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Weak solution for nonlocal thermistor problem in generalized Sobolev spaces

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Abstract

We establish by using topological degree method in the framework of generalized Sobolev spaces the existence of at least weak solution for nonlocal Dirichlet thermistor problem associated to the equation

$$\frac{\partial u}{\partial t} - \operatorname{div}\big(a(x,t,u,\nabla u)\big) = \lambda \frac{f(u)}{\big(\int_\Omega f(s)\,ds\big)^2}$$

where $-\operatorname{div}\big(a(x,t,u,\nabla u)\big)$ is a divergence operator of Leray-Lions type defined from the energy space $\mathcal{H} \subset L^{p^-}(0,T,W^{1,p(\cdot)}(\Omega))$ into its dual space and $f > 0$.

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1. Introduction

A thermistor is a solid body whose electrical conductivity is strongly influenced by temperature More precisely, is an electric circuit device made of ceramic material whose electrical conductivity decreases with increasing temperature u beyond a critical temperature u^* for type PTC and increase with the temperature for type NTC. Denote by Q the domain of $\mathbb{R}^N$ $(N \geqslant 1)$ occupied by the thermistor. The following system shows how a thermistor works

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\big(a(x,t,u,\nabla u)\big) = \lambda \dfrac{f(u)}{\big(\int_\Omega f(s)\,ds\big)^2} & \text{in } Q \\ u = 0 & \text{on } \omega = \partial\Omega \times]0,T[, \\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

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where $T > 0$ and Ω is a smooth bounded domain on $\mathbb{R}^N$ with boundary $\partial\Omega$ and closure $\overline{\Omega}$. Here $-\operatorname{div}\big(a(x,t,s,\xi)\big)$ is a classical divergence operator of Leray-Lions type defined from a space $\mathcal{H}(\Omega)$ into its dual space, λ is a real parameter and $f(u) > 0$ represents the electrical resistance of the conductor. The second term $\dfrac{f(u)}{\big(\int_\Omega f(s)\,ds\big)^2}$ represents the non-local aspect of the equation.

The problem (1.1) arises by reducing the following coupled system of two equations which model the classical thermistor problem (see [8]):

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) = \sigma(u)|\nabla\varphi|^2 \\ \nabla\cdot(\sigma(u)\nabla\varphi) = 0 \end{cases} \tag{1.2}$$

where u represents the temperature generated by the electric current flowing through a conductor, φ the electric potential, $\sigma(u)$ denotes the electric conductivity. Note that the problem (1.2) was discussed in [10] in the framework of classical Sobolev spaces.

In the particular case of (1.1) where $a(x,t,u,\nabla u) = \Delta_{p(\cdot)}(u)$ the $p(\cdot)$-laplacian operator, we obtain the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = \lambda\dfrac{f(u)}{\big(\int_\Omega f(s)\,ds\big)^2} & \text{in } Q \\ u = 0 & \text{on } \omega = \partial\Omega\times]0,T[, \\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases} \tag{1.3}$$

which was discussed in [7] in the framework of Sobolev spaces with variable exponent. Notably, the authors showed the existence of a weak solution by using topological degree method.

Our goal in this paper is the study of the existence of the weak solution of (1.1). Note that the model (1.1) is more general and more realistic than (1.3). Thus, the contribution of this work lies in its generalization and refinement of the work established in the references [8, 10, 7].

The treatment of the problem (1.1) is motivated by its physical aspect according to which the model (1.1) can be derived to describe the thermo-electric flow in the conductor (see [12] and the references therein).

This article is organized as follows: In section 2, we recall some basic results on the theory of Lebesgue-Sobolev spaces with variable exponent and also on the Berkovits topological degree. In the section 3, we state and prove our main result. In the last section 4, we provide a conclusion and some perspectives.

2. Preliminaries

2.1. Generalized Sobolev spaces

In the sequel, let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$.

We recall in this section some definitions and basic properties of the variable exponent Lebesgue and Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ (see [9, 11] for more details).

Denote

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}; \inf_{x\in\overline{\Omega}} h(x) > 1\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^- := \min\{h(x),\ x \in \overline{\Omega}\},\ \ h^+ := \max\{h(x),\ x \in \overline{\Omega}\}.$$

In this paper, we suppose that $p \in C_+(\overline{\Omega})$ such that

$$1 < p^- \le p(x) \le p^+ < \infty$$

and $p(\cdot)$ satisfies the log-Hölder continuity condition, i.e. there exists $C > 0$ such that for all $x, y \in \Omega$, $x \neq y$, one has

$$|p(x) - p(y)| \log\left(e + \frac{1}{|x-y|}\right) \leq C. \tag{2.1}$$

We define the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) = \{u;\ u : \Omega \to \mathbb{R} \text{ is measurable and } \rho_{p(\cdot)}(u) < \infty\}.$$

where

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx.$$

We define a norm, the so-called *Luxemburg norm*, on this space by

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0/\rho_{p(\cdot)}(\frac{u}{\lambda}) \leqslant 1\}.$$

$(L^{p(\cdot)}, \|\cdot\|_{p(\cdot)}$ is a Banach spaces [11, Theorem 2.5], separable, reflexive [11, Corollary 2.7] and uniformly convex.
The conjugate space of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$ where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$.
For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ Hölder inequality holds [11, Theorem 2.1]:

$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \leq 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}. \tag{2.2}$$

If $p(\cdot),\ q(\cdot) \in C_+(\overline{\Omega}),\ q(\cdot) \leq p(\cdot)$ a.e. in Ω then there exists the continuous embedding $L^{p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Omega)$.

Next, we define

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega)/|\nabla u| \in L^{p(\cdot)}(\Omega)\}$$

with the norm

$$\|u\|_{W^{1,p(\cdot)}} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$$

and $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

When the exponent satisfies the log-Hölder condition (2.1), smooth functions are dense in generalized variable exponent Sobolev space. We recall the Poincaré inequality (see [9, Theorem 2.7]) : there exists a constant $C > 0$ depending only on Ω and the function $p(\cdot)$ such that

$$\|u\|_{p(\cdot)} \leqslant C\|\nabla u\|_{p(\cdot)},\ \ \forall u \in W_0^{1,p(\cdot)}(\Omega), \tag{2.3}$$

In particular, the space $W_0^{1,p(\cdot)}(\Omega)$ has a norm given by

$$\|u\|_{1,p(\cdot)} = \|\nabla u\|_{p(\cdot)},$$

which is equivalent to the norm $\|\cdot\|_{W^{1,p(\cdot)}}$.
Moreover, the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is compact (see [11]).
The space $(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)})$ is a Banach space separable and reflexive .

The dual space of $W_0^{1,p(\cdot)}(\Omega)$, denoted $W^{-1,p'(\cdot)}(\Omega)$, is equipped with the norm

$$\|v\|_{-1,p'(\cdot)} = \inf\{\|v_0\|_{p'(\cdot)} + \sum_{i=1}^N \|v_i\|_{p'(\cdot)}\},$$

where the infinimum is taken on all possible decompositions $v = v_0 - divF$ with $v_0 \in L^{p'(\cdot)}(\Omega)$ and $F = (v_1, ..., v_N) \in (L^{p'(\cdot)}(\Omega))^N$.

Proposition 2.1. *[9] Let* $(u_n) \subset L^{p(\cdot)}(\Omega)$ *and* $u \in L^{p(\cdot)}(\Omega)$ *, we have*

1. $\|u\|_{p(\cdot)} > 1 \quad \Rightarrow \quad \|u\|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^+}$,
2. $\|u\|_{p(\cdot)} < 1 \quad \Rightarrow \quad \|u\|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^-}$,
3. $\lim\limits_{n\to\infty} \|u_n - u\|_{p(\cdot)} = 0 \quad \Leftrightarrow \quad \lim\limits_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0$,
4. $\|u\|_{p(\cdot)} \leq \rho_{p(\cdot)}(u) + 1$,
5. $\rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^-} + \|u\|_{p(\cdot)}^{p^+}$.

We introduce the following energy space

$$\mathcal{H} = \{w \in L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega)) \, / |\nabla w| \in L^{p(\cdot)}(Q)\}.$$

Following [5], this space equipped with its natural graph norm or the equivalent norm

$$\|w\|_{\mathcal{H}} = \|\nabla w\|_{L^{p(\cdot)}(Q)}$$

is a separable reflexive Banach space.
The equivalence of the norm $\|\cdot\|_{\mathcal{H}}$ and the graph norm is a consequence of the continuous embedding $L^{p(\cdot)}(Q) \hookrightarrow L^{p^-}(0,T,L^{p(\cdot)}(\Omega))$ and the Poincaré inequality (2.3). Moreover we have the following continuous dense embeddings:

$$L^{p^+}(0,T; W_0^{1,p(\cdot)}(\Omega)) \overset{d}{\hookrightarrow} \mathcal{H} \overset{d}{\hookrightarrow} L^{p^-}(0,T; W_0^{1,p(\cdot)}(\Omega))$$

In particular, since $\mathcal{D}(Q)$ is dense in $L^{p^+}(0,T; W_0^{1,p(\cdot)}(\Omega))$, it is dense in $\mathcal{H}$. For the corresponding dual space denoted by $\mathcal{H}'$, we have

$$L^{(p^-)'}(0,T; W^{-1,p'(\cdot)}(\Omega)) \hookrightarrow \mathcal{H}' \hookrightarrow L^{(p^+)'}(0,T; W^{-1,p'(\cdot)}(\Omega)).$$

2.2. Berkovits topological degree

Definition 2.2. Let X and be a real Banach spaces. We recall that a mapping $F : \Omega \subset X \to X'$ is

- Demicontinuous if for any $(u_n) \subset \Omega$, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$,
- Quasimonotone if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, it follows that $\limsup \langle Fu_n, u_n - u\rangle \geq 0$,

For any operator $F : \Omega \subset X \to X$ and any bounded operator $T : \Omega_1 \subset X \to X'$ such that $\Omega \subset \Omega_1$. We say that F satisfies the condition $(S_+)_T$, if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $\limsup \langle Fu_n, y_n - y\rangle \leq 0$, we have

$$u_n \to u.$$

Let $\mathcal{O}$ be the collection of all bounded open set in X. For any $\Omega \subset X$, we consider the following classes of operators:

$$\begin{aligned}
\mathcal{F}_1(\Omega) &:= \{F : \Omega \to X' \mid F \text{ is bounded, demicontinuous and satifies condition } (S_+)\},\\
\mathcal{F}_{T,B}(\Omega) &:= \{F : \Omega \to X \mid F \text{ is bounded, demicontinuous and satifies condition } (S_+)_T\},\\
\mathcal{F}_T(\Omega) &:= \{F : \Omega \to X \mid F \text{ is demicontinuous and satifies condition } (S_+)_T\},\\
\mathcal{F}_B(X) &:= \{F \in \mathcal{F}_{T,B}(\bar{G}) \mid G \in \mathcal{O}, \mathrm{T} \in \mathcal{F}_1(\bar{\mathrm{G}})\}.
\end{aligned}$$

Here, $T \in \mathcal{F}_1(\bar{G})$ is called an *essential inner map* to F.

Lemma 2.3. *[6, Lemmas 2.2 and 2.4] Suppose that* $T \in \mathcal{F}_1(\bar{G})$ *is continuous and* $S : D_S \subset X' \to X$ *is demicontinuous such that* $T(\bar{G}) \subset D_s$*, where* G *is a bounded open set in a real reflexive Banach space* X*. Then the following statements are true:*

(i) *If S is quasimonotone, then $I + SoT \in \mathcal{F}_T(\bar{G})$, where I denotes the identity operator.*

(ii) *If S is of class (S_+), then $SoT \in \mathcal{F}_T(\bar{G})$*

Definition 2.4. Let G be a bounded open subset of a real reflexive Banach space X, $T \in \mathcal{F}_1(\bar{G})$ be continuous and let $F, S \in \mathcal{F}_T(\bar{G})$. The affine homotopy $H : [0,1] \times \bar{G} \to X$ defined by

$$H(t,u) := (1-t)Fu + tSu \text{ for } (t,u) \in [0,1] \times \bar{G}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Following [6], we can see that the above affine homotopy satisfies condition $(S_+)_T$. We introduce the topological degree for the class $\mathcal{F}_B(X)$ ([6]).

Theorem 2.5. *There exists a unique degree function*

$$d : \{(F,G,h) | G \in \mathcal{O}, T \in \mathcal{F}_1(\bar{G}), F \in \mathcal{F}_{T,B}(\bar{G}), h \notin F(\partial G)\} \to \mathbb{Z}$$

that satisfies the following properties

1. *(Existence) if $d(F,G,h) \neq 0$, then the equation $Fu = h$ has a solution in G.*
2. *(Additivity) Let $F \in \mathcal{F}_{T,B}(\bar{G})$. If G_1 and G_2 are two disjoint open subsets of G such that $h \notin F(\bar{G} \setminus (G_1 \cup G_2))$, then we have*
$$d(F,G,h) = d(F,G_1,h) + d(F,G_2,h).$$
3. *(Homotopy invariance) If $H : [0,1] \times \bar{G} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and $h : [0,1] \to X$ is a continuous path in X such that $h(t) \notin H(t, \partial G)$ for all $t \in [0,1]$,then the value of $d(H(t,.), G, h(t))$ is constant for all $t \in [0,1]$.*
4. *(Normalization) For any $h \in G$, we have $d(I,G,h) = 1$.*

3. Main result

3.1. Basic assumptions

We assume that

- H_1: $u_0 \in L^2(\Omega)$.
- H_2: $p(\cdot)$ belongs to $C_+(\overline{\Omega})$ such that
$$1 < p^- \leqslant p(x) \leqslant p^+ < \infty$$
and $p(\cdot)$ satisfies the log-Hölder continuity condition (2.1).
- H_3: f is a function verifying: there exists a positive constant C_k , $k = 1,2$ and a function $\alpha \in L^1(Q)$ such that
$$C_1 \leqslant f(u) \leqslant C_2(|u|^{q(x)-1} + \alpha(x,t)).$$
where the exponent $q(\cdot)$ is such that $1 < q^- \leqslant q(x) \leqslant q^+ < p^-$.
- H_4: The field a is a Carathéodory function which satisfies in addition

  i) $|a(x,t,s,\xi)| \leqslant \alpha\big(b(x,t)| + |s|^{p(x)-1} + |\xi|^{p(x)-1}\big), \quad \forall (s,\xi) \in \mathbb{R}^N \times \mathbb{R}$ a.e. $(x,t) \in Q$.

  ii) $\forall \xi, \xi\prime \in \mathbb{R}^N, \ \xi \neq \xi\prime$,
$$[a(x,t,s,\xi) - a(x,t,s,\xi')] \cdot [\xi - \xi'] > 0.$$

  iii) $a(x,t,s,\xi)\xi \geqslant \theta |\xi|^{p(x)}$.

  Here α and θ are strictly positive constants and b is a positive function in $L^{p'(x)}(Q)$.

3.2. Existence result

We will prove that our problem (1.1) admits a weak solution. Let us first define the weak solution of problem (1.1).

Definition 3.1. We call that $u \in \mathcal{H}(\Omega)$ is a weak solution of (1.1) if

$$-\int_Q u\frac{\partial v}{\partial t}\,dx\,dt + \int_Q a(x,t,u,\nabla u)\nabla v\;dx\,dt = \int_Q F(u)v\;\;dx\,dt \qquad \forall v \in \mathcal{H}. \tag{3.1}$$

where $F(u) = \dfrac{f(u)}{(\int_\Omega f(s)\,ds)^2}$.

Theorem 3.2. *Under the assumptions H_1, H_2, H_3 and H_4, the problem* (1.1) *admits at least weak solution.*

In order to prove this main existence theorem by using topological degree in the sense of Berkovits, we prove some technical lemmas. We will omit the dependence of a on x and t. We define two operators S and A from $\mathcal{H}$ to $\mathcal{H}'$ as follows

$$\langle S(u), v\rangle = \int_Q F(u)v\,dx\,dt$$

$$\langle A(u), v\rangle = \int_Q a(u,\nabla u)v\,dx\,dt$$

$\forall u, v \in \mathcal{H}$.

Lemma 3.3. *Under the assumption H_3, the operator S is compact.*

Proof. By the same technic as in [7] and also in [4] we prove firstly that $F(u)$ is bounded in $L^{(p^-)'}(0,T,L^{p'(x)}(\Omega))$. Indeed, by using H_3, there is a constant C such that for all $u \in \mathcal{H}$,

$$\int_0^T \|F(u)\|^{(p^-)'}_{L^{p'(x)}(\Omega)}\,dx\,dt \leqslant C\big(\|u\|_{\mathcal{H}} + \|\alpha\|_{L^1(Q)} + 1\big).$$

Recall that $I : W_0^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$ is compact, then

$$\mathcal{I} : L^{p^-}(0,T,W_0^{1,p(x)}(\Omega)) \to L^{p^-}(0,T,L^{p(x)}(\Omega))$$

is also compact. By consequent his adjoint operator

$$\mathcal{I}^* : L^{(p^-)'}(0,T,L^{p'(x)}(\Omega)) \to L^{(p^-)'}(0,T,W^{-1,p'(x)}(\Omega))$$

is compact. Note that the embedding $L^{(p^-)'}(0,T,W^{-1,p'(x)}(\Omega)) \hookrightarrow \mathcal{H}'$ is continuous, by consequent the operator S is compact. □

Lemma 3.4. *Assume that H_4 hold. Let (u_n) be a sequence in $\mathcal{H}$ such that $u_n \rightharpoonup u$ in $\mathcal{H}$, $u_n \to u$ a.e in Q and*

$$\int_Q \big[a(u_n,\nabla u_n) - a(u_n,\nabla u)\big]\cdot\nabla(u_n - u)\,dx\,dt \to 0. \tag{3.2}$$

Then $u_n \to u$ strongly in $\mathcal{H}$.

Proof. By the same way as the proof of the Lemma 3.2 in [4] taking the weight function $\rho \equiv 1$. □

Lemma 3.5. *Under the assumptions H_2 and H_4, the operator A is bounded, continuous, coercive and of class (S_+).*

Proof. Let $u, v \in \mathcal{H}$, under the growth condition of a (the condition H_4-i) we prove easily that $\|a(u,\nabla u)\|_{L^{p'(\cdot)}(Q)} \leqslant C$ for some constant C. Thanks to Hölder inequality, we obtain

$$|\langle A(u), v\rangle| \leqslant \int_Q \|a(u,\nabla u)\|_{L^{p'(\cdot)}(Q)} \|\nabla v\|_{L^{p(\cdot)}(Q)}\, dx\, dt \leqslant C\|v\|_{\mathcal{H}}.$$

Thus, A is bounded.
To show that A is continuous, let $u_n \to u$ in $\mathcal{H}$. Then $u_n \to u$ and $\nabla u_n \to \nabla u$ in $L^{p(\cdot)}(Q)$. Hence there exist a subsequence (u_k) of (u_n) and measurable functions h in $L^{p(\cdot)}(Q)$ and g in $(L^{p(\cdot)}(Q))^N$ such that

$$u_k(x,t) \to u(x,t) \text{ and } \nabla u_k(x,t) \to \nabla u(x,t),$$

$$|u_k(x,t)| \le h(x,t) \text{ and } |\nabla u_k(x,t)| \le |g(x,t)|$$

for a.e. $(x,t) \in Q$ and all $k \in \mathbb{N}$. Since a is a Carathéodory function, we obtain

$$a(u_k(x,t), \nabla u_k(x,t)) \to a(u(x,t), \nabla u(x,t)) \text{ a.e. } (x,t) \in Q.$$

It follows from H_4 that

$$|a(u_k(x,t), \nabla u_k(x,t))| \le b(x,t) + c(|h(x,t)|^{p(x)-1} + |g(x,t)|^{p(x)-1})$$

for a.e. $(x,t) \in Q$ and for all $k \in \mathbb{N}$.
Since

$$b + c(|h|^{p(x)-1} + |g|^{p(x)-1}) \in L^{p'(\cdot)}(Q),$$

and taking into account the equality

$$\rho_{p'(\cdot)}(a(u_k,\nabla u_k) - a(u,\nabla u)) = \int_\Omega |a(u_k(x),\nabla u_k(x)) - a(u(x),\nabla u(x))|^{p'(x)} dx.$$

The dominated convergence theorem and the equivalence (3) of Proposition 2.1 imply that

$$a(u,\nabla u_k) \to a(u,\nabla u) \text{ in } L^{p'(\cdot)}(Q).$$

Thus the entire sequence $a(u,\nabla u_n)$ converges to $a(u,\nabla u)$ in $L^{p'(\cdot)}(Q)$.
Then $\langle Au_n, v\rangle \to \langle Au, v\rangle$, which implies that the operator A is continuous.

Let show that A is coercive. Let $v \in \mathcal{H}$, from the assumption H_4 and the implications (1) and (2) of proposition 2.1

$$\frac{< A(v), v >}{\|v\|_{\mathcal{H}}} \geqslant \frac{c' \rho_{p(\cdot)}(\nabla v)}{\|v\|_{\mathcal{H}}} \geqslant c'\|v\|_{\mathcal{H}}^r$$

for some $r > 1$. Therefore

$$\frac{< A(v), v >}{\|v\|_{\mathcal{H}}} \longrightarrow \infty \quad \text{as } \|v\|_{\mathcal{H}} \to \infty$$

so, A is coercive.

It remains to show that the operator A is of class (S_+). Let $(u_n)_n$ be a sequence in domain of A such that

$$\begin{cases} u_n \rightharpoonup u \quad \text{in } \mathcal{H} \\ \limsup_{k\to\infty} \langle Au_n - Au, u_n - u\rangle \leqslant 0. \end{cases} \tag{3.3}$$

We will prove that

$$u_n \longrightarrow u \text{ in } \mathcal{H}.$$

Since (u_n) is a bounded sequence in $\mathcal{H}$, then by the reflexivity of $\mathcal{H}$ and the compact embedding $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \hookrightarrow L^{p^-}(0,T;L^{p(\cdot)}(\Omega))$, there is a subsequence still denoted by (u_n) such that

$$u_n \rightharpoonup u \text{ in } \mathcal{H},$$

$$u_n \to u \text{ in } L^{p^-}(0,T;L^{p(\cdot)}(\Omega)) \text{ and a.e in } Q.$$

Since a is a Carathéodory function,

$$a(u_n, \nabla u) \to a(u, \nabla u) \text{ in } (L^{p'(\cdot)}(Q))^N. \tag{3.4}$$

We have

$$\int_Q [a(u_n, \nabla u_n) - a(u_n, \nabla u)] \cdot [\nabla u_n - \nabla u]\, dxdt =$$
$$\langle Au_n - Au, u_n - u\rangle + \int_Q [a(u, \nabla u) - a(u_n, \nabla u)] \cdot [\nabla u_n - \nabla u]\, dxdt,$$

and from (3.3) and (3.4), we have

$$\limsup_{n\to\infty} \int_Q (a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)) \cdot (\nabla u_n - \nabla u)\, dx\, dt \leqslant 0. \tag{3.5}$$

Thanks to assumption H_4 we have

$$\liminf_{n\to\infty} \int_Q (a(u_n, \nabla u_n) - a(u_n, \nabla u)) \cdot (\nabla u_n - \nabla u)\, dx\, dt \geqslant 0. \tag{3.6}$$

Then, by combining (3.5) and (3.6), we get

$$\lim_{n\to\infty} \int_Q (a(u_n, \nabla u_n) - a(u_n, \nabla u)) \cdot (\nabla u_n - \nabla u)\, dx\, dt = 0.$$

This implies, by using the lemma 3.4, that

$$u_n \longrightarrow u \quad \text{ in } \quad \mathcal{H}.$$

So the operator A is of type (S_+). □

Theorem 3.6. *Under the assumptions H_1, H_2, H_3 and H_4, the problem* (1.1) *admits at least weak solution.*

Proof. $u \in \mathcal{H}$ is a weak solution of (1.1) if and only if

$$Au = -Su \tag{3.7}$$

Thanks to the assumption H_4 and the properties of operator A presented in Lemma 3.5 and in view of Minty-Browder Theorem (see [15]), the inverse operator of A denoted by $T := A^{-1} : \mathcal{H}' \to \mathcal{H}$ is also bounded, continuous and of type (S_+). The equation (3.7) is equivalent to

$$u = Tv \text{ and } v + SoTv = 0. \tag{3.8}$$

To solve equation (3.8), we will apply the Theorem 2.5. For this end, we will first show the boundedness of the following set

$$B := \{v \in \mathcal{H}' |\ v + r\, SoTv = 0 \text{ for some } r \in [0,1]\}.$$

Let $v \in B$ and set $u = Tv$, then $\|Tv\|_{\mathcal{H}} = \|\nabla u\|_{L^{p(x)}(Q)}$.

If $\|\nabla u\|_{p(\cdot)} \leqslant 1$, then $\|Tv\|_{\mathcal{H}}$ is bounded.

If $\|\nabla u\|_{p(\cdot)} \geqslant 1$, then we have by proposition 2.1

$$\|Tv\|_{\mathcal{H}}^{p^-} = \|\nabla u\|_{L^{p(x)}(Q)}^{p-} \leqslant \int_0^T \rho_{p(x)}(\nabla u)\, dt. \tag{3.9}$$

By the assumption H_4, we have

$$a(u, \nabla u) \cdot \nabla u \geqslant \theta |\nabla u|^{p(x)},$$

which implies that

$$\begin{aligned} \int_0^T \rho_{p(x)}(\nabla u)\, dt &= \int_Q |\nabla u|^{p(x)} dx\, dt \\ &\leqslant \frac{1}{c'} \int_Q a(x, u, \nabla u).\nabla u\, dx\, dt \\ &= \frac{1}{\theta} \langle Au, u\rangle \\ &= \frac{1}{\theta} \langle v, Tv\rangle \\ &= \frac{-r}{\theta} \langle SoTv, Tv\rangle. \end{aligned}$$

This implies, in particular, that

$$\rho_{p(\cdot)}(\nabla u) \leqslant \frac{r}{\theta} \int_Q \lambda F(u) u\, dx\, dt. \tag{3.10}$$

We get, by the inequalities (3.9), (3.10), the growth condition H_3, the Hölder inequality (2.2), the inequality (5) of proposition 2.1 and Young's inequality, the following estimate for some positive constant C

$$\begin{aligned} \|Tv\|_{\mathcal{H}}^{p^-} &\leqslant C\big(\int_Q |\alpha(x,t) u(x,t)|\, dx\, dt + \int_Q |\nabla u|^{q(x)-1} |u|\, dx\, dt\big) \\ &\leqslant C(\|\alpha\|_{L^{p'(\cdot)}(Q)} \|u\|_{L^{p(\cdot)}(Q)} + \|u\|_{L^{q(\cdot)}(Q)}^{q^-} + \|u\|_{L^{q(\cdot)}(Q)}^{q^+}) \\ &\leqslant C(\|u\|_{L^{p(\cdot)}(Q)} + \|u\|_{L^{q(\cdot)}(Q)}^{q^-} + \|u\|_{L^{q(\cdot)}(Q)}^{q^+}). \end{aligned}$$

From the following continuous embedding $L^{p(x)} \hookrightarrow L^{q(x)}$, we can deduce the following estimate

$$\|Tv\|_{1,p(\cdot)}^{p^-} \leqslant C(\|Tv\|_{L^{p(\cdot)}(Q)}^{q^+} + \|Tv\|_{L^{p(\cdot)}(Q)}).$$

It follows that $\{Tv | v \in B\}$ is bounded.
Since the operator S is bounded, it is obvious from (3.8) that the set B is bounded in $\mathcal{H}'$. Consequently, there exists $R > 0$ such that

$$\|v\|_{\mathcal{H}'} < R \text{ for all } v \in B.$$

This says that

$$v + rSoTv \neq 0 \quad \text{ for all } v \in \partial B_R(0) \text{ and all } r \in [0, 1].$$

From Lemma 2.3 it follows that

$$I + SoT \in \mathcal{F}_T(\overline{B_R(0)}) \text{ and } I = LoT \in \mathcal{F}_T(\overline{B_R(0)}).$$

Since the operators I, S and T are bounded, $I + SoT$ is also bounded. We conclude that

$$I + SoT \in \mathcal{F}_{T,B}(\overline{B_R(0)}) \text{ and } I \in \mathcal{F}_{T,B}(\overline{B_R(0)}).$$

Consider the homotopy $H : [0, 1] \times \overline{B_R(0)} \to \mathcal{H}'$ given by

$$H(r, v) := v + rSoTv \text{ for } (r, v) \in [0, 1] \times \overline{B_R(0)}.$$

Applying the homotopy invariance and normalization property of the degree stated in Theorem 2.5, we get

$$d(I + SoT, B_R(0), 0) = d(I, B_R(0), 0) = 1,$$

and hence there exists a point $v \in B_R(0)$ such that

$$v + SoTv = 0.$$

We conclude finally that $u = Tv$ is a weak solution of (1.1) in the sense of the definition 3.1. □

4. Conclusion and some perspectives

The assumptions $H_1 - H_4$ guarantees that the problem (1.1) admits at least solution which lies in the energy space $\mathcal{H}$. These assumptions can be improved to ensure the existence of a capacity solution [14, 10].

The uniqueness of weak solution to problem (1.1) is not discussed in this paper. it will be necessary to impose some assumptions on f.

Drawing inspiration from the work of [2, 3, 13], studies of optimal control and also fractional optimal control of problem (1.1) are planned for future work.

The following strongly nonlinear parabolic problem associated with problem (1.1)

$$\frac{\partial u}{\partial t} - \operatorname{div}\big(a(x, t, u, \nabla u)\big) + \alpha|u|^{q(x)-2}u = \lambda \frac{f(u)}{\big(\int_\Omega f(s)\, ds\big)^2}$$

can be studied by drawing inspiration from the work of [1].

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