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ON RANDOM COVERING OF A CIRCLE

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ABSTRACT. Let $X_j, j = 1, 2, ..., n$ be the independent and identically distributed random vectors which take the values on the unit circumference. Let S_n be the area of the convex polygon having X_j as vertices. The paper by Nagaev and Goldfield (1989) has proved the asymptotic normality of random variableS_n. Our main aim is to show that the random variableS_n can be represented as a sum of functions of uniform spacings. This allows us to apply known results related to uniform spacings for the analysis of S_n.

1. INTRODUCTION

Consider a circle of radius r and let $X_j, j = 1, 2, ..., n$ be the random points taken from uniform distribution on it. Denote by S_n the area of the convex polygon having $X_j, j = 1, 2, ..., n$ as vertices. Obviously $S_n \to \pi r^2$,almost surely, as$n \to \infty$. Investigation of the random variable $S_n$related to the coverage by random arcs of a circle has been done by several authors, we refer to Holst (1981) and references contained therein. Conditions under which a given sequence of arcs lengths will almost surely cover the circle were studied by Dvoretzki (1956).The problem of covering the circle by random arcs has been considered also by Siegel, A. (1978, 1979). During the discussion of the covering of a circle several random variables of interest can be represented as a sum of specific functions, for example, the number of uncovered segments or gaps on the circumference, for more details see Rao, J. S. (1976) see also L.Holst and Hustler (1984).
Put $\Delta_n(r) = \pi r^2 - S_n$. Nagaev, A.V. and Goldfield, S.M. (1989) has proved that the random variable $\Delta_n(1)$ has asymptotically normal distribution with mean $4\pi^3 n^{-2}$ and variance$160\pi^6/n^5$. They represented the random variable

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S_n as a sum of sine functions and analyzed it by a method known as "ε-approximation" method which is complicated and long. In this paper we wish to show that $\Delta_n(r)$ can be represented as a sum of special function of uniform spacings. This allow us to use the well known limit theorems for the sum of functions of uniform spacings, see for example, Pyke (1965,1972), Holst and Rao (1981), Does et al (1987).

2. Preliminaries

Let $z_1, z_2, ..., z_n$ be the ordered sample from uniform$(0,1)$ distribution and $D_{i,n} = z_{i,n} - z_{i-1,n}$, $i = 1, ..., n$ be their spacings, $D = (D_{1,n}, ..., D_{n,n})$. Let $f_{m,n}(y)$, $m = 0, 1, ..., n$ be the measurable functions of non-negative arguments y. Consider the statistic

$$R_n(D) = \sum_{m=1}^{n} f_{m,n}(nD_{m,n}).$$

Let $Y_1, Y_2, ...$ be independent exponentially distributed random variables (r. v.) with parameter 1 and let $Y = (Y_1, ..., Y_n)$, $b_n = \sum_{m=1}^{n}(Y_m - 1)$, $R_n(Y) = \sum_{m=1}^{n} f_{mn}(Y_m)$. Then it is well known that

$$\mathcal{L}(R_n(D)) = \mathcal{L}(R_n(Y)/b_n = 0),$$

i.e. $R_n(D)$ has the same distribution as a sum of independent special r. v. given another sum of independent r. v. We suppose that the moments used below exist.

Let $\rho_n = corr(R_n(Y), b_n)$, $g_m(u) = f_{mn}(u) - E\, f_{m,n}(Y_m) - \rho_n\sqrt{varR_n(Y)/n}\,(u-1)$, $H_n(D) = \sum_{m=1}^{n} g_m(nD_{mn})$ and $H_n(Y) = \sum_{m=1}^{n} g_m(Y_m)$.

Note that $\sigma_n^2 \equiv varH_n(Y) = \left(1-\rho_n^2\right) varR_n(Y)$ and $E\,H_n(Y) = 0$, $cov(H_n(Y), b_n) = 0$. Obviously $H_n(D) = R_n(D) - ER_n(Y)$. Therefore, without loss of generality, we may consider the statistics $H_n(D)$ instead of $R_n(D)$. From the definition of σ_n^2, it is clear that $\sigma_n^2 = 0$ if and only if $f_{mn}(y) = C\,y + a_m$, where a_m are arbitrary constants and C does not depend on m for all m=1,…,n. We suppose that $\sigma_n^2 > 0$, for all n=1,…. Let $\bar{H}_n = H_n(Y_n)/\sigma_n$, $\bar{Y}_n = (Y_n - 1)/\sqrt{n}$, $\beta_{k,n} = \sum_{m=1}^{n} E\left|\bar{H}_n\right|^k$, $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt$ and $A_n(x) = P\{H_n(D) < x\,\sigma_n\}$. Then by the Corollary 2 of Mirakhmedov (2005) it follows that there exist a constant C such that

$$\sup_x |A_n(x) - \Phi(x)| \le C\beta_{2+\delta,n}$$

for arbitrary $\delta \in (0,1]$ and n>2.

3. MAIN RESULTS

Put $F_n(x) = P\left\{\left(2r^{-2}\Delta_n(r) - \alpha_n\right)\beta_n^{-1} < x\right\}$

Theorem 1. *There exist a constant $C > 0$ such that*

$$|F_n(x) - \Phi(x)| \le \frac{C}{\sqrt{n}}.$$

Theorem 2. *The random variable $2r^{-2}\Delta_n(r)$ has asymptotically normal distribution with mean $\alpha_n = 8\pi^3/n^2$ and variance$\beta_n^2 = 19\left(64\pi^6/n^5\right)$, as $n \to \infty$.*

We will demonstrate the proof of Theorem 1 only, Theorem 2 readily follows from Theorem 1.

Proof. We suppose that $X_1, X_{2,}..., X_n$ are arranged on the circumference in such a way that X_{i+1} comes after X_i in the counterclockwise direction and such that $X_n = 2\pi r$. Let the successive arc lengths which are the spacings between these points be denoted by $D_1, D_2, ..., D_n$ that is $D_i = X_i - X_{i-1}, \quad i = 1, 2, ..., n$ with $X_0 = 0$. We have

$$S_n = \frac{1}{2}\sum_{m=1}^{n} r^2 \sin\frac{1}{r}D_m \tag{1}$$

Note that $D_1 + D_2 + ... + D_n = 2\pi r$, and $D_1, D_2, ..., D_n$ can be considered as spacings formed from sample drawn from uniform distribution in the interval $[0, 2\pi r]$. Let $U_1, U_2, ..., U_{n-1}$ be the ordered sample from uniform $(0, 1)$ distribution and $D_i' = U_i - U_{i-1}$, $i = 1, 2, ..., n$, with $U_0 = 0$, $U_n = 1$, $i = 1, 2, ..., n$ be their spacings. From (1) we have

$$\frac{2}{r^2}\Delta_n(r) = \sum_{m=1}^{n}\left(\frac{1}{r}D_m - \sin\frac{1}{r}D_m\right) = \sum_{m=1}^{n}\left(2\pi D_m' - \sin 2\pi D_m'\right) = \sum_{m=1}^{n} f\left(nD_m'\right) \tag{2}$$

where $f(x) = \frac{2\pi}{n}x - \sin\frac{2\pi}{n}x$.

Relation (2) allows us to use known results for the sum functions of uniform spacings for the analysis of the random variable $2r^{-2}\Delta_n(r)$. Let $f(x)$ be a real-valued measurable function defined on $[0, \infty)$ and statistic R_n be defined as $R_n = \sum_{m=1}^{n} f\left(nD_m'\right)$. Denote $g(x)$ and σ^2 as under

$$g(x) = f(x) - \mu - (x-1)\rho, \sigma^2 = Var\, g(Y) = Var\, f(Y) - \rho^2$$

where $\mu = Ef(Y)$, $\rho = cov(f(Y), Y)$

with $Y$to be random variable having an exponential distribution with expectation 1. Obviously

$$R_n - n\mu = \sum_{m=1}^{n} g(nD'_m).$$

Put

$$\mathrm{P}_n(x) = \mathrm{P}\left\{R_n < x\sigma\sqrt{n} + n\mu\right\}.$$

Assertion . There exist a positive constant C such that

$$\max_{x\in\mathbb{R}} |\mathrm{P}_n(x) - \Phi(x)| \le C\frac{E\,|g(Y)|^3}{\sigma^3\sqrt{n}}.$$

Assertion follows from Corollary 2 of Mirakhmedov (2005).
By direct calculations we find

$$\mu = Ef\,(Y) = \left(\frac{2\pi}{n}\right)^3 + O\left(n^{-5}\right)$$

$$\rho = Cov\,(Y, f\,(Y)) = E\,(Y-1, f\,(Y)) = 3\,(2\pi/n)^3 + O\left(n^{-5}\right)$$

$$Var\,f(Y) = Ef^2\,(Y) - (Ef\,(Y))^2 = 19\,(2\pi/n)^6 + O\left(n^{-8}\right)$$

$$\sigma^2 = Var f\,(Y) - \rho^2 = 10\,(2\pi/n)^6 + O\left(n^{-8}\right) \tag{3}$$

$$Eg^4(Y) = E\left\{f\,(Y) - \mu - (Y-1)\,\rho\right\}^4 = 224136\,(2\pi/n)^{12} + O\left(n^{-16}\right) \tag{4}$$

Using Holder's inequality we get $\sigma^{-3}E\,|g(Y)|^3 \le \left(\sigma^{-4}Eg^4(Y)\right)^{1/2}$. Hence Theorem follows from Assertion and relations (3) and (4). Theorem 2 immediately implies Theorem1. □

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