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Oscillation of Higher-Order Dynamic Equations with Advanced Arguments in Non-Canonical Form

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Abstract

In this paper, we examine the oscillatory behavior of solutions to nth-order neutral dynamic equations with advanced arguments in the non-canonical form. Unlike previous studies, we introduce new criteria to ensure the oscillation of the considered equation. The results presented here generalize and enhance several existing findings, applicable to arbitrary time scales. To demonstrate the applicability and effectiveness of our results, An illustrative example is provided.

Keywords: Higher-order; Nonlinear dynamic equations; Oscillation; Non-Canonical ; Riccati transformation.

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1. Introduction

In this article, we introduce new oscillation criteria for the higher-order dynamic equations of the form

$$(r(\vartheta)(\varkappa^{\alpha}(\vartheta)+p(\vartheta)\varkappa(\eta(\vartheta)))^{\Delta^{n-1}})^{\Delta}+q(\vartheta)f(\varkappa(\varrho(\vartheta)))=0, \quad \vartheta\geq\vartheta_0>0, \tag{1.1}$$

where $n\geq 2$ is even , the following assumptions will be essential

(H1) $\alpha>1$ is a ratio of two odd natural numbers;

(H2) $r\in \mathrm{C}^1_{rd}([\vartheta_0,\infty)_{\mathbb{T}},(0,\infty)), r^{\Delta}(\vartheta)>0$, $p(\varsigma)\in \mathrm{C}_{rd}([\vartheta_0,\infty)_{\mathbb{T}},(0,1))$, $q(\vartheta)\in \mathrm{C}_{rd}([\vartheta_0,\infty)_{\mathbb{T}},(0,\infty))$, p,q are not identically zero for large ϑ;

(H3) $\eta,\varrho\in \mathrm{C}^1_{rd}([\vartheta_0,\infty)_{\mathbb{T}},(0,\infty)), \eta(t)\leq\vartheta, \varrho(\vartheta)\geq\vartheta, \varrho^{\Delta}(\vartheta)>0$ and $\lim_{\vartheta\to\infty}\eta(\vartheta)=\infty$;

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(H4) $f \in C(\mathbb{T}, \mathbb{T}), \varkappa f(\varkappa) > 0$ for all $\varkappa \neq 0$, and there exists a positive constant k such that $\frac{f(\varkappa)}{\varkappa^\alpha} \geq k$ for all $\varkappa \neq 0$.

Inspired by Trench [1], we define Eq. (1.1) as being in canonical form if the following condition holds:

$$\int_{\vartheta_0}^{\infty} \frac{\Delta s}{r(s)} = \infty. \tag{1.2}$$

Conversely, Eq. (1.1) is considered to be in non-canonical form if:

$$\int_{\vartheta_0}^{\infty} \frac{\Delta s}{r(s)} < \infty. \tag{1.3}$$

A solution of (1.1) is understood to be a real-valued function $\varkappa \in C\left([\vartheta_\varkappa, \infty)_\mathbb{T}, \mathbb{R}\right)$, defined for all $\vartheta_\varkappa \geq \vartheta_0$, that satisfies (1.1) on $[\vartheta_\varkappa, \infty)_\mathbb{T}$. Additionally, $\varkappa^\alpha(\vartheta) + p(\vartheta)\varkappa(\eta(\vartheta))$ and $r(\vartheta)\left(\varkappa^\alpha(\vartheta) + p(\vartheta)\varkappa(\eta(\vartheta))\right)$ must be continuously differentiable for $t \in [\vartheta_x, \infty)_\mathbb{T}$. We concentrate on solutions $\varkappa(\vartheta)$ of (1.1) that satisfy $\sup\{|\varkappa(\vartheta)| : \vartheta \geq \vartheta_a\} > 0$ for all $\vartheta_a \geq \vartheta_\varkappa$, and we assume the existence of such solutions.

A solution of (1.1) is termed oscillatory if it has an infinite number of zeros on $[\vartheta_0, \infty)_\mathbb{T}$; otherwise, it is considered nonoscillatory. Equation (1.1) is classified as oscillatory if all its solutions are oscillatory.

The initial development of dynamic equations on time scales is attributed to Hilger [2] as a unifying framework for differential and difference equations. Over the years, significant theoretical advancements have been made in this field. For foundational insights into time scales, refer to [3, 4].

A time scale, denoted by $\mathbb{T}$, is defined as a nonempty closed subset of the real numbers, inheriting the standard topology of $\mathbb{R}$. Intervals on $\mathbb{T}$ are represented as the intersection of conventional intervals with $\mathbb{T}$. Key operators include the forward jump operator $\sigma(\vartheta) := \inf(\vartheta, \infty)_\mathbb{T}$ and the backward jump operator $\rho(\vartheta) := \sup(-\infty, \vartheta)_\mathbb{T}$, which describe the evolution of points on $\mathbb{T}$. The graininess function $\mu(\vartheta) := \sigma(\vartheta) - \vartheta$ quantifies the step size between consecutive points. Points on $\mathbb{T}$ are classified as right-dense if $\sigma(\vartheta) = \vartheta$ (or $\mu(\vartheta) = 0$) and right-scattered otherwise, with analogous definitions for left-dense and left-scattered points using $\rho(\vartheta)$. $C_{rd}(\mathbb{T}, \mathbb{R})$ represents the set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}.C^1_{rd}(\mathbb{T}, \mathbb{R})$ denotes the subset of $C_{rd}(\mathbb{T}, \mathbb{R})$ containing functions with rd-continuous derivatives.

Taylor monomials $h_n : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $n \in \mathbb{N}_0$ (see [3], Sec. 1.6), are recursively defined as:

$$h_0(\vartheta, s) = 1, \quad h_1(\vartheta, s) = \vartheta - s, \quad \text{and} \quad h_{n+1}(\vartheta, s) = \int_s^\vartheta h_n(\delta, s)\Delta\delta,$$

for $\vartheta, s \in \mathbb{T}$ and $n \geq 1$. Computing $h_n(\vartheta, s)$ for $n \geq 2$ is generally nontrivial, but for specific time scales like $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, these functions can be easily determined:

$$h_n(\vartheta, s) = \frac{(\vartheta - s)^n}{n!} \quad \text{for all} \quad s, \vartheta \in \mathbb{R},$$
$$h_n(\vartheta, s) = \frac{(\vartheta - s + n - 1)^{(n)}}{n!} \quad \text{for all} \quad s, \vartheta \in \mathbb{Z}.$$

The following results are essential for subsequent analysis:

Theorem 1.1. *[3] Let $\Omega : \mathbb{T} \to \mathbb{R}$ be strictly increasing, and let $\tilde{\mathbb{T}} := \Omega(\mathbb{T})$ be a time scale. If $\Psi : \tilde{\mathbb{T}} \to \mathbb{R}$ satisfies $\Psi^{\tilde{\Delta}}[\Omega(\vartheta)]$ and $\Omega^\Delta(\vartheta)$ exist for $\vartheta \in \mathbb{T}_k$, then:*

$$(\Psi[\Omega(\vartheta)])^\Delta = \Psi^{\tilde{\Delta}}[\Omega(\vartheta)]\Omega^\Delta(\vartheta).$$

Lemma 1.2. *[5] Let $n \in \mathbb{N}$, $\Lambda \in C^n_{rd}(\mathbb{T}, \mathbb{R})$, and $\sup \mathbb{T} = \infty$. Suppose Λ is either positive or negative, Λ^{Δ^n} is not identically zero, and is either nonnegative or nonpositive on $[\vartheta_0, \infty)_\mathbb{T}$ for some $\vartheta_0 \in \mathbb{T}$. Then, there exist $\vartheta_1 \in [\vartheta_0, \infty)_\mathbb{T}$ and $m \in [0, n)_\mathbb{Z}$ such that:*

- $\Lambda(\vartheta)\Lambda^{\Delta^{\jmath}}(\vartheta) > 0$ *for all* $\vartheta \in [\vartheta_0, \infty)_{\mathbb{T}}$ *and* $\jmath \in [0, m)_{\mathbb{Z}}$,
- $(-1)^{m+\jmath}\Lambda(\vartheta)\Lambda^{\Delta^{\jmath}}(\vartheta) \geq 0$ *for all* $\vartheta \in [\vartheta_0, \infty)_{\mathbb{T}}$ *and* $\jmath \in [m, n)_{\mathbb{Z}}$.

Lemma 1.3. *[6] Let* $\sup \mathbb{T} = \infty$, $n \in \mathbb{N}$, *and* $\Lambda \in C^n_{rd}([\vartheta_0, \infty), \mathbb{R}^+_0)$ *with* $\Lambda^{\Delta^n} \leq 0$ *on* $[\vartheta_0, \infty)_{\mathbb{T}}$. *If Lemma 1.2 holds with* $m \in [0, n)_{\mathbb{Z}}$ *and* $s \in [\vartheta_0, \infty)_{\mathbb{T}}$, *then:*

$$\Lambda(\vartheta) \geq h_m(\vartheta, s)\Lambda^{\Delta^m}(\vartheta) \quad \text{for all} \quad \vartheta \in [s, \infty)_{\mathbb{T}}. \tag{1.4}$$

Lemma 1.4. *[7] Let* $\sup \mathbb{T} = \infty$ *and* $\Lambda \in C^n_{rd}(\mathbb{T}, \mathbb{R}^+)$, $(n \geq 2)$. *Suppose Kneser's theorem holds with* $m \in [1, n)_{\mathbb{N}}$ *and* $\Lambda^{\Delta^n}(\vartheta) \leq 0$ *on* $\mathbb{T}$. *Then, there exists a sufficiently large* $\vartheta_1 \in \mathbb{T}$ *such that:*

$$\Lambda^{\Delta}(\vartheta) \geq h_{m-1}(\vartheta, \vartheta_1)\Lambda^{\Delta^m}(\vartheta) \quad \text{for all} \quad \vartheta \in [\vartheta_1, \infty)_{\mathbb{T}}.$$

Recent years have seen significant progress in the study of oscillatory behavior in delay differential equations. Works such as [8, 9, 10, 11, 12] have established criteria for the oscillation of neutral second-order differential equations. Similarly, [13, 14, 15, 16, 17] have contributed novel results on third-order equations. Higher-order equations have also been explored in [28, 18, 19, 20, 21, 22], with improved techniques for analyzing their oscillatory behavior. Extending these findings to dynamic equations remains an active area of research.

Much of the literature on the oscillation of dynamic equations has relied on comparison, integral averaging, and Riccati transformation techniques [7, 28, 23, 24, 25, 26, 27, 29]. However, there is a noticeable gap in the study of higher-order dynamic equations.

For Example, Grace [30] introduced new criteria for the oscillation of even-order dynamic equations in canonical form:

$$\left(r(\vartheta)\left(\varkappa^{\Delta^{n-1}}(\vartheta)\right)^{\alpha}\right)^{\Delta} + q(\vartheta)(\varkappa(\vartheta))^{\alpha} = 0.$$

Recently, El-sheikh et al. [31] provided sufficient conditions for the oscillation of solutions to nonlinear neutral delay dynamic equations:

$$\left(r(\vartheta)\left((\varkappa(\vartheta) + p(\vartheta)\varkappa(\tau(\vartheta)))^{\Delta^{n-1}}(\vartheta)\right)^{\alpha}\right)^{\Delta} + q(\vartheta)f(\varkappa(\delta(\vartheta))) = 0, \quad \vartheta \in [\vartheta_0, \infty)_{\mathbb{T}},$$

where $\alpha \geq 1$ is a quotient of odd positive integers, and $0 \leq p(\vartheta) \leq p_0 < \infty$, in both canonical and noncanonical cases.

Grace et al. [32] recently presented oscillation criteria for noncanonical higher-order nonlinear neutral dynamic equations:

$$\left(a(\vartheta)\left((\varkappa(\vartheta) + p(\vartheta)\varkappa^{\gamma}(\delta(\vartheta)))^{\Delta^{n-1}}(\vartheta)\right)^{\alpha}\right)^{\Delta} + q(\vartheta)\varkappa^{\beta}(\tau(\vartheta)) = 0,$$

covering both sublinear and superlinear neutral terms.

In [33], Mert established oscillation conditions for n^{th}-order neutral dynamic equations with distributed deviating arguments:

$$[\varkappa^{\alpha}(\vartheta) + p(\vartheta)\varkappa(\tau(\vartheta))]^{\Delta^n} + \int_c^d \sum_{i=1}^{2} \lambda_i q_i(\vartheta, \xi) f_i\left(\varkappa\left(\eta_i(\vartheta, \xi)\right)\right) \Delta\xi = g(\vartheta), \quad \vartheta \in [\vartheta_0, \infty),$$

where $\alpha \geq 1$ is a quotient of odd positive integers, $\lambda_1, \lambda_2 \in \{-1, 0, 1\}$, and $c, d, \vartheta_0 \in \mathbb{T}$.

Recent work by Shi et al. [34] focused on second-order half-linear neutral differential equations with advanced arguments:

$$\left(r(\vartheta)\left((\varkappa(\vartheta) + p(\vartheta)\varkappa(\tau(\vartheta)))'\right)^{\alpha}\right)' + q(\vartheta)\varkappa^{\alpha}(\sigma(\vartheta)) = 0, \quad \vartheta \geq \vartheta_0,$$

under the conditions $\pi(\vartheta) = \int_{\vartheta}^{\infty} r^{-\frac{1}{\alpha}}(s)\mathrm{d}s < \infty$, $\sigma(\vartheta) \geq \vartheta$, and $\inf_{\vartheta \geq \vartheta_0}\left(1 - p(\vartheta)\frac{\pi(\tau(\vartheta))}{\pi(\vartheta)}\right) > 0$.

Muhib [35] extended the techniques in [34] to derive oscillation criteria for second-order differential equations:

$$\left(r(\vartheta)\left(\varkappa^{\alpha}(\vartheta)+q(\vartheta)\varkappa(\lambda(\vartheta))\right)'\right)'+\sum_{i=1}^{m}h_i(\vartheta)g\left(\varkappa\left(\sigma_i(\vartheta)\right)\right)=0,$$

where $\alpha>1$ is a ratio of odd natural numbers, under the conditions $\int_{\vartheta_0}^{\infty}\frac{1}{r(s)}\mathrm{d}s<\infty$, $0<1-q(\vartheta)\frac{\pi(\lambda(\vartheta))}{\pi(\vartheta)}<1$, and $\inf_{\theta\geq\vartheta_1}\left(1-q(\vartheta)\frac{\pi(\lambda(\vartheta))}{\pi(\vartheta)}\right)>0$.

Motivated by these advancements, we establish new sufficient conditions for the oscillation of all solutions of (1.1), where $\inf_{\vartheta\geq\vartheta_0}\frac{p(\vartheta)}{\varphi^{1-\frac{1}{\alpha}}(\eta(\vartheta))}\neq 0$ and $\varphi(\vartheta)$ is a positive decreasing function converges to zero as $\vartheta\to\infty$. These results generalize and improve upon previous findings, even for special cases such as $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$.

2. Our Main Result

To simplify our analysis of nonoscillatory solutions of equation (1.1), we consider only those that are eventually positive, without loss of generality. The following notation is defined:

$$\begin{aligned}
&F(\vartheta)=\varkappa^{\alpha}(\vartheta)+p(\vartheta)\varkappa(\eta(\vartheta)),\\
&\xi(\vartheta)=\int_{\vartheta}^{\infty}\frac{\Delta s}{r(s)},\\
&\overline{\xi}(\vartheta)=\xi(\vartheta)+k\int_{\vartheta}^{\infty}q(s)\left(1-\frac{p(\varrho(s))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(s)))}\right)h_{n-2}(\varrho(s),\vartheta_1)\xi(\varrho(s))\xi(\sigma(s))\Delta s.
\end{aligned}$$

To demonstrate our findings, we shall provide the following Lemma.

Lemma 2.1. *Assume that $\varkappa(\vartheta)$ is an eventually positive solution of* (1.1) *on $[\vartheta_1,\infty)_{\mathbb{T}}$, where $\vartheta_1\geq\vartheta_0$. Furthermore, suppose that* (1.3) *holds and*

$$\int_{\vartheta_0}^{\infty}q(s)\Delta s=\infty. \tag{2.1}$$

Then,

$$F(\vartheta)>0, F^{\Delta}(\vartheta)>0,\cdots F^{\Delta^{n-2}}(\vartheta)>0,\ F^{\Delta^{n-1}}(\vartheta)<0, \tag{2.2}$$

and $\left(\frac{F^{\Delta n-2}(\vartheta)}{\xi(\vartheta)}\right)^{\Delta}\geq 0$ *on* $[\vartheta_1,\infty)_{\mathbb{T}}$.

Proof. Suppose that (1.1) has a nonoscillatory solution on $[\vartheta_0,\infty)_{\mathbb{T}}$. We suppose, without loss of generality, that there exists a $\vartheta_1\in[\vartheta_0,\infty)_{\mathbb{T}}$ such that $\varkappa(\vartheta)$, $\varkappa(\eta(\vartheta))$, $\varkappa(\varrho(\vartheta))>0$ on $[\vartheta_1,\infty)_{\mathbb{T}}$. From (1.1), we can obtain

$$(r(\vartheta)F^{\Delta^{n-1}}(\vartheta))^{\Delta}=-q(\vartheta)f(\varkappa(\varrho(\vartheta)))\leq -kq(\vartheta)\varkappa^{\alpha}(\varrho(\vartheta))\leq 0. \tag{2.3}$$

Then, the function $(r(\vartheta)F^{\Delta^{n-1}}(\vartheta))$ is strictly decreasing. Consequently, $F^{\Delta^{n-1}}(\vartheta)$ is either eventually positive or eventually negative for $\vartheta\geq\vartheta_2\geq\vartheta_1$ such that.

1. $F^{\Delta^{n-1}}(\vartheta)>0$ and $(r(\vartheta)F^{\Delta^{n-1}}(\vartheta))^{\Delta}\leq 0$;
2. $F^{\Delta^{n-1}}(\vartheta)<0$ and $(r(\vartheta)F^{\Delta^{n-1}}(\vartheta))^{\Delta}\leq 0$.

The preceding cases apply to all values of n, while the subcases involving $F^{\Delta^{n}}\leq 0$ on $[\vartheta_1,\infty)_{\mathbb{T}}$ are excluded because of the condition $\lim_{\vartheta\to\infty}\varkappa(\vartheta)>0$.

Consider the firs case (i) holds, we have $F^{\Delta} > 0$ for $\vartheta \geq \vartheta_2$. Then

$$\varkappa^{\alpha}(\vartheta) = F(\vartheta) - p(\vartheta)\varkappa(\eta(\vartheta)) \geq F(\vartheta) - p(\vartheta)F^{\frac{1}{\alpha}}(\eta(\vartheta)) = F(\vartheta) - p(\vartheta)\frac{F(\vartheta)}{F(\vartheta)}F^{\frac{1}{\alpha}}(\eta(\vartheta))$$

$$\geq F(\vartheta) - p(\vartheta)\frac{F(\vartheta)}{F(\eta(\vartheta))}F^{\frac{1}{\alpha}}(\eta(\vartheta)) = F(\vartheta)\left[1 - \frac{p(\vartheta)}{F^{1-\frac{1}{\alpha}}(\eta(\vartheta))}\right], \quad \text{for } \vartheta \geq \vartheta_2.$$

The positivity and increasing nature of $F(\vartheta)$ imply the existence of $\vartheta_3 \geq \vartheta_2$ and a positive decreasing function $\varphi(\vartheta) = F(\vartheta)\frac{\xi(\vartheta)}{\xi(\vartheta_3)}$ converges to zero as ϑ tends to ∞ such that $F(\vartheta) \geq \varphi(\vartheta)$ for $\vartheta \geq \vartheta_3$ and consequently, we have

$$\varkappa^{\alpha}(\varrho(\vartheta)) \geq F(\varrho(\vartheta))\left[1 - \frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right], \quad \vartheta \geq \vartheta_3. \tag{2.4}$$

Using (2.4) into (2.3), we obtain

$$(r(\vartheta)F^{\Delta^{n-1}}(\vartheta))^{\Delta} \leq -kq(\vartheta)F(\varrho(\vartheta))\left[1 - \frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right]. \tag{2.5}$$

Define the Riccati transformation as follows:

$$w(\vartheta) := \frac{r(\vartheta)F^{\Delta^{n-1}}(\vartheta)}{F(\varrho(\vartheta))}. \tag{2.6}$$

Clearly, $w(\vartheta) > 0$ and

$$\begin{aligned}
w^{\Delta}(\vartheta) &= \frac{(r(\vartheta)F^{\Delta^{n-1}}(\vartheta))^{\Delta}}{F(\varrho(\vartheta))} + (r(\sigma(\vartheta))F^{\Delta^{n-1}}(\sigma(\vartheta)))\left(\frac{1}{F(\varrho(\vartheta))}\right)^{\Delta} \\
&= \frac{(r(\vartheta)F^{\Delta^{n-1}}(\vartheta))^{\Delta}}{F(\varrho(\vartheta))} - \frac{F^{\Delta}(\varrho(\vartheta))\varrho^{\Delta}(\vartheta)}{F(\varrho(\vartheta))F(\varrho(\sigma(\vartheta)))}(r(\sigma(\vartheta))F^{\Delta^{n-1}}(\sigma(\vartheta))) \\
&\leq -kq(\vartheta)\left[1 - \frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right] - \frac{F^{\Delta}(\varrho(\vartheta))\varrho^{\Delta}(\vartheta)}{F(\varrho(\vartheta))}w(\sigma(\vartheta)) \\
&\leq -kq(\vartheta)\left[1 - \frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right].
\end{aligned}$$

Upon integrating the inequality from ϑ_3 to ϑ, we get

$$\begin{aligned}
w^{\Delta}(\vartheta) &\leq w(\vartheta_2) - k\int_{\vartheta_2}^{\vartheta} q(s)\left[1 - \frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right]\Delta s \\
&\leq -k\inf_{\vartheta \geq \vartheta_3}\left[1 - \frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right]\int_{\vartheta_3}^{\vartheta} q(s)\Delta s.
\end{aligned}$$

Calling the assumption (2.1), we deduce that $w(\vartheta) < 0$ for all large enough ϑ, leading to a contradiction.. As a result, the case $F^{\Delta^{n-1}} > 0$ is ruled out and $F^{\Delta^{n-1}} < 0$. By virtue of Lemma 1.2 and Lemma 1.3, we obtain

$$F(\vartheta) > 0, F^{\Delta}(\vartheta) > 0, \cdots, F^{\Delta^{n-2}}(\vartheta) > 0,\ F^{\Delta^{n-1}}(\vartheta) < 0, \quad \text{for } \vartheta \geq \vartheta_3.$$

Furthermore, because $r(\vartheta)F^{\Delta n-1}(\vartheta)$ is monotonic, it follows that

$$\begin{aligned}
F^{\Delta^{n-2}}(\vartheta) &\geq -\int_{\vartheta}^{\infty} \frac{r(s)F^{\Delta^{n-1}}(s)}{r(s)}\Delta s \\
&\geq -r(\vartheta)F^{\Delta^{n-1}}(\vartheta)\int_{\vartheta}^{\infty} \frac{\Delta s}{r(s)} \\
&\geq -r(\vartheta)F^{\Delta^{n-1}}(\vartheta)\xi(\vartheta).
\end{aligned} \tag{2.7}$$

Hence,

$$\left(\frac{F^{\Delta^{n-2}}(\vartheta)}{\xi(\vartheta)}\right)^{\Delta} = \frac{\xi(\vartheta)F^{\Delta^{n-1}}(\vartheta) - F^{\Delta^{n-2}}(\vartheta)\xi^{\Delta}(\vartheta)}{\xi(\vartheta)\xi(\sigma(\vartheta))}$$
$$\bigvee = \frac{\xi(\vartheta)r(\vartheta)F^{\Delta^{n-1}}(\vartheta) + F^{\Delta^{n-2}}(\vartheta)}{r(\vartheta)\xi(\vartheta)\xi(\sigma(\vartheta))} \geq 0.$$

This completes the proof. □

Theorem 2.2. *Assume that conditions* (1.3) *and* (2.1) *hold. If there exists a positive decreasing function* $\varphi(\vartheta)$ *converging to zero as* ϑ *tends to* ∞ *such that*

$$0 < 1 - \frac{p(\vartheta)}{\varphi^{1-\frac{1}{\alpha}}(\eta(\vartheta))} < 1, \qquad \inf_{\vartheta \geq \vartheta_1} \left(1 - \frac{p(\vartheta)}{\varphi^{1-\frac{1}{\alpha}}(\eta(\vartheta))}\right) > 0 \tag{2.8}$$

and

$$\int_{\vartheta_0}^{\infty} \left(\frac{1}{r(u)} \int_{\vartheta_0}^{u} q(s)h_{n-2}(\varrho(s), \vartheta_1)\xi(\varrho(s))\Delta s\right) \Delta u = \infty, \tag{2.9}$$

then (1.1) *oscillates.*

Proof. Assume that (1.1) is non-oscillatory . Without loss of generality, we may assume that $\varkappa(\vartheta) > 0$, $\varkappa(\eta(\vartheta)) > 0$ and $\varkappa(\varrho(\vartheta)) > 0$ on $[\vartheta_1, \infty)_{\mathbb{T}}$. By virtue of (2.1), (1.3) and fact that $\xi^{\Delta}(\vartheta) = -\frac{1}{r} < 0$, it follows that the integral

$$\int_{\vartheta_0}^{\infty} q(s)h_{n-2}(\varrho(s), \vartheta_1)\xi(\varrho(s))\Delta s$$

is unbounded. Since $F^{\Delta}(\vartheta) > 0$, following the proof of Lemma 2.1, we obtain (2.4) for $\vartheta \geq \vartheta_2 \geq \vartheta_1$. Moreover, by Lemma 2.1, there exists $\vartheta_3 \in [\vartheta_2, \infty)_{\mathbb{T}}$ such that $F(\vartheta)$ satisfies (2.2). Finally, applying Lemma 1.3 with $m = n - 2$, we deduce that

$$F(\vartheta) \geq h_{n-2}(\vartheta, \vartheta_1)F^{\Delta^{n-2}}(\vartheta), \quad \text{for } \vartheta \geq \vartheta_3. \tag{2.10}$$

Using (2.4) and (2.10) in (2.3), we get

$$\begin{aligned}(r(\vartheta)F^{\Delta^{n-1}}(\vartheta))^{\Delta} &\leq -kq(\vartheta)\left(1 - \frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right)F(\varrho(\vartheta)) \\ &\leq -kq(\vartheta)\left(1 - \frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right)h_{n-2}(\varrho(\vartheta), \vartheta_1)F^{\Delta^{n-2}}(\varrho(\vartheta)).\end{aligned} \tag{2.11}$$

Using the nondecreasing fact of $\frac{F^{\Delta^{n-2}}(\vartheta)}{\xi(\vartheta)}$, then there exists a constant $c > 0$ such that $F^{\Delta^{n-2}}(\vartheta) \geq c\xi(\vartheta)$. Hence, (2.11) takes the form

$$(r(\vartheta)F^{\Delta^{n-1}}(\vartheta))^{\Delta} \leq -ckq(\vartheta)\left(1 - \frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right)h_{n-2}(\varrho(\vartheta), \vartheta_1)\xi(\varrho(\vartheta)). \tag{2.12}$$

Integrating (2.12) form ϑ_3 to ϑ, we get

$$\begin{aligned}r(\vartheta)F^{\Delta^{n-1}}(\vartheta) &\leq r(\vartheta)F^{\Delta^{n-1}}(\vartheta) - r(\vartheta_1)F^{\Delta^{n-1}}(\vartheta_1) \\ &\leq -ck\int_{\vartheta_1}^{\vartheta} q(s)\left(1 - \frac{p(\varrho(s))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(s)))}\right)h_{n-2}(\varrho(s), \vartheta_1)\xi(\varrho(s))\Delta s.\end{aligned}$$

I.e.,

$$F^{\Delta^{n-1}}(\vartheta) \leq -\frac{ck}{r(\vartheta)} \int_{\vartheta_1}^{\vartheta} q(s) \left(1 - \frac{p(\varrho(s))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(s)))}\right) h_{n-2}(\varrho(s), \vartheta_1)\xi(\varrho(s))\Delta s. \tag{2.13}$$

Integrating (2.13) once more from ϑ_3 to ϑ yields

$$\begin{aligned} F^{\Delta^{n-2}}(\vartheta) &\leq F^{\Delta^{n-2}}(\vartheta_3) - \int_{\vartheta_3}^{\vartheta} \left(\frac{ck}{r(u)} \int_{\vartheta_3}^{u} q(s) \left(1 - \frac{p(\varrho(s))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(s)))}\right) h_{n-2}(\varrho(s), \vartheta_1)\xi(\varrho(s))\Delta s\right) \Delta u \\ &\leq F^{\Delta^{n-2}}(\vartheta_3) - ck \inf_{\vartheta \geq \vartheta_3} \left(1 - \frac{p(\varrho(s))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(s)))}\right) \int_{\vartheta_3}^{\vartheta} \left(\frac{1}{r(u)} \int_{\vartheta_3}^{u} q(s)h_{n-2}(\varrho(s), \vartheta_1)\xi(\varrho(s))\Delta s\right) \Delta u. \end{aligned} \tag{2.14}$$

According to assumptions (2.8) and (2.9), the last inequality leads to $F^{\Delta^{n-2}} \to -\infty$ as $\vartheta \to \infty$, which is a contradiction. The proof is complete. □

Theorem 2.3. *Assume that* (1.3) *and* (2.1) *hold. If there exists a positive decreasing function* $\varphi(\vartheta)$ *converging to zero as* ϑ *approaches* ∞ *satisfies* (2.8) *and*

$$\limsup_{\vartheta \to \infty} k\xi(\varrho(\vartheta)) \int_{\vartheta_1}^{\vartheta} q(s)h_{n-2}(\varrho(s), \vartheta_1) \left(1 - \frac{p(\varrho(s))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(s)))}\right) \Delta s > 1, \tag{2.15}$$

then, (1.1) *is oscillatory.*

Proof. Suppose on the contrary that (1.1) is non-oscillatory and let $\varkappa(\vartheta)$ is an eventually positive solution of (1.1) on $[\vartheta_0, \infty)_{\mathbb{T}}$. We could assume, without loss of generality, that $\varkappa(\vartheta) > 0$, $\varkappa(\eta(\vartheta)) > 0$ and $\varkappa(\varrho(\vartheta)) > 0$ for $\vartheta \geq \vartheta_1 \geq \vartheta_0$. It follows from (2.1) that $F(\vartheta)$ satisfies (2.2). Integrating (1.1) from ϑ_1 to ϑ and taking into account (2.10), we get

$$\begin{aligned} -r(\vartheta)F^{\Delta^{n-1}}(\vartheta) &= -r(\vartheta_1)F^{\Delta^{n-1}}(\vartheta_1) + k\int_{\vartheta_1}^{\vartheta} q(s)\varkappa^{\alpha}(\varrho(s))\Delta s \geq k\int_{\vartheta_1}^{\vartheta} q(s)\varkappa^{\alpha}(\varrho(s))\Delta s \\ &\geq k\int_{\vartheta_1}^{\vartheta} q(s) \left(1 - \frac{p(\varrho(s))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(s)))}\right) F(\varrho(s))\Delta s \\ &\geq k\int_{\vartheta_1}^{\vartheta} q(s) \left(1 - \frac{p(\varrho(s))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(s)))}\right) h_{n-2}(\varrho(s), \vartheta_1)F^{\Delta^{n-2}}(s)\Delta s \\ &\geq kF^{\Delta^{n-2}}(\varrho(\vartheta)) \int_{\vartheta_1}^{\vartheta} q(s) \left(1 - \frac{p(\varrho(s))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(s)))}\right) h_{n-2}(\varrho(s), \vartheta_1)\Delta s. \end{aligned}$$

Taking into account (2.7); it follows that

$$-r(\vartheta)F^{\Delta^{n-1}}(\vartheta) \geq -kr(\varrho(\vartheta))F^{\Delta^{n-1}}(\varrho(\vartheta))\xi(\varrho(\vartheta)) \int_{\vartheta_1}^{\vartheta} q(s) \left(1 - \frac{p(\varrho(s))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(s)))}\right) h_{n-2}(\varrho(s), \vartheta_1)\Delta s.$$

By virtue the decreasing fact of $r(\vartheta)F^{\Delta^{n-1}}(\vartheta)$ and $\varrho(\vartheta) \geq \vartheta$, we get

$$-r(\vartheta)F^{\Delta^{n-1}}(\vartheta) \geq -kr(\vartheta)F^{\Delta^{n-1}}(\vartheta)\xi(\varrho(\vartheta)) \int_{\vartheta_1}^{\vartheta} q(s) \left(1 - \frac{p(\varrho(s))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(s)))}\right) h_{n-2}(\varrho(s), \vartheta_1)\Delta s. \tag{2.16}$$

It follows that

$$k\xi(\varrho(\vartheta))\int_{\vartheta_1}^{t} q(s)\left(1-\frac{p(\varrho(s))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(s)))}\right)h_{n-2}(\varrho(s),\vartheta_1)\Delta s\le 1,\quad \vartheta\ge\vartheta_1 \tag{2.17}$$

and

$$\limsup_{t\to\infty} k\xi(\varrho(\vartheta))\int_{\vartheta_1}^{\vartheta} q(s)\left(1-\frac{p(\varrho(s))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(s)))}\right)h_{n-2}(\varrho(s),\vartheta_1)\Delta s\le 1.$$

This is a contradiction. As a result of that, the proof is complete. □

Theorem 2.4. *Assume that* (1.3) *and* (2.1) *hold. If there exists a positive decreasing function* $\varphi(\vartheta)$ *converging to zero as* ϑ *approaches* ∞ *satisfies* (2.8) *and*

$$\liminf_{\vartheta\to\infty}\int_{\vartheta}^{\varrho(\vartheta)} q(s)\left(1-\frac{p(\varrho(s))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(s)))}\right)h_{n-2}(\varrho(s),\vartheta_1)\overline{\xi}(\varrho(s))\Delta s>\frac{1}{ek}, \tag{2.18}$$

then (1.1) *is oscillatory.*

Proof. Let $\varkappa(\vartheta)$ be a non-oscillatory solution of (1.1). We can assume, without loss of generality, that $\varkappa(\vartheta)>0$, $\varkappa(\eta(\vartheta))>0$ and $\varkappa(\varrho(\vartheta))>0$ for $\vartheta\ge\vartheta_1\ge\vartheta_0$. By virtue of (2.1) and Lemma 2.1, we wee that $F(\vartheta)$ satisfies (2.2). Now, define $\psi(\vartheta)=F^{\Delta^{n-2}}(\vartheta)+r(\vartheta)F^{\Delta n-1}(\vartheta)\xi(\vartheta)\ge 0$, then

$$\begin{aligned}\psi^{\Delta}(\vartheta)&=F^{\Delta^{n-1}}(\vartheta)+\xi^{\Delta}(\vartheta)(r(\vartheta)F^{\Delta^{n-1}}(\vartheta))+\xi(\sigma(\vartheta))(r(\vartheta)F^{\Delta^{n-1}}(\vartheta))^{\Delta}\\&=(r(\vartheta)F^{\Delta^{n-1}}(\vartheta))^{\Delta}\xi(\sigma(\vartheta))\le 0.\end{aligned} \tag{2.19}$$

From (1.1) and (2.11); (2.19) takes the form

$$\psi^{\Delta}(\vartheta)\le -kq(\vartheta)\left(1-\frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right)h_{n-2}(\varrho(\vartheta),\vartheta_1)F^{\Delta^{n-2}}(\varrho(\vartheta))\xi(\sigma(\vartheta)). \tag{2.20}$$

Integrating (2.20) from ϑ to ∞ and taking in our account (2.7), we conclude that

$$\begin{aligned}\psi(\vartheta)&\ge k\int_{\vartheta}^{\infty} q(s)\left(1-\frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right)h_{n-2}(\varrho(s),\vartheta_1)F^{\Delta^{n-2}}(\varrho(s))\xi(\sigma(s))\Delta s\\&\ge k\int_{\vartheta}^{\infty} q(s)\left(1-\frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right)h_{n-2}(\varrho(s),\vartheta_1)(-r(\varrho(s))F^{\Delta^{n-1}}(\varrho(s)))\xi(\varrho(s))\xi(\sigma(s))\Delta s\\&\ge k\int_{\vartheta}^{\infty} q(s)\left(1-\frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right)h_{n-2}(\varrho(s),\vartheta_1)(-r(s)F^{\Delta^{n-1}}(s))\xi(\varrho(s))\xi(\sigma(s))\Delta s\\&\ge k(-r(\vartheta)F^{\Delta^{n-1}}(\vartheta))\int_{\vartheta}^{\infty} q(s)\left(1-\frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right)h_{n-2}(\varrho(s),\vartheta_1)\xi(\varrho(s))\xi(\sigma(s))\Delta s.\end{aligned} \tag{2.21}$$

Using the definition of $\psi(\vartheta)$ and (2.21), we have

$$F^{\Delta^{n-2}}(\vartheta)\ge -r(\vartheta)F^{\Delta^{n-1}}(\vartheta)\overline{\xi}(\vartheta). \tag{2.22}$$

Substituting into (2.11), we obtain

$$-(r(\vartheta)F^{\Delta^{n-1}}(\vartheta))^{\Delta}\ge kq(\vartheta)\left(1-\frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right)h_{n-2}(\varrho(\vartheta),\vartheta_1)(-r(\varrho(\vartheta))F^{\Delta^{n-1}}(\varrho(\vartheta))\overline{\xi}(\varrho(\vartheta)).$$

Consider $\Upsilon(\vartheta) := -r(\vartheta)F^{\Delta^{n-1}}(\vartheta)$. As a result , $\Upsilon(\vartheta)$ is an eventually positive solution of the first-order advanced dynamic inequality

$$\Upsilon^{\Delta}(\vartheta) - kq(\vartheta)\left(1 - \frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right) h_{n-2}(\varrho(\vartheta), \vartheta_1)\overline{\xi}(\varrho(\vartheta))\Upsilon(\varrho(\vartheta)) \geq 0.$$

According to Theorem 2.4 in [36], (1.1) cannot admit an eventually positive solution. On the other hand (2.18) guarantees the oscillation of (1.1). This contradiction establishes that the assumption of a non-oscillatory solution is invalid, thereby completing the proof. □

The following Theorem introduce a new oscillation criterion using Riccati substitution

Theorem 2.5. *Assume that* (1.3) *and* (2.1) *hold. If there exists a positive decreasing function* $\varphi(\vartheta)$ *tends to zero as* ϑ *tends to* ∞ *satisfies* $0 < 1 - \frac{p(\vartheta)}{\varphi^{1-\frac{1}{\alpha}}(\eta(\vartheta))} < 1$ *and*

$$\limsup_{\vartheta\to\infty} \int_{\vartheta_0}^{\vartheta} \left(kq(s)\left(1 - \frac{p(\varrho(s))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(s)))}\right) h_{n-2}(\varrho(s), \vartheta_1)\xi(\sigma(s)) - \frac{\xi(s))}{4r(s)\xi^2(\sigma(s))}\right) \Delta s = \infty, \tag{2.23}$$

then (1.1) is oscillatory.

Proof. Assume that $\varkappa(\vartheta)$ is a non-oscillatory solution of (1.1) on $[\vartheta_0, \infty)_{\mathbb{T}}$ such that $\varkappa(\vartheta) > 0$, $\varkappa(\eta(\vartheta)) > 0$ and $\varkappa(\varrho(\vartheta)) > 0$ for $\vartheta \in [\vartheta_1, \infty)_{\mathbb{T}}$, so that $\varkappa(\vartheta)$ fulfills the conditions outlined of Lemma 2.1 on $[\vartheta_0, \infty)_{\mathbb{T}}$. Define the following Riccati substitution

$$\upsilon(\vartheta) = \frac{r(\vartheta)F^{\Delta n-1}(\vartheta)}{F^{\Delta^{n-2}}(\vartheta)}, \qquad \vartheta \geq \vartheta_1. \tag{2.24}$$

Differentiating (2.24), we obtain

$$\begin{aligned}
\upsilon^{\Delta}(\vartheta) &= \frac{(r(\vartheta)F^{\Delta^{n-1}}(\vartheta))^{\Delta}}{F^{\Delta^{n-2}}(\vartheta)} + (r(\sigma(\vartheta))F^{\Delta^{n-1}}(\sigma(\vartheta)))\left(\frac{1}{F^{\Delta^{n-2}}(\vartheta)}\right)^{\Delta} \\
&\leq \frac{(r(\vartheta)F^{\Delta^{n-1}}(\vartheta))^{\Delta}}{F^{\Delta^{n-2}}(\vartheta)} - \frac{r(\vartheta)(F^{\Delta^{n-1}}(\vartheta))^2}{F^{\Delta^{n-2}}(\vartheta)F^{\Delta^{n-2}}(\sigma(\vartheta))} \\
&\leq \frac{(r(\vartheta)F^{\Delta^{n-1}}(\vartheta))^{\Delta}}{F^{\Delta^{n-2}}(\vartheta)} - \frac{r(\vartheta)(F^{\Delta^{n-1}}(\vartheta))^2}{(F^{\Delta^{n-2}}(\vartheta))^2}.
\end{aligned} \tag{2.25}$$

Using (2.11) and (2.24) leads to

$$\upsilon^{\Delta}(\vartheta) \leq \frac{-kq(\vartheta)\left(1 - \frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right) h_{n-2}(\varrho(\vartheta), \vartheta_1)F^{\Delta^{n-2}}(\varrho(\vartheta))}{F^{\Delta^{n-2}}(\vartheta)} - \frac{\upsilon^2(\vartheta)}{r(\vartheta)}. \tag{2.26}$$

Applying the increasing fact of $\frac{F^{\Delta^{n-2}}(\vartheta)}{\xi(\vartheta)}$; (2.26) takes the form

$$\upsilon^{\Delta}(\vartheta) \leq -kq(\vartheta)\left(1 - \frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right) h_{n-2}(\varrho(\vartheta), \vartheta_1)\frac{\xi(\varrho(\vartheta))}{\xi(\vartheta)} - \frac{\upsilon^2(\vartheta)}{r(\vartheta)}. \tag{2.27}$$

Multiplying both sides of (2.27) by $\xi(\vartheta)$, yields

$$\begin{aligned}
&(\upsilon(\vartheta)\xi(\vartheta))^{\Delta} + \frac{\upsilon(\sigma(\vartheta))}{r(\vartheta)} = \upsilon^{\Delta}(\vartheta)\xi(\vartheta) \\
&\leq -kq(\vartheta)\left(1 - \frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right) h_{n-2}(\varrho(\vartheta), \vartheta_1)\xi(\sigma(\vartheta)) - \frac{\upsilon^2(\vartheta)\xi(\vartheta)}{r(\vartheta)}.
\end{aligned}$$

I.e.,

$$(\upsilon(\vartheta)\xi(\vartheta))^{\Delta} \leq -kq(\vartheta)\left(1-\frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right)h_{n-2}(\varrho(\vartheta),\vartheta_1)\xi(\sigma(\vartheta)) - \frac{\upsilon^2(\vartheta)\xi(\vartheta)}{r(\vartheta)} - \frac{\upsilon(\sigma(\vartheta))}{r(\vartheta)}. \tag{2.28}$$

Since $r(\vartheta)F^{\Delta^{n-1}}(\vartheta)$ is negative and decreasing, and $\frac{F^{\Delta^{n-2}}(\vartheta)}{\xi(\vartheta)}$ is positive and increasing with $\sigma(\vartheta) \geq \vartheta$, then we have

$$\upsilon^2(\vartheta) = \frac{(r(\vartheta)F^{\Delta^{n-1}}(\vartheta))^2}{(F^{\Delta^{n-2}}(\vartheta))^2} \geq \frac{(r(\sigma(\vartheta))F^{\Delta^{n-1}}(\sigma(\vartheta)))^2}{(F^{\Delta^{n-2}}(\vartheta))^2} \geq \upsilon^2(\sigma(\vartheta))\left(\frac{\xi(\sigma(\vartheta))}{\xi(\vartheta)}\right)^2.$$

Substituting into (2.28), we get

$$\begin{aligned}(\upsilon(\vartheta)\xi(\vartheta))^{\Delta} &\leq -kq(\vartheta)\left(1-\frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right)h_{n-2}(\varrho(\vartheta),\vartheta_1)\xi(\sigma(\vartheta)) - \frac{\upsilon^2(\sigma(\vartheta))\xi^2(\sigma(\vartheta))}{r(\vartheta)\xi(\vartheta)} - \frac{\upsilon(\sigma(\vartheta))}{r(\vartheta)}\\ &\leq -kq(\vartheta)\left(1-\frac{p(\varrho(\vartheta))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(\vartheta)))}\right)h_{n-2}(\varrho(\vartheta),\vartheta_1)\xi(\sigma(\vartheta)) + \frac{\xi(\vartheta)}{4r(\vartheta)\xi^2(\sigma(\vartheta))}.\end{aligned}$$

Performing the definite integral from ϑ_1 to ϑ, we obtain

$$\begin{aligned}\upsilon(\vartheta)\xi(\vartheta) - \upsilon(\vartheta_1)\xi(\vartheta_1) \leq &-k\int_{\vartheta_1}^{\vartheta} q(s)\left(1-\frac{p(\varrho(s))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(s)))}\right)h_{n-2}(\varrho(s),\vartheta_1)\xi(\sigma(s))\Delta s\\ &+\int_{\vartheta_1}^{\vartheta}\frac{\xi(s)}{4r(s)\xi^2(\sigma(s))}\Delta s.\end{aligned} \tag{2.29}$$

From (2.7) and (2.24), we get

$$-1 \leq \upsilon(\vartheta)\xi(\vartheta) < 0, \quad \vartheta \geq \vartheta_2. \tag{2.30}$$

Combining (2.29) with (2.30), we conclude that

$$1+\upsilon(\vartheta_1)\xi(\vartheta_1) \geq \int_{\vartheta_1}^{\vartheta}\left[kq(s)\left(1-\frac{p(\varrho(s))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(s)))}\right)h_{n-2}(\varrho(s),\vartheta_1)\xi(\sigma(s)) - \frac{\xi(s)}{4r(s)\xi^2(\sigma(s))}\right]\Delta s.$$

Letting $\vartheta \to \infty$, we obtain a contradiction with (2.23). This completes the proof. □

Example 2.6. Consider for $\mathbb{T} = \mathbb{R}$, the second-order differential equation with $q_0 > 0$

$$\left(\vartheta^2\left(\varkappa^3(\vartheta) + \frac{1}{3\vartheta}\varkappa(\vartheta-2)\right)'\right)' + q_0\vartheta\varkappa(3\vartheta) = 0, \quad \vartheta > 2. \tag{2.31}$$

Here, $n = 2$, $r(\vartheta) = \vartheta^2$, $\alpha = 3$, $p(\vartheta) = \frac{1}{3\vartheta}$, $\eta(\vartheta) = \vartheta - 2$, $q(\vartheta) = q_0\vartheta$ and $\varrho(\vartheta) = 3\vartheta$. It is known that for $\mathbb{T} = \mathbb{R}$; $h_n(\vartheta, s) = \frac{(\vartheta-s)^n}{n!}$. Choosing $\varphi(\vartheta) = \frac{1}{\vartheta^{\frac{3}{2}}}$, we observe that

$$\inf_{\vartheta\geq\vartheta_0}\frac{p(\vartheta)}{\varphi^{1-\frac{1}{\alpha}}(\eta(\vartheta))} = \inf_{\vartheta\geq\vartheta_0}\frac{\vartheta-2}{3\vartheta} \neq 0,$$

$$0 < 1 - \frac{p(\vartheta)}{\varphi^{1-\frac{1}{\alpha}}(\eta(\vartheta))} = 1 - \frac{\vartheta-2}{3\vartheta} = \frac{2}{3} + \frac{2}{3\vartheta} < 1$$

and

$$\xi(\vartheta) = \int_{\vartheta}^{\infty}\frac{\Delta s}{r(s)} = \int_{\vartheta}^{\infty}\frac{ds}{s^2} = \frac{1}{\vartheta} < \infty.$$

It follows that condition (2.9) becomes

$$\int_{\vartheta_0}^{\infty}\left(\frac{1}{r(u)}\int_{\vartheta_0}^{u} q(s)h_{n-2}(\varrho(s),\vartheta_1)\xi(\varrho(s))\Delta s\right)\Delta u$$
$$=\int_{\vartheta_0}^{\infty}\left(\frac{1}{u^2}\int_{\vartheta_0}^{u}\frac{q_0 s}{3s}ds\right)du=\int_{\vartheta_0}^{\infty}\left(\frac{1}{u^2}\int_{\vartheta_0}^{u}\frac{2q_0}{3}ds\right)du=\infty.$$

Using Theorem 2.2, every solution of Eq. (2.31) is oscillatory. In addition, we have

$$\int_{\vartheta_0}^{\infty} q(s)\Delta s=\int_{\vartheta_0}^{\infty} q_0 s ds=\infty$$

and

$$\limsup_{\vartheta\to\infty}\int_{\vartheta_0}^{\vartheta}\left(kq(s)\left(1-\frac{p(\varrho(s))}{\varphi^{1-\frac{1}{\alpha}}(\eta(\varrho(s)))}\right)h_{n-2}(\varrho(s),t_1)\xi(\sigma(s))-\frac{\xi(s))}{4r(s)\xi^2(\sigma(s))}\right)\Delta s$$
$$=\limsup_{\vartheta\to\infty}\int_{\vartheta_0}^{\vartheta}\left(q_0 s\left(\frac{2}{3}+\frac{2}{9s}\right)\frac{1}{s}-\frac{\frac{1}{s}}{4s^2\frac{1}{s^2}}\right)ds=\limsup_{\vartheta\to\infty}\int_{\vartheta_0}^{\vartheta}\left(\frac{2q_0}{3}+\frac{2q_0}{9s}-\frac{1}{4s}\right)ds=\infty.$$

Using Theorem 2.5, we have the same coclusion that every solution of Eq. (2.31) is oscillatory. Figure 1 presents some numerical solutions of Eq.(2.31)

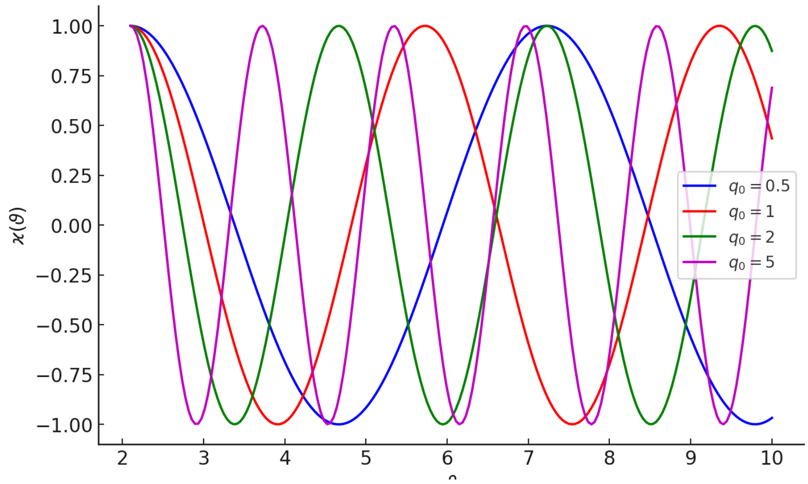

Figure 1: Solutions of (2.31) for different values of q_0

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