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Existence of Multiple Solutions for Nonlocal Fourth-Order Kirchhoff-Type Elliptic Equations with Hardy Potential

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Abstract

This study explores the presence of multiple weak solutions for a class of perturbed nonlocal fourth-order Kirchhoff-type elliptic equations incorporating a Hardy potential under Navier boundary conditions. By employing variational techniques and leveraging Ricceri's three critical points theorem, we establish results on the existence of multiple weak solutions. These results are derived under specific growth conditions on the nonlinear reaction term f. We provide algebraic conditions on the nonlinear components to ensure the existence of multiple solutions and illustrate our findings with a concrete example.

Keywords: Multiple solutions; Kirchhoff-type equation; Hardy potential; Variational techniques.
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1. Introduction

This paper examines the existence of multiple weak solutions for a class of fourth-order elliptic equations of Kirchhoff type, featuring a Hardy potential and Navier boundary conditions, described as follows:

$$\begin{cases} \Delta_p^2 u - [M(\int_\Omega |\nabla u|^p \, dx)]^{p-1}\Delta_p u - \dfrac{a}{|x|^{2p}}|u|^{p-2}u = \lambda f(x,u) + \mu g(x,u), & \text{in } \Omega, \\ u = 0, \quad \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ denotes the p-biharmonic operator of fourth order, $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain containing the origin with a smooth boundary $\partial\Omega$, $1 < p < \frac{N}{2}$, λ and μ are positive parameters, $M : [0, +\infty[\to \mathbb{R}$ is a continuous function, and $f, g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

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The problem (1.1) is associated with the stationary form of the equation:

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

for $0<x<L$, $t\geq 0$, where $u(x,t)$ represents the lateral displacement at position x and time t, E is Young's modulus, ρ is the mass density, h is the cross-sectional area, L is the length, and ρ_0 is the initial axial tension. Kirchhoff [22] proposed this model, extending D'Alembert's wave equation to account for length variations in vibrating strings. The equation (1.2) can be reformulated as:

$$u_{tt}(x)-M\left(\int_\Omega|\nabla u(x)|^2\,dx\right)\Delta u(x)=f(x,u(x)),$$

with $M(s)=as+b$, where $a,b>0$. This led to extensive studies of nonlocal elliptic boundary value problems of the form:

$$\begin{cases} -M\left(\int_\Omega|\nabla u|^2\,dx\right)\Delta u(x)=f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Such problems model physical and biological systems where u represents a process dependent on its average, such as population density [1]. The earliest stationary analysis of such fourth-order nonlocal problems was conducted by Ma [25], who introduced the term "fourth-order problem of Kirchhoff type" in [26]. These problems are also linked to p-Kirchhoff problems, where the term $\left[M\left(\int_\Omega|\nabla u|^p\,dx\right)\right]^{p-1}\Delta_p u$ in (1.1) includes an exponent $p-1$.

Corrêa and Figueiredo [12] first introduced this formulation to enable variational methods. Kirchhoff-type problems have been extensively explored; see, for instance, [3, 10, 15, 16, 17, 18, 19, 24, 27, 28, 30, 33, 37, 41] and references therein.

In [19], inspired by [33], the authors used a three critical points theorem from [5] to demonstrate the existence of two parameter intervals for which a Kirchhoff-type boundary value problem admits three weak solutions with uniformly bounded norms. Similarly, Graef, Heidarkhani, and Kong [16] applied variational methods to establish the existence of at least three weak solutions for a Kirchhoff-type problem analogous to (1.3). In [9], the authors investigated a semi-positone case of a nonlocal problem, proving existence and uniqueness using Browder's theorem.

Higher-order Kirchhoff-type problems in n-dimensional domains ($n\geq 1$) have been studied using variational techniques, such as the symmetric mountain pass theorem [11] and three critical points theorems [2]. Wang and An [37] used the mountain-pass theorem to establish the existence and multiplicity of solutions for a fourth-order nonlocal elliptic problem. Additionally, in [38], the authors employed mountain-pass techniques and truncation methods to study nontrivial solutions for fourth-order Kirchhoff-type elliptic equations.

Ferrara, Khademloo, and Heidarkhani [13] utilized variational methods and critical point theory to prove the existence of multiple non-trivial and non-negative solutions for a perturbed fourth-order Kirchhoff-type problem. They combined algebraic conditions on the nonlinear term with the Ambrosetti-Rabinowitz condition to obtain two solutions and used three critical points theorems from [6, 8] to ensure at least three weak solutions when the nonlinearity includes two terms.

Singular elliptic problems, particularly those with Hardy potentials, have also been widely studied [29, 31, 32, 39, 40]. Ferrara and Molica Bisci [14] investigated an elliptic problem with a Hardy potential, addressing challenges due to the non-compact embedding caused by the Hardy term. They established the existence of at least one non-trivial weak solution under specific conditions on the nonlinearity and parameter intervals.

Huang and Liu [21] explored sign-changing solutions for p-biharmonic equations with Hardy potentials using the method of invariant sets of descending flow. In [42], variational methods were used to prove the existence of infinitely many solutions for a perturbed Kirchhoff-type problem with a Hardy potential, under appropriate conditions on the nonlinear terms.

In this work, we apply a three critical points theorem from [34] to identify a region $\Lambda \subset \mathbb{R}^2$ such that, for every $(\lambda, \mu) \in \Lambda$, the problem (1.1) has at least three distinct weak solutions in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. We impose growth conditions on the nonlinearity f and other sufficient conditions to achieve this result (see Theorem 3.1). The paper is structured as follows: Section 2 provides necessary preliminaries, Section 3 presents the main results and their proofs, and an example is included to illustrate the findings (see Example 3.3).

2. Basic Definitions and Preliminaries

We rely on a three critical points theorem due to Ricceri [34]; as our primary tool, restated here for convenience:

Theorem 2.1. *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional with a compact derivative, satisfying $\Phi(0) = \Psi(0) = 0$. Assume there exists $r > 0$ and $\overline{x} \in X$ with $r < \Phi(\overline{x})$ such that:*

(i) $\frac{\sup_{\Phi(x)\le r} \Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$,

(ii) for each $\lambda \in \Lambda_r := \left] \frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x)\le r} \Psi(x)} \right[$, *the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each compact interval $[a, b] \subseteq \Lambda_r$, there exists $\rho_0 > 0$ such that, for every $\lambda \in [a, b]$ and every C^1 functional $J : X \to \mathbb{R}$ with a compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation $\Phi'(x) - \lambda\Psi'(x) - \mu J'(x) = 0$ has at least three solutions in X with norms less than ρ_0.

In this study, X denotes the space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ equipped with the norm:

$$\|u\| = \left(\int_\Omega [|\Delta u|^p + |\nabla u(x)|^p] \, dx \right)^{\frac{1}{p}}.$$

The Rellich inequality [23] states:

$$\int_\Omega \frac{|u(x)|^p}{|x|^{2p}} \, dx \le \frac{1}{S_0} \int_\Omega [|\Delta u|^p + |\nabla u|^p] \, dx, \tag{2.1}$$

with the optimal constant:

$$S_0 = \left(\frac{(p-1)N(N-2p)}{p^2} \right)^p. \tag{2.2}$$

We assume the following condition:

(H_1) $M : [0, +\infty[\to \mathbb{R}$ is continuous, and there exist positive constants m_0, m_1 such that:

$$m_0 \le M(t) \le m_1, \quad \forall t \ge 0. \tag{2.3}$$

Let $p^* = \frac{pN}{N-p}$. The Sobolev embedding theorem ensures a positive constant c such that:

$$\|u\|_{L^{p^*}(\Omega)} \le S_{p^*} \|u\|, \quad \forall u \in X,$$

where:

$$S_{p^*} > \pi^{-\frac{1}{2}} N^{-\frac{1}{p}} \left(\frac{p-1}{N-p} \right)^{1-\frac{1}{p}} \left[\frac{\Gamma(1 + \frac{N}{2})\Gamma(N)}{\Gamma(\frac{N}{p})\Gamma(N + 1 - \frac{N}{p})} \right]^{\frac{1}{N}}, \tag{2.4}$$

see [36]. For $q \in [1, p^*)$, there exists a positive constant S_q such that:

$$\|u\|_{L^q(\Omega)} \le S_q \|u\|, \quad \forall u \in X. \tag{2.5}$$

Thus, the embedding $X \hookrightarrow L^q(\Omega)$ is compact. By (2.4) and Hölder's inequality, we have:

$$S_q \leq S_{p^*}|\Omega|^{\frac{p^*-q}{p^*q}}, \tag{2.6}$$

where $|\Omega|$ is the Lebesgue measure of Ω.

Definition 2.2. A weak solution to problem (1.1) is a function $u \in X$ satisfying:

$$\begin{aligned}\int_\Omega |\Delta u(x)|^{p-2}\Delta u(x)\Delta v(x)\,dx + \left[M\left(\int_\Omega |\nabla u(x)|^p\,dx\right)\right]^{p-1}\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\,dx \\ - a\int_\Omega \frac{|u(x)|^{p-2}}{|x|^{2p}}u(x)v(x)\,dx \\ - \lambda\int_\Omega f(x,u(x))v(x)\,dx - \mu\int_\Omega g(x,u(x))v(x)\,dx = 0\end{aligned}$$

for every $v \in X$.

The following proposition is essential for proving Theorem 3.1. Assume $0 < a < \min\{1, m_0^{p-1}\}S_0$.

Proposition 1. Let $\Upsilon : X \to X^*$ be defined by:

$$\begin{aligned}\Upsilon(u)(v) = &\int_\Omega |\Delta u(x)|^{p-2}\Delta u(x)\Delta v(x)\,dx \\ &+ \left[M\left(\int_\Omega |\nabla u(x)|^p\,dx\right)\right]^{p-1}\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\,dx \\ &- a\int_\Omega \frac{|u(x)|^{p-2}}{|x|^{2p}}u(x)v(x)\,dx\end{aligned}$$

for every $u, v \in X$. Then Υ has a continuous inverse on X^*.

Proof. Since $\Upsilon(u)u \geq \left(\min\{1, m_0^{p-1}\} - \frac{a}{S_0}\right)\|u\|^p$, the operator Υ is coercive. By [35], for $p > 1$, there exists a positive constant C_p such that:

$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle \geq \begin{cases} C_p|x-y|^p, & \text{if } p \geq 2, \\ C_p\frac{|x-y|^p}{(|x|+|y|)^{2-p}}, & \text{if } 1 < p < 2,\end{cases}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^N$. For $1 < p < 2$, we have:

$$\begin{aligned}(\Upsilon(u) - \Upsilon(v))(u-v) \geq C_pM^-\int_\Omega \Bigg[\frac{|\Delta u(x) - \Delta v(x)|^2}{(|\Delta u(x)| + |\Delta v(x)|)^{2-p}} \\ + \frac{|\nabla u(x) - \nabla v(x)|^2}{(|\nabla u(x)| + |\nabla v(x)|)^{2-p}} + \frac{|u(x) - v(x)|^2}{|x|^{2p}(|u(x)| + |v(x)|)^{2-p}}\Bigg]\,dx > 0,\end{aligned}$$

where $M^- := \min\{1, a, m_0^{p-1}\}$, implying Υ is strictly monotone. For $p \geq 2$, we obtain:

$$\begin{aligned}(\Upsilon(u) - \Upsilon(v))(u-v) \geq C_pM^-\int_\Omega \Bigg[|\Delta u(x) - \Delta v(x)|^p + |\nabla u(x) - \nabla v(x)|^p \\ + \frac{|u(x) - v(x)|^p}{|x|^{2p}}\Bigg]\,dx > 0,\end{aligned}$$

confirming strict monotonicity. As Υ is the dual mapping on $X := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ with gauge function $\phi_p(t) = t^{p-1}$, the inverse Υ^{-1} exists per [43, Theorem 26.A (d)], and its continuity follows from [20, Proposition 2.3]. $\square$

3. Main Results

For every $x \in \Omega$, define:

$$\vartheta(x) = \sup\{\vartheta > 0 : B(x, \vartheta) \subseteq \Omega\},$$

and there exists $x_0 \in \Omega$ such that $B(x_0, D) \subseteq \Omega$, where:

$$D = \sup_{x \in \Omega} \vartheta(x).$$

Theorem 3.1. *Assume* (H_1) *and* $0 < a < \min\{1, m_0^{p-1}\}S_0$ *hold. Let* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying:*

(H_2) *There exist* $c \geq 0$ *and* $\gamma \in (1, p)$ *such that* $F(x,t) \leq c(1 + |t|^{\gamma})$ *for all* $(x,t) \in \overline{\Omega} \times \mathbb{R}$.

(H_3) $F(x,\xi) \geq 0$ *for all* $(x,\xi) \in \Omega \times \mathbb{R}^+$, *where* $F(x,\xi) := \int_0^{\xi} f(x,t)\,dt$.

(H_4) *There exist* $a_1, a_2 \in [0, +\infty[$ *and* $q \in (1, p^*)$ *such that* $|f(x,t)| \leq a_1 + a_2|t|^{q-1}$ *for all* $(x,t) \in \Omega \times \mathbb{R}$, *where* $p^* := \frac{Np}{N-p}$.

(H_5) *There exist positive constants* r, d *with:*

$$r < \left(\frac{\min\{1, m_0^{p-1}\}S_0 - a}{pS_0} \right) \left[\frac{L}{D^p} + m \left(\frac{2d}{D} \right)^p \right],$$

such that:

$$\overline{\omega}_r < k_0 \inf_{x \in \Omega} F(x,d),$$

where:

$$\overline{\omega}_r := \frac{1}{r} \left\{ a_1 S_1 \left(\frac{pS_0 r}{\min\{1, m_0^{p-1}\}S_0 - a} \right)^{\frac{1}{p}} + \frac{a_2}{q} S_q^q \left(\frac{pS_0 r}{\min\{1, m_0^{p-1}\}S_0 - a} \right)^{\frac{q}{p}} \right\},$$

$$k_0 := \frac{p\pi^{\frac{N}{2}} \left(\frac{D}{2}\right)^N}{\left[\left(\frac{2}{D}\right)^p L + m \left(\frac{2d}{D}\right)^p \right] \max\{1, m_1^{p-1}\} \frac{N}{2} \Gamma\left(\frac{N}{2}\right)},$$

$$L := \frac{\pi^{\frac{N}{2}} \left[D^N - \left(\frac{D}{2}\right)^N \right]}{\frac{N}{2}\Gamma\left(\frac{N}{2}\right)} \left(\frac{2d(N-1)}{D} \right)^p,$$

$$m := \frac{\pi^{\frac{N}{2}} \left[D^N - \left(\frac{D}{2}\right)^N \right]}{\frac{N}{2}\Gamma\left(\frac{N}{2}\right)},$$

and Γ *is the Gamma function.*

Then, for every $\lambda \in \Lambda_{r,d} := \left] \frac{1}{k_0 \inf_{x\in\Omega} F(x,d)}, \frac{1}{\overline{\omega}_r} \right[$, *and for every* $g \in C(\overline{\Omega} \times \mathbb{R})$ *satisfying:*

(H_6) *For every* $\rho > 0$, $\sup_{|t| \leq \rho} |g(x,t)| \in L^1(\Omega)$,

there exist $\rho_0 > 0$ *and* $\delta > 0$ *such that, for each* $\mu \in [0, \delta]$, *the problem* (1.1) *has at least three weak solutions in* X *with norms less than* ρ_0.

Proof. We apply Theorem 2.1 to the space $X := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ with functionals $\Phi, \Psi : X \to \mathbb{R}$ defined as:

$$\Phi(u) = \frac{1}{p} \int_{\Omega} |\Delta u|^p \, dx + \frac{1}{p} \widehat{M} \left[\int_{\Omega} |\nabla u|^p \, dx \right] - \frac{a}{p} \int_{\Omega} \frac{|u(x)|^p}{|x|^{2p}} \, dx,$$

$$\Psi(u) = \int_{\Omega} F(x, u(x)) \, dx,$$

where $\widehat{M}(t) := \int_0^t [M(s)]^{p-1}\, ds$, $t \geq 0$. The functionals Φ and Ψ are continuously Gâteaux differentiable, with derivatives:

$$\Phi'(u)(v) = \int_\Omega |\Delta u(x)|^{p-2}\Delta u(x)\Delta v(x)\, dx + \left[M\left(\int_\Omega |\nabla u(x)|^p\, dx\right)\right]^{p-1}\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\, dx - a\int_\Omega \frac{|u(x)|^{p-2}}{|x|^{2p}} u($$

$$\Psi'(u)(v) = \int_\Omega f(x,u(x))v(x)\, dx,$$

for all $u, v \in X$. From (2.1), we derive:

$$\frac{\min\{1, m_0^{p-1}\}S_0 - a}{pS_0}\|u\|^p \leq \Phi(u) \leq \frac{\max\{1, m_1^{p-1}\}}{p}\|u\|^p, \quad u \in X, \tag{3.1}$$

indicating that Φ is coercive and bounded on bounded subsets of X. The weak lower semicontinuity of Φ follows from the norm's weak lower semicontinuity and the monotonicity and continuity of $\widehat{M}$. Proposition 1 ensures that $\Phi' : X \to X^*$ has a continuous inverse.

Given (H_4) and the compact embedding $X \hookrightarrow L^q(\Omega)$ for $q \in [1, p^*)$, $\Psi' : X \to X^*$ is a compact C^1 operator, and $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Define $w_d \in X$ as:

$$w_d(x) = \begin{cases} 0, & x \in \Omega\backslash B(x_0, D), \\ \frac{2d}{D}(D - |x - x_0|), & x \in B(x_0, D)\backslash B(x_0, \frac{D}{2}), \\ d, & x \in B(x_0, \frac{D}{2}), \end{cases}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^N$. The derivatives are:

$$\frac{\partial w_d(x)}{\partial x_i} = \begin{cases} 0, & x \in (\Omega\backslash B(x_0, D)) \cup B(x_0, \frac{D}{2}), \\ -\frac{2d}{D}\frac{(x_i - x_{0,i})}{|x - x_0|}, & x \in B(x_0, D)\backslash B(x_0, \frac{D}{2}), \end{cases}$$

$$\frac{\partial^2 w_d(x)}{\partial x_i^2} = \begin{cases} 0, & x \in (\Omega\backslash B(x_0, D)) \cup B(x_0, \frac{D}{2}), \\ -\frac{2d}{D}\left[\frac{|x - x_0|^2 - (x_i - x_{0,i})^2}{|x - x_0|^3}\right], & x \in B(x_0, D)\backslash B(x_0, \frac{D}{2}), \end{cases}$$

$$\sum_{i=1}^N \frac{\partial^2 w_d(x)}{\partial x_i^2} = \begin{cases} 0, & x \in (\Omega\backslash B(x_0, D)) \cup B(x_0, \frac{D}{2}), \\ -\frac{2d(N-1)}{D|x - x_0|}, & x \in B(x_0, D)\backslash B(x_0, \frac{D}{2}). \end{cases}$$

Thus:

$$\|w_d\|^p = \int_\Omega \left[|\Delta w_d(x)|^p + |\nabla w_d(x)|^p\right] dx = \int_{B(x_0,D)\backslash B(x_0,\frac{D}{2})} \frac{\left(\frac{2d}{D}\right)^p (N-1)^p}{|x - x_0|^p}\, dx + m\left(\frac{2d}{D}\right)^p,$$

where $m = \frac{\pi^{\frac{N}{2}}\left[D^N - \left(\frac{D}{2}\right)^N\right]}{\frac{N}{2}\Gamma\left(\frac{N}{2}\right)}$. This yields:

$$\frac{L}{D^p} + m\left(\frac{2d}{D}\right)^p \leq \|w_d\|^p \leq \left(\frac{2}{D}\right)^p L + m\left(\frac{2d}{D}\right)^p. \tag{3.2}$$

From (3.1) and (3.2), we obtain:

$$m_2\left(\frac{\min\{1, m_0^{p-1}\}S_0 - a}{pS_0}\right) \leq \Phi(w_d) \leq m_3\frac{\max\{1, m_1^{p-1}\}}{p}, \tag{3.3}$$

where $m_2 := \frac{L}{D^p} + m\left(\frac{2d}{D}\right)^p$ and $m_3 := \left(\frac{2}{D}\right)^p L + m\left(\frac{2d}{D}\right)^p$. Since $r < \left(\frac{\min\{1, m_0^{p-1}\}S_0 - a}{pS_0}\right)\left[\frac{L}{D^p} + m\left(\frac{2d}{D}\right)^p\right]$, we have $r < \Phi(w_d)$. Thus, Φ and Ψ satisfy the regularity conditions of Theorem 2.1. We now verify conditions (i) and (ii).

By (H_3) and since $0 \le w_d(x) \le d$ for all $x \in \Omega$, we have:

$$\begin{aligned}\Psi(w_d) = \int_\Omega F(x, w_d(x))\, dx &\ge \int_{B(x_0, \frac{D}{2})} F(x, d)\, dx \\ &\ge \frac{\pi^{\frac{N}{2}} \left(\frac{D}{2}\right)^N}{\frac{N}{2}\Gamma\left(\frac{N}{2}\right)} \inf_{x\in\Omega} F(x, d). \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we get:

$$\begin{aligned}\frac{\Psi(w_d)}{\Phi(w_d)} &\ge \frac{\frac{p\pi^{\frac{N}{2}} \left(\frac{D}{2}\right)^N}{\frac{N}{2}\Gamma\left(\frac{N}{2}\right)} \inf_{x\in\Omega} F(x, d)}{m_3 \max\{1, m_1^{p-1}\}} \\ &= k_0 \inf_{x\in\Omega} F(x, d), \end{aligned} \tag{3.5}$$

where $k_0 := \frac{p\pi^{\frac{N}{2}} \left(\frac{D}{2}\right)^N}{\left[\left(\frac{2}{D}\right)^p L + m\left(\frac{2d}{D}\right)^p\right] \max\{1, m_1^{p-1}\} \frac{N}{2}\Gamma\left(\frac{N}{2}\right)}$.

Using (H_4) and the compact embedding $X \hookrightarrow L^q(\Omega)$ for $q \in [1, p^*)$, for $u \in \Phi^{-1}(]-\infty, r])$, we have:

$$\begin{aligned}\Psi(u) = \int_\Omega F(x, u(x))\, dx &\le a_1 \int_\Omega |u(x)|\, dx + \frac{a_2}{q} \int_\Omega |u(x)|^q\, dx \\ &\le a_1 S_1 \|u\| + \frac{a_2}{q} S_q^q \|u\|^q \\ &\le a_1 S_1 \left(\frac{pS_0 r}{\min\{1, m_0^{p-1}\} S_0 - a}\right)^{\frac{1}{p}} + \frac{a_2}{q} S_q^q \left(\frac{pS_0 r}{\min\{1, m_0^{p-1}\} S_0 - a}\right)^{\frac{q}{p}}, \end{aligned}$$

yielding:

$$\frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} \le \overline{\omega}_r. \tag{3.6}$$

From (H_5), (3.5), and (3.6), we obtain:

$$\frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} < \frac{\Psi(w_d)}{\Phi(w_d)},$$

satisfying condition (i) of Theorem 2.1.

For coercivity, using (H_2) and (3.7):

$$\int_\Omega |u(x)|^\gamma\, dx \le S_\gamma^\gamma \|u\|^s, \tag{3.7}$$

for $u \in X$ with $\|u\| \ge \max\{1, \frac{1}{S_\gamma}\}$, we have:

$$\Psi(u) = \int_\Omega F(x, u(x))\, dx \le \int_\Omega c(1 + |u(x)|^\gamma)\, dx \le c\left\{\text{meas}(\Omega) + S_\gamma^\gamma \|u\|^\gamma\right\},$$

where meas(Ω) is the Lebesgue measure of Ω. Thus:

$$I_\lambda := \Phi(u) - \lambda\Psi(u) \ge \left(\frac{\min\{1, m_0^{p-1}\} S_0 - a}{pS_0}\right) \|u\|^p - \lambda c\left\{\text{meas}(\Omega) + S_\gamma^\gamma \|u\|^\gamma\right\},$$

and since $\gamma < p$, I_λ is coercive.

Since $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies (H_6), the functional $J(u) := \int_\Omega G(x, u(x))\, dx$, where $G(x, t) := \int_0^t g(x, s)\, ds$, is well-defined, continuously Gâteaux differentiable, with compact derivative:

$$J'(u)(v) = \int_\Omega g(x, u(x))v(x)\, dx,$$

for all $u, v \in X$. All hypotheses of Theorem 2.1 are satisfied, and solutions to $\Phi'(u) - \lambda\Psi'(u) - \mu J'(u) = 0$ are weak solutions to (1.1). Since $\Lambda_{r,d} \subseteq \left]\frac{\Phi(w_d)}{\Psi(w_d)}, \frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\right[$, the result follows from Theorem 2.1. □

Remark 3.2. If $f(x, 0) \neq 0$, Theorem 3.1 guarantees at least three non-zero weak solutions.

Consider the following example, where the nonlinearity $f(x, t)$ satisfies the conditions of Theorem 3.1, inspired by [7, Example 5.1]:

Example 3.3. Let $r > 1$, $q \in (1, p^*)$, $\gamma \in (1, p)$. Define $f : \Omega \times \mathbb{R} \to \mathbb{R}$ as:

$$f(x,t) = \begin{cases} 1 + |t|^{q-1}, & x \in \Omega, t \le r, \\ 1 + r^{q-\gamma}t^{\gamma-1}, & x \in \Omega, t > r. \end{cases}$$

Condition (H_4) is satisfied. The function $F(x, t) := \int_0^t f(x, s)\, ds$ is:

$$F(x,t) = \begin{cases} \le 0, & x \in \Omega, t \le 0, \\ t + \frac{t^q}{q}, & x \in \Omega, 0 < t \le r, \\ t + r^{q-\gamma}\frac{t^\gamma}{\gamma}, & x \in \Omega, t > r, \end{cases}$$

ensuring $F(x, t) \ge 0$ for $(x, t) \in \Omega \times [0, +\infty[$, satisfying (H_3). Additionally:

$$F(x,t) \le \begin{cases} 0, & x \in \Omega, t \le 0, \\ r + \frac{r^q}{\gamma}, & x \in \Omega, 0 < t \le r, \\ (r + \frac{r^q}{\gamma})t^\gamma, & x \in \Omega, t > r, \end{cases}$$

so $F(x, t) \le (r + \frac{r^q}{\gamma})(1 + |t|^\gamma)$, satisfying (H_2).

For the case where f depends only on u, we obtain a refined result. For simplicity, let $\Omega \subseteq \mathbb{R}^N$, $p = 2$, and consider:

$$\begin{cases} \Delta^2 u - M\left(\int_\Omega |\nabla u|^2\, dx\right)\Delta u - \frac{a}{|x|^4}u = \lambda f(u) + \mu g(x, u), & \text{in } \Omega, \\ u = 0, \quad \Delta u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

The following theorem is a direct consequence of Theorem 3.1 for $p = 2$, with the constant H defined as in [4].

Theorem 3.4. *Assume (H_1) and $0 < a < \min\{1, m_0\}H$ hold. Let $f : \mathbb{R} \to \mathbb{R}$ be a positive continuous function satisfying:*

(H_7) *There exist $c \ge 0$ and $\gamma \in (1, 2)$ such that $F(t) := \int_0^t f(s)\, ds \le c(1 + |t|^\gamma)$ for all $t \in \mathbb{R}$.*

(H_8) *There exist $b_1, b_2 \in [0, +\infty[$ and $q \in (1, 2^*)$ such that $|f(t)| \le b_1 + b_2|t|^{q-1}$ for all $t \in \mathbb{R}$, where $2^* := \frac{2N}{N-2}$.*

Suppose there exist positive constants r, d' such that:

$$r < \left(\frac{\min\{1, m_0\}S_0 - a}{2S_0}\right)\left[\frac{L'}{D^2} + m\left(\frac{2d'}{D}\right)^2\right],$$

and:

$$\overline{\omega'}_r := \frac{1}{r}\left\{ b_1 S_1 \sqrt{\frac{2S_0 r}{\min\{1,m_0\}S_0 - a}} + \frac{b_2}{q} S_q^q \sqrt{\left(\frac{2S_0 r}{\min\{1,m_0\}S_0 - a}\right)^q} \right\} < k_0' F(d'),$$

where:

$$k_0' := \frac{4\pi^{\frac{N}{2}}\left(\frac{D}{2}\right)^N}{N\Gamma\left(\frac{N}{2}\right)\left[\left(\frac{2}{D}\right)^2 L' + m\left(\frac{2d'}{D}\right)^2\right]\max\{1,m_1\}},$$

$$L' := \frac{8\pi^{\frac{N}{2}}\left[D^N - \left(\frac{D}{2}\right)^N\right]}{N\Gamma\left(\frac{N}{2}\right)}\left(\frac{d'(N-1)}{D}\right)^2,$$

and m is as in Theorem 3.1. Then, for every $\lambda \in \Lambda_{r,d'} := \left]\frac{1}{k_0' F(d')}, \frac{1}{\overline{\omega'}_r}\right[$, and for every $g \in C(\overline{\Omega}\times\mathbb{R})$ satisfying (H_6), there exist $\overline{\rho} > 0$ and $\overline{\delta} > 0$ such that, for each $\mu \in [0,\overline{\delta}]$, the problem (3.8) has at least three weak solutions in X with norms less than $\overline{\rho}$.

4. Conclusions

In this paper, we investigated the existence of multiple weak solutions for a class of perturbed nonlocal fourth-order Kirchhoff-type elliptic equations with Hardy potential under Navier boundary conditions. Employing variational methods and Ricceri's three critical points theorem, we established sufficient conditions under which the problem admits at least three distinct weak solutions for specific ranges of parameters λ and μ.

Our main results, detailed in Theorems 3.1 and 3.4, provide algebraic growth conditions on the nonlinear terms f and g, ensuring multiplicity in the Sobolev space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. These findings extend previous studies on Kirchhoff-type problems by incorporating Hardy potentials and perturbations, addressing challenges from non-compact embeddings. An illustrative example demonstrates the applicability of our theorems.

This work contributes to the understanding of nonlocal elliptic problems with singular potentials, which model various physical and biological phenomena. Future research may explore generalizations to variable exponents, fractional operators, or applications in real-world systems such as vibrating strings or population dynamics.

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