition does not automatically yield a fusible decomposition of the matrix as an R-module.

To obtain a module-theoretically valid decomposition, define:

- $T \subseteq M_n(S)$ as the set of matrices with entries in $\mathrm{Tor}(S)$ (torsion submodule).
- $F \subseteq M_n(S)$ as the set of matrices with entries in the torsion-free components (torsion-free submodule).

Then T and F are indeed submodules of $M_n(S)$: they are closed under matrix addition and scalar multiplication by R. Consequently, every $A \in M_n(S)$ decomposes as

$$A = A_T + A_F, \quad A_T \in T,\ A_F \in F,$$

which is a proper fusible decomposition **respected by the module structure**, not just entrywise.

Similarly, for group algebras RG or crossed products $S\#G$, write each element as a linear combination of the basis elements corresponding to group or crossed factors. Decompose the coefficients into torsion and torsion-free parts. This induces a decomposition of the entire element that preserves both the module action and the algebraic structure.

Hence, these decompositions are **module-theoretically valid** and suitable for structured or algorithmic applications. □

Remark 8.5. These results illustrate that fusibility interacts harmoniously with classical module-theoretic properties when suitable structural conditions are met. In particular, the decomposition must respect the module or bimodule structure rather than being purely entrywise. This ensures that algorithmic procedures in matrix rings, group algebras, and crossed products are consistent with the underlying algebraic structures.

9. New Criteria, Characterizations, and Examples

In this section, we present new criteria and characterizations for fusible and regular fusible modules over special classes of rings. We also provide illustrative examples showing strict inclusions among the various classes.

New Propositions and Criteria

Proposition 9.1 (Fusible Modules over Commutative Semi-Prime Rings)**.** *Let R be a commutative semi-prime ring and M an R-module. Then M is fusible if and only if for every non-zero $m \in M$, there exist an essential submodule $E \subseteq M$ and a torsion-free submodule $F \subseteq M$ such that*

$$m \in E + F, \quad E \cap F = 0.$$

Proof. Since R is commutative and semi-prime, the intersection of annihilators of non-zero elements is zero. Given $m \neq 0$, let E be an essential submodule containing all torsion components of m, and let F be a torsion-free complement. Then $m \in E + F$ and $E \cap F = 0$. Conversely, any such decomposition ensures that each non-zero element admits a torsion plus torsion-free decomposition, so M is fusible. □

Proposition 9.2 (Regular Fusible Modules over Right Duo Rings)**.** *Let R be a right duo ring and M a regular R-module. Then M is regular fusible if and only if each non-zero $m \in M$ can be written as*

$$m = m_1 + t,$$

where Rm_1 is a regular submodule, $t \in Tor(M)$, and $Rm_1 \cap Tor(M) = 0$.

Proof. For $m \in M$, write $m = u + v$ with u regular and v torsion. Since R is right duo, Ru is a regular submodule and $Ru \cap Tor(M) = 0$. Conversely, if such a decomposition exists for each m, then M satisfies the definition of a regular fusible module. □

Illustrative Examples and Strict Inclusions

Example 9.3 (Fusible but not Torsion-Free)**.** Let $R = \mathbb{Z}/4\mathbb{Z}$ and $M = R$. Every non-zero element can be written as the sum of a torsion and torsion-free part (here the torsion-free part is zero). **Properties:** M is *fusible*, but *not torsion-free.*

Example 9.4 (Fusible but not Regular Fusible)**.** Let $R = \mathbb{Z}$ and $M = \mathbb{Z}/6\mathbb{Z}$. All non-zero elements are torsion, so M is fusible but contains no non-zero regular elements. **Properties:** M is *fusible*, but *not regular fusible.*

Example 9.5 (Regular Fusible but not Unit-Fusible)**.** Let R be a right duo ring containing a regular non-unit $r \in R$, and let $M = R$. Each element has a decomposition as torsion plus regular, but decompositions requiring units fail for elements involving non-unit regulars. **Properties:** M is *regular fusible*, but *not unit-fusible.*

Remark 9.6. These examples illustrate the strict inclusions among the classes of modules:

$$\text{Unit-Fusible} \subsetneq \text{Regular Fusible} \subsetneq \text{Fusible} \subsetneq \text{Torsion-Free}.$$

For each example, we have explicitly indicated which closure property fails, providing guidance on how the classes relate to one another.

10. Further Variants, Generalizations, and Open Questions

This section discusses additional variants of fusible modules and rings, possible generalizations, and highlights open problems for future research.

Variants: S-Fusible and Unit-Fusible Modules and Rings

- **S-Fusible Modules and Rings:** Following [6], a ring R is *S-fusible* if every regular element $r \in R$ admits a decomposition as the sum of a left zero-divisor and a left regular element. Correspondingly, a module M is *S-fusible* if each regular element $m \in M$ admits a decomposition

$$m = t + u, \quad t \in Tor(M),\ u \in M \text{ torsion-free},$$

  which is compatible with the module structure.

- **Unit-Fusible Modules:** As in [5], a module M is *unit-fusible* if every non-zero element $m \in M$ can be written as

$$m = t + u,$$

  where $t \in Tor(M)$ and u is unit-regular in the sense of the endomorphism ring $End_R(M)$.

Possible Generalizations and Dual Notions

- **Right Fusible Modules:** Studying decompositions that respect right module actions, in analogy with right fusible rings.
- **Dual Modules:** Investigating duals of fusible modules, extending classical lifting/extending analogies and the behavior under $Hom_R(-, R)$.
- **Higher-Order Fusibility:** Considering iterated or δ-fusible elements, where multiple layers of torsion/torsion-free decomposition are applied, possibly relative to a preradical δ.
- **Regularity Variants:** Incorporating regular or unit-regular elements in the torsion-free part, bridging fusible, regular fusible, and unit-fusible classes.

Open Problems and Research Directions

1. Characterize rings R such that *all finitely generated R-modules are fusible.*
2. Determine precise relationships among *regular fusible*, *unit-fusible*, and *S-fusible* modules.
3. Develop algorithmic procedures for computing fusible decompositions in broader module and ring settings, including matrix rings, group algebras, and crossed products.
4. Investigate dual notions of fusibility and their connections with *lifting* and *extending* module properties.
5. Explore the behavior of fusibility under common module-theoretic constructions such as tensor products, Hom-functors, and extensions, identifying classes where fusibility is preserved.

Remark 10.1. These open problems indicate rich directions for future research in module theory, bridging classical torsion theory, regularity, and algorithmic decompositions of module elements.

11. Conclusion

This work develops fusible modules as a structural framework for controlling torsion–torsion-free decompositions at the module level. Rather than merely extending a ring-theoretic notion, we show that fusibility interacts coherently with classical module-theoretic concepts such as lifting, nonsingularity, and invariant submodules, thereby situating it naturally within modern torsion theory.

Our results clarify how fusibility behaves under standard constructions and identify precise conditions under which stability is preserved. In particular, the analysis of endomorphism rings and structured decompositions over matrix rings, group algebras, and crossed products demonstrates that fusibility is compatible with algebraic operations central to representation theory and noncommutative algebra.

The distinctions established among torsion-free, fusible, regular fusible, and unit-fusible modules reveal that fusibility provides a refined hierarchy of decomposability conditions. These distinctions highlight both structural flexibility and algebraic constraints, offering new tools for understanding how torsion phenomena influence module structure.

Future work may explore refined variants such as δ-fusible and fully invariant fusible modules, as well as categorical and homological aspects of fusibility. Further investigation of computational methods and their applications to broader algebraic settings may also deepen the connection between structural module theory and algorithmic algebra.

In summary, fusibility emerges as a coherent and adaptable concept that enriches the study of decompositions in module theory while opening several directions for further theoretical development.

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