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ALMOST PERIODIC FUNCTIONS DEFINED ON $\mathbb{R}^n$ WITH VALUES IN LOCALLY CONVEX SPACES

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ABSTRACT. In this paper we develop the theory of almost periodic functions defined on $\mathbb{R}^n$ with values in locally convex spaces and Fréchet spaces.

Key words : Almost periodic functions, locally convex spaces, Fréchet spaces, relatively dense set.
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1. Introduction

The theory of almost periodic functions was started between 1923 and 1926 by Harald Bohr (1887-1951) (13). The theory has attracted many mathematicians for decades. Some generalizations of the concept have been introduced successfully by V.V. Stepanov and A.S Besicovitch (4). Almost periodic functions with values in Banach spaces was first studied by S. Bochner (16) and developed by many mathematicians like C. Corduneanu (8), S. Zaidman (17) , J.A. Goldstein, L. Amerio, G. Prouse (15), and A.M. Fink (2) etc. The theory of almost periodic functions defined on $\mathbb{R}^n$ with values in a locally convex space or in a Fréchet space was first studied by G.M. N Guérékata e.g see (9, 10, 11, 12). In the monograph of S. Zaidman (17) the theory of almost periodic functions defined on $\mathbb{R}^n$ with values in Banach spaces is elaborated but the theory of almost periodic functions defined on $\mathbb{R}^n$ with values in Locally convex spaces or Fréchet spaces was not developed yet. In this paper we define almost periodicity of functions which are defined on $\mathbb{R}^n$ and have values in a locally convex space or in a Fréchet space and give some results.

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Now we recall the following facts about the Euclidean $\mathbb{R}^n$, locally convex spaces and Fréchet spaces

2. Preliminaries

Let $\mathbb{R}^n$ be the usual Euclidean n-dimensional space. The elements x of $\mathbb{R}^n$ are the n-tuples $x = (x_1, x_2, ..., x_n)$ and a norm of $x \in \mathbb{R}^n$ is given by

$$\|x\| = (x_1^2 + x_2^2 + ... + x_n^2)^{\frac{1}{2}}$$

A closed ball $\overline{B}(x_0; r)$ in $\mathbb{R}^n$ with center x and radius $r > 0$ is defined by the set

$$\overline{B}(x_0; r) = \{x \in \mathbb{R}^n : \|x - x_0\| \leq r\}$$

Definition 1. *A set P is said to be relatively dense in $\mathbb{R}^n$ if there exists a number $r > 0$ such that $P \cap \overline{B}(x; r) \neq \emptyset$, for all $x \in \mathbb{R}^n$.*

We also have the following two important theorems for the sequel. For the detailed proofs of these theorems see (17).

Theorem 1. *A subset P of $\mathbb{R}^n$ is relatively dense in $\mathbb{R}^n$ if and only if, for some $r > 0$, we have the relation $\mathbb{R}^n = \underset{p \in P}{\cup} \overline{B}(p; r)$.*

Theorem 2. *A subset P of $\mathbb{R}^n$ is relatively dense if and only if there exists a compact set $K \subset \mathbb{R}^n$ such that $K + P = \mathbb{R}^n$ (vector sum of K and P)*

Definition 2. *Let X be a vector space over the field $\mathbb{R}$ or $\mathbb{C}$. Then X is said to be topological vector space, which we denote $X = X(\tau)$, if X is equipped with topology τ which is compatible to the algebraic structure of X.*

Remark 1. *The translation $f_a : \mathbb{R}^n \longrightarrow X$ defined by $f_a(x) = f(x + a)$ for a fixed $a \in \mathbb{R}^n$ is a homeomorphism. Thus if Φ is a base of neighborhoods of the origin 0 of X then $\Phi + a$ is a base of a. Therefore the whole topological structure of X can be determined by the base of neighborhoods of the origin 0 of X. Also $\forall\ \lambda \in \mathbb{R}$ or $\mathbb{C}$, the mapping $f_\lambda : \mathbb{R}^n \longrightarrow X$ defined by $f_\lambda(x) = f(\lambda x)$ is homeomorphism, so that if U is neighbourhood of the origin 0 of X then λU is also neighborhood of the origin 0 of X. For more about base of neighborhoods we refer the readers (1). A topological vector space is said to be locally convex topological space or simply locally convex space if it possesses a base of convex neighborhoods.*

Definition 3. *Any complete locally convex space whose topology is induced by an invariant metric is called a Fréchet space.*

To give the idea of Fréchet spaces in terms of semi-norms we have the following:

Definition 4. *Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A function $p : X \longrightarrow \mathbb{R}^+$ is called a a semi-norm if*

(i) $p(x) \geq 0, \forall\ x \in X$;
(ii) $p(\lambda x) = |\lambda|\, p(x), \forall\ x \in X$ *and* $\forall\ \lambda \in \mathbb{R}$ *or* $\mathbb{C}$;
(iii) $p(x+y) \leq p(x) + p(y), \forall\ x, y \in X$.

Remark 2. *Fréchet spaces are also characterized by the existence of a countable,sufficient and increasing family of seminorms* $(p_i)_{i\in\mathbb{N}}$ *(that is* $p_i(x) = 0$*, for all* $i \in \mathbb{N}$ *implies* $x = 0$*, and* $p_i(x) \leq p_{i+1}(x)$*, for all* $x \in X$*,* $i \in \mathbb{N}$ *), which define the pseudonorm:*

$$\|x\| = \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{p_i(x)}{1+p_i(x)},\ \ x \in X,$$

and the metric $d(x,y) = \|x-y\|$ *is invariant with respect to translations, such that* d *generates a complete (by sequences) topology equivalent to that of locally convex space. That is,* d *has the properties:* $d(x,y) = 0$ *iff* $x = y$*,* $d(x,y) = d(y,x)$*,* $d(x,y) \leq d(x,z) + d(z,y)$*,* $d(x+z, y+z) = d(x,y)$ *for all* $z, y, z \in X$*. Also we notice that since* $\frac{p_i(x)}{1+p_i(x)} \leq 1$ *and* $\sum\limits_{i=0}^{\infty} \frac{1}{2^i} = 1$*, it follows that* $\|x\| \leq 1$ *for all* $x \in X$*.*

Moreover d has also the properties given in the following

Theorem 3. (i) $d(cx, cy) \leq d(x,y)$ *for* $|c| \leq 1$*,*
(ii) $d(x+u, y+v) \leq d(x,y) + d(u,v)$*,*
(iii) $d(kx, ky) \leq d(rx, ry)$ *for* $k, r \in \mathbb{R}$*,* $0 < k \leq r$*,*
(iv) $d(kx, ky) \leq kd(x,y)$*,* $\forall k \in \mathbb{N}$*,* $k \geq 2$*,*
(v) $d(cx, cy) \leq (|c|+1)d(x,y)$*,* $\forall\ c \in \mathbb{R}$*.*

3. Almost Periodic Function From $\mathbb{R}^n$ Into A Locally Convex Space

Definition 5. *Let* $X = X(\tau)$ *be a complete Hausdorff locally convex space. A continuous function* $f : \mathbb{R}^n \longrightarrow X$*, is called almost periodic function, if for every neighborhood* U *of the origin* 0 *of* X *we can find a relatively dense set, which we denote by* $T(f;U)$*, and that is contained in* $\mathbb{R}^n$ *such that*

$$f(x+y) - f(x) \in U, \forall y \in T(f;U), \forall x \in \mathbb{R}^n$$

Hence to any neighborhood U *of the origin* 0*, we may associate a number* $r = r(U) > 0$*, in such manner that in any ball* $\overline{B}(x;r)$ *there exists at least one element of the set* $T(f;U)$*. The elements of set* $T(f;U)$ *are called* U*-translation vectors.*

Theorem 4. *If* f *is almost periodic, then the functions* λf*, (* λ *is any scalar) and* $\overset{\vee}{f} : \mathbb{R}^n \longrightarrow X$ *defined by* $\overset{\vee}{f}(x) = f(-x)$ *are also almost periodic.*

Proof. (i) Since $f : \mathbb{R}^n \longrightarrow X$ is almost periodic function so for every neighborhood U of the origin 0 we may find a relatively dense set $T(f;U)$ such that

$$f(x+y) - f(x) \in U, \forall\ y \in T(f;U), \forall x \in \mathbb{R}^n$$

Now

$$\lambda f(x+y) - \lambda f(x) \in \lambda U, \forall\ y \in T(f;U), \forall\ x \in \mathbb{R}^n$$

i.e. for every neighborhood U of the origin 0 we can find a relatively dense set $T(f;\lambda U)$ such that

$$\lambda f(x+y) - \lambda f(x) \in \lambda U, \forall\ y \in T(f;U), \forall\ x \in \mathbb{R}^n$$

This shows that f is almost periodic.

(ii) Since f almost periodic function, it follows that for every neighborhood U of the origin 0 we can find a relatively dense set $T(f;U)$ such that

$$f(x+y) - f(x) \in U, \forall\ y \in T(f;U), \forall\ x \in \mathbb{R}^n$$

Let $-x = x^{'}$, therefore we have

$$f(-x^{'} + y) - f(-x^{'}) \in U, \forall\ y \in T(f;U), \forall\ x^{'} \in \mathbb{R}^n$$

or

$$f(-(x^{'} - y) - f(-x^{'}) \in U, \forall\ y \in T(f;U), \forall\ x^{'} \in \mathbb{R}^n$$

Replacing $x^{'}$ by x we get

$$\overset{\vee}{f}(x-y) - \overset{\vee}{f}(x) \in U, \forall\ y \in T(f;U), \forall\ x \in \mathbb{R}^n$$

This proves that f is almost periodic with $-y$ as U-translation vectors.

Theorem 5. *Let X be a complete Hausdorff locally convex space and let $f : \mathbb{R}^n \longrightarrow X$ be almost periodic, then the range of f is bounded in X.*

Proof. Let $U(\varepsilon, p_i, 1 \leq i \leq n)$ be any neighborhood of the origin 0 of X and let the associated relatively dense set be $T(f;U)$. By Theorem 2 we have $\mathbb{R}^n = \underset{y \in T(f;U)}{\cup} \overline{B}(y;r)$ for some $r = r(U)$. Therefore for any $x \in \mathbb{R}^n$ $\exists$ $y \in T(f;U)$ such that $\|x-y\| \leq r$. Then, if $x^{'} = x - y$, we have $\left\|x^{'}\right\| \leq r$, $y \in T(f;U)$. Now

$$f(x) = f(x^{'} + y) = f(x^{'} + y) - f(x^{'}) + f(x^{'})$$

therefore for any seminorms p_i we have

$$p_i\left(f(x)\right) = p_i\left(f(x^{'} + y) - f(x^{'}) + f(x^{'})\right) \leq p_i\left(f(x^{'} + y) - f(x^{'})\right) + p_i\left(f(x^{'})\right)$$

Since $x' \longmapsto p_i\left(f(x')\right)$ is continuous function on the compact set $\overline{B}(0;r)$, hence it is bounded there in. Thus from the above inequality we have

$$p_i\left(f(x)\right) \leq \varepsilon + \sup_{x' \in \overline{B}(0;r)} p_i\left(f(x')\right), \forall x \in \mathbb{R}^n.$$

This completes the proof of the theorem. □

Theorem 6. *Let X be a complete Hausdorff locally convex space.*

(i) *If $f : \mathbb{R}^n \longrightarrow X$ is almost periodic, then f is uniformly continuous over $\mathbb{R}^n$.*

(ii) *If $\{f_k\}_{k=1}^{\infty}$ is sequence of almost periodic functions, $f_k : \mathbb{R}^n \longrightarrow X$, $1 \leq k < \infty$. If $f_k \longrightarrow f$ uniformly to f on $\mathbb{R}^n$ then f is almost periodic.*

Proof. (i) Let U be an arbitrary neighborhood of the origin 0 and let V be the symmetric neighborhood of the origin such that $V + V + V \subset U$. Let $r = r(V) > 0$ in the definition of an almost periodic function then by Theorem 2 any $x' \in \mathbb{R}^n$ can be written as $x' = x + y$, $y \in T(f;U), \|x\| \leq r$, where $T(f;U)$ is the relatively dense set corresponding to f. Now by the uniform continuity of f on the closed ball $\overline{B}(0;2r) = \{x \in \mathbb{R}^n : \|x\| \leq 2r\}$ we can find $\delta = \delta(V) > 0$ such that $f(x_1) - f(x_2) \in V$ whenever $\|x_1\| \leq 2r, \|x_1\| \leq 2r$ and $\|x_1 - x_2\| < \delta$. Take now any pair $x_1', x_2' \in \mathbb{R}^n$ such that $\left\|x_1' - x_2'\right\| < \delta$. Then for any given $y \in T(f;V)$, we obtain the decomposition $x_1' = x_1 + y$, where $\|x_1\| \leq r, \|x_1\| \leq r$ in view of Theorem 2. It follows that

$$\|x_1 - x_2\| = \left\|(x_1' - y) - (x_2' - y)\right\| = \left\|x_1' - x_2'\right\| < \delta.$$

Therefore we have

$$f(x_1) - f(x_2) \in V, , \forall\, x_1, x_2 \in \mathbb{R}^n, \|x_1\| \leq r, \|x_2\| \leq r$$

Now any for $y \in T(f;V)$ and $\forall\, x_1', x_2' \in \mathbb{R}^n$ such that $\left\|x_1' - x_2'\right\| < \delta$, we get $f(x_1') - f(x_2') = f(x_1+y) - f(x_2+y) = [f(x_1+y) - f(x_1)] + [f(x_1) - f(x_2)] + [f(x_2) - f(x_2+y)] \in V + V + V \subset U$.
This proves that f is uniformly continuous over $\mathbb{R}^n$.

(ii) Let U be the neighborhood of the origin 0 and let V be the symmetric neighborhood of the origin 0 such that $V + V + V \subset U$. Since $f_k(x) \longrightarrow f(x)$ uniformly over $\mathbb{R}^n$ as $k \longrightarrow \infty$ so we can find a natural number k_0 such that

$$\forall\, k \geq k_0 \Rightarrow f_k(x) - f(x) \in V$$

Since $f_k : \mathbb{R}^n \longrightarrow X$ is almost periodic for $k = 1, 2, 3, ...$, so for the symmetric neighborhood V of the origin 0 we can find a relatively dense set $T(f_k;V)$ such that

$$f_k(x+y) - f_k(x) \in V, \forall\, y \in T(f_k;V), \forall\, x \in \mathbb{R}^n, k = 1, 2, 3....$$

Now $f(x+y)-f(x)=[f(x+y)-f_k(x+y)]+[f_k(x+y)-f_k(x)]+[f_k(x)-f(x)]\subset V+V+V\subset U, \forall\ y\in T(f_k;V), \forall\ x\in\mathbb{R}^n$. Hence for the relatively dense $T(f;U)$ we have

$$f(x+y)-f(x)\in U, \forall\ y\in T(f;U), \forall\ x\in\mathbb{R}^n$$

f is proved to be almost periodic. □

Theorem 7. *Let $X=X(\tau)$ be a complete Hausdorff locally convex space. If $f:\mathbb{R}^n\longrightarrow X$ is almost periodic function then given any neighborhood U of the origin 0 we can find two positive numbers $r=r(U)$ and $\delta=\delta(U)$ such that any ball $\overline{B}(a;r)$ in $\mathbb{R}^n$ contains a ball of radius δ which is contained in $T(f;U)$.*

Proof. Let V be the symmetric neighborhood and U be any neighborhood of the origin such that $V+V\subset U$. Since f is almost periodic function therefore there exists relatively dense set $T(f;V)$ in $\mathbb{R}^n$ and the associated number $R(V)=R$ such that

$$\overline{B}(x;R)\cap T(f;V)\neq\emptyset,\ \forall x\in\ \mathbb{R}^n$$

By the uniform continuity of f over $\mathbb{R}^n$ we can find $\delta(V)=\delta$ such that if $h\in\mathbb{R}^n$ and $\|h\|<\delta$ then

$$f(x+h)-f(x)\subset V,\ \forall x\in\ \mathbb{R}^n$$

We shall now prove that $r(U)=R(V)+2\delta(V)$ and $\delta(U)$ are our desired numbers. In fact given $a\in\mathbb{R}^n$, take $z\in\mathbb{R}^n$ with $\|z\|=\delta$. Then $\exists\ y\in T(f;V)\cap\overline{B}(z+a;R)$. Hence $\|y-a\|\leq R+\delta<r$, so that $y\in\overline{B}(a;r)$. Furthermore, $\forall$ $h\in\mathbb{R}^n$, $\|h\|<\delta$, $\|y+h-a\|\leq R+\delta+\delta=r$, hence $y+h\in\overline{B}(a;r)$. Therefore the whole ball $\overline{B}(y;\delta)$ is contained in the ball $\overline{B}(a;r)$. Finally, any vector in this ball belongs to $T(f;U)$; this is because, if $y+h$ with $\|h\|\leq\delta$ is such a vector that $x\in\mathbb{R}^n$, we have
$f(x+y+h)-f(x)=[f(x+h+y)-f(x+h)]+[f(x+h)-f(x)]\subset V+V\subset U$
where we have used the facts that $y\in T(f;V)$, $\|h\|<\delta$ and the uniform continuity of f over $\mathbb{R}^n$. This proves the result. □

Theorem 8. *Let X be a complete Hausdorff locally convex space. If $f:\mathbb{R}^n\longrightarrow X$ is almost periodic, then the range $\{f(x):x\in\mathbb{R}^n\}$ of f is totally bounded in X.*

Proof. Let U be any neighborhood of the origin 0 and V be a symmetric neighborhood of the origin 0 such that $V+V\subset U$. Let $r=r(V)>0$ be any real number. By the continuity of f, the set $\left\{f(x):x\in\overline{B}(0;r)\right\}$ is compact in X. But in a locally convex space, every compact set is totally bounded, therefore there exist $x_1,x_2,...,x_k$ in X such that for every $x\in\overline{B}(0;r)$ we have

$$f(x)\in\bigcup_{i=1}^{k}(x_i+V)$$

Take now an arbitrary $x \in \mathbb{R}^n$ and consider $y \in T(f;V)$ then we have

$$f(x+y) - f(x) \in V$$

Choose x_l among $x_1, x_2, ..., x_k$ such that

$$f(x) \in x_l + V \Longrightarrow f(x) - x_l \in V, l = 1, 2, 3, ..., k$$

Now $f(x) - x_l = [-f(x+y) + f(x)] + [f(x+y) - x_l] \in V + V \subset U$ or $f(x) \in x_l + U$. Since x is an arbitrary number, we conclude that

$$\{f(x) : x \in \mathbb{R}^n\} \subset \bigcup_{i=1}^{k} (x_i + U)$$

This completes the proof the theorem. □

Remark 3. *If $f : \mathbb{R}^n \longrightarrow X$ is almost periodic with X a Fréchet space then its range is relatively compact in X, since in every complete metric space, relatively compactness and total boundedness are equivalent notions. We conclude in this case that every sequence $\{f(x_k)\}_{k=1}^{\infty}$ contains a convergent subsequence $\{f(x_k^{'})\}_{k=1}^{\infty}$.*

Theorem 9. *Let X be a Fréchet space. Then the function $f : \mathbb{R}^n \longrightarrow X$ is almost periodic iff for every sequence $\{x_k\}_{k=1}^{\infty}$ there exists a subsequence $\left\{x_k^{'}\right\}_{k=1}^{\infty}$, in $\mathbb{R}^n$ such that $\left\{f(x + x_k^{'})\right\}_{k=1}^{\infty}$ converges uniformly over $\mathbb{R}^n$.*

Proof. Let $\{x_k\}_{k=1}^{\infty}$ be a given sequence in $\mathbb{R}^n$ and consider the sequence $\{f_{x_k}\}$ functions $f_{x_k} : \mathbb{R}^n \longrightarrow X$ defined by $f_{x_k}(x) = f(x + x_k), 1 \leq k < \infty$. Let $S = \{\eta_k\}_{k=1}^{\infty}$ be a countable dense set in $\mathbb{R}^n$. Since the range $\{f(x) : x \in \mathbb{R}^n\}$ of f is relatively compact therefore we can extract from $\{f(\eta_1 + x_k)\}_{k=1}^{\infty}$ a convergent subsequence. Let $\{f_{x_1,k}\}_{k=1}^{\infty}$ be a subsequence of the sequence $\{f_{x_k}\}_{k=1}^{\infty}$ which is convergent at η_1. We apply the same argument to the sequence $\{f_{x_1,k}\}_{k=1}^{\infty}$ to choose a subsequence $\{f_{x_2,k}\}_{k=1}^{\infty}$ which converges at η_2. We continue the process, and consider the diagonal sequence $\{f_{x_k,k}\}_{k=1}^{\infty}$ which converges at each η_k in S. Call this last sequence $\{f_{x_k^{'}}\}_{k=1}^{\infty}$. We shall prove that this last sequence is uniformly convergent over $\mathbb{R}^n$, that is, for every neighborhood U of the origin 0, there exists natural number $k_0 = k_0(U)$ such that

$$f(x + x_k^{'}) - f(x + x_l^{'}) \in U, \forall\ k, l \geq k_0, \forall\ x \in \mathbb{R}^n$$

Consider now an arbitrary neighborhood U of the origin 0 and a symmetric neighborhood V of the origin 0 such that $V + V + V + V + V \subset U$. Let $r = r(V) > 0$, then by the uniform continuity of f over $\mathbb{R}^n$ we can find $\delta = \delta(V) > 0$, such that

$$\left\| x - x^{'} \right\| < \delta \Longrightarrow f(x) - f(x^{'}) \in V, \forall x, x^{'} \in \mathbb{R}^n \tag{1}$$

Since $\overline{B}(0;r)$ is compact in $\mathbb{R}^n$ so we can suppose that $\overline{B}(0;r)$ is contained in the union of finite number of υ balls (say) of radii smaller than δ and choose from each ball a point of S, we obtain $S_0 = \{\xi_1, \xi_2, ...\xi_\upsilon\}$. Since S_0 is finite set, $\left\{f_{x'_k}\right\}_{k=1}^{\infty}$ is uniformly convergent over S_0 ; therefore there exists a natural number $k_0 = k_0(V)$ such that

$$f(\xi_i + x_k^{'}) - f(\xi_i + x_l^{'}) \in V, \forall\ i, 1 \leq i \leq \nu \text{ and } k, l \geq k_0 \tag{2}$$

Let $x \in \mathbb{R}^n$ be arbitrary and $y \in T(f;V)$ then $f(x+y) - f(x) \in V$ this is because of almost periodicity of f. Let us choose η_i such that $\|x + y + \xi_i\| < \delta$ then

$$f(x + y + x_k^{'}) - f(\xi_i + x_k^{'}) \in V\ \ , \forall\ k \text{ (by(1))} \tag{3}$$

Now $f(x+x_k^{'}) - f(x+x_l^{'}) = [f(x+x_k^{'}) - f(x+x_k^{'}+y)] + [f(x+x_k^{'}+y) - f(\xi_i+x_k^{'})] + [f(\xi_i+x_k^{'}) - f(\xi_i+x_l^{'})] + [f(\xi_i+x_l^{'}) - f(x+y+x_l^{'})] + [f(x+y+x_l^{'}) - f(x+x_l^{'})]$. Then it appears that $f(x + x_k^{'}) - f(x + x_l^{'}) \in V + V + V + V + V \subset U, \forall k, l \geq k_0$ [By (2), (3) and almost periodicity of f]. Which proves the uniform convergence of the sequence $\{f(x + x_k)\}_{k=1}^{\infty}$ over $\mathbb{R}^n$.

We now prove the sufficiency of the Theorem. Suppose, by contradiction, that f is not almost periodic, so there exists a neighborhood of the origin U such that for every real number $r > 0$, there exists a closed ball $\overline{B}(a;r)$ which contains no element of $T(f;U)$. Consider now an arbitrary vector $x_1 \in \mathbb{R}^n$ and take $r_2\ \|x_1\| > 0$ so $\exists$ a ball $\overline{B}(x_2;r_2)$ which is disjoint of $T(f;U)$. Note that $x_2 - x_1 \in \overline{B}(x_2;r_2)$ hence $x_2 - x_1 \notin T(f;U)$. Next take $r_3 > \|x_1\| + \|x_2\|$ and find a ball $\overline{B}(x_3;r_3)$ which is disjoint of $T(f;U)$. Now both the $x_3 - x_1$ and $x_2 - x_3$ belong to $\overline{B}(x_3;r_3)$ but $x_3 - x_1, x_2 - x_3 \notin T(f;U)$. Continuing this procedure, we can find an infinite sequence $\{x_k\}_{k=1}^{\infty} \subset \mathbb{R}^n$ such that

$$\forall k, l \in \mathbb{N}\ , k \neq l \Longrightarrow x_k\ - x_l \notin T(f;U)$$

It also follows that

$$f(x + x_k + x_l) - f(x) \notin U, \forall k, l \in \mathbb{N}, k \neq l$$

If we put $y = x - x_l$ we get

$$f(y + x_k) - f(y + x_l) \notin U \forall k, l \in \mathbb{N}, k \neq l ...(_*)$$

Suppose there exists a subsequence $\{x_k^{'}\}_{k=1}^{\infty}$ of $\{x_k\}_{k=1}^{\infty}$ such that $\{f(x + x_k^{'})\}_{k=1}^{\infty}$ converges uniformly over $\mathbb{R}^n$. Now for every neighborhood U of the origin 0, there exists a natural number $k_0 = k_0\ (U)$ such that

$$\forall k, l\ \geq k_0\ \Rightarrow f(x + x_k^{'}) - f(x + x_l^{'}) \in U, \forall x \in \mathbb{R}^n$$

This contradicts $(_*)$ and thus the sufficiency of the theorem is established. This also completes the proof of the theorem. $\square$

Theorem 10. *Let X be a Fréchet space and let f, g, f_1, f_2 be almost periodic functions from $\mathbb{R}^n$ into X. Then*

(i) *$f+g$ is almost periodic*
(ii) *The function $F : \mathbb{R}^n \longrightarrow X \times X$ defined by $F(x) = (f_1(x), f_2(x))$ is almost periodic*

Proof. (i) Let $\{x_k\}_{k=1}^{\infty}$ be sequence in $\mathbb{R}^n$. By almost periodicity of f and g there exists a subsequence $\{x_k^{'}\}_{k=1}^{\infty} \subset \{x_k\}_{k=1}^{\infty}$ such that both $\{f(x+x_k^{'})\}_{k=1}^{\infty}$ and $\{g(x+x_k^{'})\}_{k=1}^{\infty}$are uniformly convergent x in $\mathbb{R}^n$ in view of Theorem 9. Consequently $\{(f+g)(x+x_k^{'})\}_{k=1}^{\infty}$ is also uniformly convergent x in $\mathbb{R}^n$. This completes the proof again by Theorem 9.

(ii) Let $\{x_k\}_{k=1}^{\infty}$ be sequence in $\mathbb{R}^n$. One can find a subsequence $\{x_k^{'}\}_{k=1}^{\infty} \subset \{x_k\}_{k=1}^{\infty}$ such that both $\{f(x+x_k^{'})\}_{k=1}^{\infty}$ and $\{g(x+x_k^{'})\}_{k=1}^{\infty}$are uniformly convergent x in $\mathbb{R}^n$ in view of Theorem 9. Thus $\{F(x+x_k^{'})\}_{k=1}^{\infty} = \{(f_1(x+x_k^{'}), f_2(x+x_k^{'}))\}_{k=1}^{\infty}$ is uniformly convergent x in $\mathbb{R}^n$, which proves the almost periodicity of F. □

Corollary 11. *If X is a Fréchet space and f_1, f_2 are almost periodic functions, then for every neighborhood U of the origin 0 of X the functions f_1 and f_2 have common U-translation vectors.*

Proof. Let U be a neighborhood of the origin 0 of X then by theorem 10 (ii) the function $F(x) = (f_1(x), f_2(x))$ from $\mathbb{R}^n$ to $X \times X$ is almost periodic. Consider now y a U-translation vector of F, then

$$F(x+y) - F(x) \in U \times U \ \forall, x \in \mathbb{R}^n$$

and therefore

$$f_i(x+y) - f_i(x) \in U, i = 1,2 \ \forall, x \in \mathbb{R}^n$$

This proves that y is a U-translation vector for each of $f_i, i = 1,2$. □

Remark 4. *Theorem 10 (ii) and corollary 11 hold true even for k function $k > 2$.*

Theorem 12. *Let X be a Fréchet space. Then the set of all almost periodic functions from $\mathbb{R}^n$ to X is a Fréchet space.*

Proof. Let $C_b(\mathbb{R}^n, X)$ denote the linear space of all continuous bounded functions $\mathbb{R}^n \longrightarrow X$ and $\{q_k\}$, $k \in \mathbb{N}$, the family of seminorms which generates the topology of X. Without loss of generality we may assume that $q_{k+1} \geq q_k$, pointwise, for $k \in \mathbb{N}$. Define

$$q_k^{'}(f) = \sup_{x \in \mathbb{R}^n} q_k(f(x)), \ k \in \mathbb{N}$$

Then $\{q_k^{'}\}$ is a family of seminorms of $C(\mathbb{R}^n, X)$. Moreover it is clear that $q_{k+1}^{'} \geq q_k^{'}, k \in \mathbb{N}$. Now define

$$\|f\| = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{q_k^{'}(f)}{1 + q_k^{'}(f)}.$$

Obviously $C_b(\mathbb{R}^n, X)$ with the above defined norm is a Fréchet space. Since $AP(\mathbb{R}^n, X)$ the set of all almost periodic functions from $\mathbb{R}^n$ to X is a linear subspace of $C_b(\mathbb{R}^n, X)$ and by Theorem 6 (ii), it is closed. This proves that $AP(\mathbb{R}^n, X)$ is a Fréchet space. □

Definition 6. *A continuous function $f : \mathbb{R}^n \longrightarrow X$, where X is Fréchet space, is called a normal function if any sequence $\{x_k^{'}\}_{k=1}^{\infty}$ in $\mathbb{R}^n$ contains a subsequence $\{x_k\}_{k=1}^{\infty}$ such that the sequence $\{f(x + x_k)\}_{k=1}^{\infty}$ is uniformly convergent over $\mathbb{R}^n$.*

Remark 5. *From Theorem 9 it is clear that any function $f : \mathbb{R}^n \longrightarrow X$, where X is Fréchet space, is normal iff it is almost periodic.*

Now we give the following interesting theorem.

Theorem 13. *Let X be a complete locally convex space and let $f : \mathbb{R}^n \longrightarrow X$ and almost periodic function. Then for every sequence $\{x_k\}_{k=1}^{\infty}$ of real numbers there exists a subsequence $\{x_k^{'}\}_{k=1}^{\infty}$ such that for every neighborhood of the origin U, we have $f(x + x_k) - f(x + x_l) \in U$, $\forall\ x \in \mathbb{R}^n$, $\forall\ k, l \in \mathbb{N}$.*

Proof. Let U be the neighbourhood of the origin and let V be a symmetric neighborhood of the origin such that $V + V + V \subset U$. Now by Theorem 2 any vector $x_k \in \mathbb{R}^n$ can be written as $x_k = z_k + y_k$ where $y_k \in T(f;V)$ and $||z|| \leq r$. Moreover, using the continuity of f over $\mathbb{R}^n$, we can find $\delta > 0$ such that

$$||x_1 + x_2|| < 2\delta \Rightarrow f(x_1) - f(x_2) \in V.$$

Note that in the ball $\{x \in \mathbb{R}^n : ||x - z|| < \delta\}$ we find an infinite sequence of $z_k\prime s$ which we denote by $\{z_{k_i}\}_{i=1}^{\infty}$. Now take two vectors x_{i_j} and x_{i_k} where $x_{i_l} = y_{i_l} + z_{i_l}$, $y_{i_l} \in T(f, V)$, $||z_{i_l}|| \leq r, l = j, k$ then $f(x + x_{i_k}) - f(x + x_{i_j})$ $= f(x + y_{i_k} + z_{i_k}) - f(x + y_{i_j} + z_{i_j}) = [f(x + y_{i_k} + z_{i_k}) - f(x + z_{i_k})] + [f(x + z_{i_k}) - f(x + z_{i_j})] + [f(x + z_{i_j}) - f(x + y_{i_j} + z_{i_j})] \in V + V + V \subset U, x \in \mathbb{R}^n$ and $i, j \in \mathbb{N}...(_*)$ $(_*)$ is a consequence of the fact that $f(x+y_{i_k}+z_{i_k})-f(x+z_{i_k}) \in V$, $f(x + z_{i_k}) - f(x + z_{i_j}) \in V$, $f(x + z_{i_j}) - f(x + y_{i_j} + z_{i_j}) \in V$, $\forall\ x \in \mathbb{R}^n$ and $z_{i_k} - z_{i_j} \in T(f;V), \forall\ i, j \in \mathbb{N}$ while $||z_{i_k} - z_{i_j}|| = ||z_{i_k} - z + z - z_{i_j}|| \leq ||z_{i_k} - z|| + ||z_{i_j} - z|| \leq \delta + \delta = 2\delta$. If we put $x_{i_k} = x_k$ and $x_{i_j} = x_j$, $k, j \in \mathbb{N}$ then we get

$$f(x + x_k) - f(x + x_j) \in U \forall x \in \mathbb{R}^n, \forall k, j \in \mathbb{N}.$$

This completes the proof of the Theorem. □

Remark 6. *In terms of the family of seminorms* $\{p_i\}_{i=1}^{\infty}$*, Definition 5 can be rewritten as follows: for any* $\varepsilon > 0$*, there exists a relatively dense set* $T(f,\varepsilon)$ *in* $\mathbb{R}^n$*, such that* $\sup\limits_{x\in\mathbb{R}^n} p_i(f(x+y) - f(x)) \leq \varepsilon$*,* $\forall\ i \in \mathbb{N}$ *where* $y \in T(f,\varepsilon)$ *is called and* $\varepsilon-$*almost period of* f*. Similar to the case of Banach spaces (see e.g. 15, pp. 7-11), we can develop a theory of Bochner's transform for a Fréchet space as follows: we know that* $C_b(\mathbb{R}^n, X)$ *is a Fréchet space of all bounded continuous functions from* $\mathbb{R}^n$ *to* X*, endowed with the countably family of increasing seminorms* $q_i(f) = \sup\limits_{x\in\mathbb{R}^n} p_i(f(x))$ *,*$\forall\ k \in \mathbb{N}$*. Note that* $f \in C_b(\mathbb{R}^n, X)$ *is bounded if* $f(\mathbb{R}^n)$ *is bounded in the metric space* (X,d) *i.e.* $f \in C_b(\mathbb{R}^n, X)$ *is bounded set if* $\sup\limits_{x\in\mathbb{R}^n} p_i(f(x)) < +\infty$*,*$\forall\ k \in \mathbb{N}$*. The Bochner transform on* $C_b(\mathbb{R}^n, X)$ *is defined as in the case of Banach space,* $\widetilde{f} : \mathbb{R}^n \longrightarrow X$ *by* $\widetilde{f}(s)(x) = f(x+s)$*,* $\forall x \in \mathbb{R}^n$ *and we write* $\widetilde{f} = B(f)$*.*

□

The properties of Bochner transform in a Fréchet space are given by the following theorem.

Theorem 14. (i) $q_i(\widetilde{f}(s)) = q_i(\widetilde{f}) = q_i(\widetilde{f}(0))$, $\forall s \in \mathbb{R}^n$, $\forall\ i \in \mathbb{N}$

(ii) $q_i(\widetilde{f}(s+y) - \widetilde{f}(s)) = q_i(\widetilde{f}(y) - \widetilde{f}(0))$, $\forall s \in \mathbb{R}^n$, $\forall\ i \in \mathbb{N}$

(iii) *f is almost periodic iff for any $\forall s \in \mathbb{R}^n$, $\widetilde{f}(s)$ is almost periodic, with the same set of $\varepsilon-$ almost periods.*

(iv) *$\widetilde{f}(s)$ is almost periodic iff there exists a relatively dense sequence in $\mathbb{R}^n$, denoted by $\{x_p\}_{p=1}^{\infty}$, such that the set $\left\{\widetilde{f}(s_p) : n \in \mathbb{N}\right\}$ is relatively compact in the complete metric space (X,d)*

(v) *$\widetilde{f}(s)$ is almost periodic iff the range $R_{\widetilde{f}(s)}$ is relatively compact in the complete metric space (X,d)*

(vi) *f is almost periodic iff $R_{\widetilde{f}(s)}$ is relatively compact in the complete metric space (X,d)*

Proof. (i) $q_i(\widetilde{f}(s)) = \sup\limits_{x\in\mathbb{R}^n} p_i(f(s+x)) = \sup\limits_{x\in\mathbb{R}^n} p_i(f(x)) = q_i(\widetilde{f}) = \sup\limits_{x\in\mathbb{R}^n} p_i(f(x+0)) = q_i(\widetilde{f}(0))$

(ii) $\forall s \in \mathbb{R}^n$, $\forall\ i \in \mathbb{N}.\forall\ y \in \mathbb{R}^n$ $q_i(\widetilde{f}(s+y) - \widetilde{f}(s)) = \sup\limits_{x\in\mathbb{R}^n} p_i(f(x+s+y) - f(x+s))= \sup\limits_{x\in\mathbb{R}^n} p_i(f(x+y) - f(x))= q_i(\widetilde{f}(y) - \widetilde{f}(0))$, $\forall s \in \mathbb{R}^n$, $\forall\ i \in \mathbb{N}$

(iii) This is an immediate consequence of (ii)

(iv) If $\widetilde{f}(s)$ is almost periodic, it follows by Remark 3 that for every sequence $\{x_p\}_{p=1}^{\infty}$ in $\mathbb{R}^n$, the set $\{\widetilde{f}(x_p) : n \in \mathbb{N}\}$ is relatively compact in the complete metric space (X,d). Conversely suppose that there exists a relatively dense sequence $\{x_p\}_{p=1}^{\infty}$ in $\mathbb{R}^n$ such that the set $\{\widetilde{f}(x_p) : p \in \mathbb{N}\}$ is relatively compact in the complete metric space (X,d). This is equivalent with the fact that $\{\widetilde{f}(x_p) : n \in \mathbb{N}\}$ is totally bounded in (X,d). Therefore due to total boundedness of the set $\left\{\widetilde{f}(x_p) : n \in \mathbb{N}\right\}$ in $C_b(\mathbb{R}^n, X)$, it is possible to fined ν vectors $\{x_{1,0}, x_{2,0}, ...x_{\nu,0}\}$ in $\{x_k\}_{k=1}^{\infty}$ such that $\widetilde{f}(x_n) \in \bigcup\limits_{j=1}^{\nu} \overline{B}(\widetilde{f}(x_{j,0}), \varepsilon)$, $\forall$ $n = 1, 2, 3, ...$ We divide the whole sequence $\left\{\widetilde{f}(x_n) : n \in \mathbb{N}\right\}$ into ν subsequences $\left\{\widetilde{f}(x_{j,n}) : n \in \mathbb{N}\right\}$, $j = 1, 2, 3, ..., \nu$, where $\{x_{j,p}\}_{p=1}^{\infty} \subset \{x_p\}_{p=1}^{\infty}$ is such that $q_i(\widetilde{f}(x_{j,p} - x_{j,0}) - \widetilde{f}(0)) < \varepsilon$. It follows that $y_{j,p} = x_{j,p} - x_{j,0} \in T(\widetilde{f}, \varepsilon)$. Now we show that the set $\bigcup\limits_{j=1}^{\nu} \{y_{j,p} : n \in \mathbb{N}\}$ is relatively dense in $\mathbb{R}^n$. Let $r > 0$ be such that any ball $\overline{B}(x,r)$ in $\mathbb{R}^n$contains some x_k. We say that in any ball $\overline{B}(x, r + \overline{r})$,where $\overline{r} = \sup\limits_{1 \le j \le \nu} \{\|x_{j,0}\|\}$, contains some $y_{j,p}$. In fact, in $B(x,r)$ there is some x_k which is an $x_{j,p}$ for some j and p. Hence $\|x_{j,p} - x\| \le r$ and it follows that $\|x_{j,p} - x_{j,0} - x\| \le \|x_{j,p} - x\| + \|x_{j,0}\| \le r + \overline{r}$, so that $x_{j,p} - x_{j,0} \in \overline{B}(x, r + \overline{r})$. We note that the relatively dense set contained in $T(\widetilde{f}, \varepsilon)$ was obtained by taking differences of elements in $\{x_p\}_{p=1}^{\infty}$. Also assume that $f \in C_b(\mathbb{R}^n, X)$ is almost periodic then $\left\{\widetilde{f}(x) : x \in \mathbb{R}\right\}$ is relatively compact and in particular $\left\{\widetilde{f}(w_p) : w_p \in \mathbb{Z}^n\right\}$ is relatively compact sequence in $C_b(\mathbb{R}^n, X)$. It follows that, $\forall$ $\varepsilon > 0$, the set $T(f, \varepsilon)$ contains relatively dense set formed from the elements in $\mathbb{Z}^n$. This proves the almost periodicity of $\widetilde{f}$.

(v) The necessity follows from Theorem 9. The sufficiency is a direct consequence of (iv).

(vi) It is a consequence of (iii) and (v) □

The following theorem is also a sufficient criterion for almost periodicity.

Theorem 15. *Let $f \in C_b(\mathbb{R}^n, X)$ and $\{f(x_k) : n \in \mathbb{N}\}$ be a relatively compact sequence in X for a relatively dense sequence $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{R}^n$. Assume that for every $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ and a constant $C_{i,j} > 0$ such that $C_{i,j} \sup \{p_i(f(x + x_p) - f(x + x_q)) : x \in \mathbb{R}^n\} \le p_j(f(x_p) - f(x_q))$, $\forall$ $p, q \in \mathbb{N}$. Then f is almost periodic.*

Proof. The inequality in the statement together with the Theorem 14 (i) obviously imply that

$$C_{i,j} q_i(\widetilde{f}(x_p) - \widetilde{f}(x_q)) \le p_j(f(x_p) - f(x_q)), \forall p, q \in \mathbb{N}.$$

Since the set $\{f(x_k) : n \in \mathbb{N}\}$ is relatively compact, it has a convergent subsequence $\left\{f(x_k^{'}) : n \in \mathbb{N}\right\}$ which is a Cauchy sequence in the complete metric space (X, d), so it must be convergent. With this together with Theorem 14, (iv), it follows that $\widetilde{f}(s)$ is almost periodic, which by Theorem 14, (ii), implies that f is almost periodic. This completes the proof of the theorem. $\square$

Corollary 16. *(A criterion of Bochner) Let the function $f : \mathbb{R}^n \longrightarrow X$ have relatively compact range and also assume that for every $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ and a constant $C_{i,j} > 0$ such that*
$C_{i,j} \sup \{p_i(f(x+y) - f(x)) : x \in \mathbb{R}^n\} \leq \inf \{p_j(f(x+y) - f(x)) : x \in \mathbb{R}^n\}$.
Then f is almost periodic.

Proof. Let $\{x_k\}_{k=1}^{\infty}$ be any relatively dense sequence in $\mathbb{R}^n$ and let $y = x_p - x_q$, we get from the given inequality
$C_{i,j} \sup\{p_i(f(x + x_p - x_q) - f(x)) : x \in \mathbb{R}^n\} \leq \inf\{p_j(f(x + x_p - x_q) - f(x)) : x \in \mathbb{R}^n\}$
Take $u = x - x_q$ then
$C_{i,j} \sup\{p_i(f(u + x_p) - f(u + x_q)) : u \in \mathbb{R}^n\} \leq \inf\{p_j(f(u + x_p) - f(u + x_q)) : u \in \mathbb{R}^n\} \leq \{p_j(f(x_p) - f(x_q)) : u \in \mathbb{R}^n\}$.
From Theorem 13 the almost periodicity is proved. $\square$

Definition 7. *Let $f : \mathbb{R}^n \longrightarrow X$ be an almost periodic function. A sequence $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{R}^n$ is said to be regular with respect to f if the sequence of translates $\{f(x + x_k)\}_{k=1}^{\infty}$ is uniformly convergent on $\mathbb{R}^n$ which is equivalent to the convergence of $\left\{\widetilde{f}(x_k)\right\}_{k=1}^{\infty}$ in $C_b(\mathbb{R}^n, X)$.*

Theorem 17. *Let $f : \mathbb{R}^n \longrightarrow X$ be an almost periodic function and assume that for a sequence $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{R}^n$ there exists the limit $\lim_{k\to\infty} f(\eta_p + x_k) = g_p \in X$ for all elements in the sequence $\{\eta_p\}_{p=1}^{\infty}$ which is dense in $\mathbb{R}^n$ then $\{x_k\}_{k=1}^{\infty}$ is regular to f.*

Proof. Suppose that $\left\{\widetilde{f}(x_k)\right\}_{k=1}^{\infty}$ is not convergent then there exists $\varepsilon_0 > 0$ and one may extract sequences $\{p_l\}_{l=1}^{\infty}$ and $\{p_l\}_{l=1}^{\infty}$ from $\{p\}_{p=1}^{\infty}$, such that $q_i(\widetilde{f}(x_{p_k}) - \widetilde{f}(x_{q_k})) \geq \varepsilon_0$, $k = 1, 2, 3....$ As the range of $\widetilde{f}$ is relatively compact, we may assume that $\lim_{k\to\infty} \widetilde{f}(x_{p_k}) = \widetilde{g}_1$, $\lim_{k\to\infty} \widetilde{f}(x_{q_k}) = \widetilde{g}_2$, where $q_i(\widetilde{g}_1 - \widetilde{g}_2) \geq \varepsilon_0$. Hence $\lim_{k\to\infty} f(x + x_{p_k}) = g_1(x)$, $\lim_{k\to\infty} f(x + x_{q_k}) = g_2(x)$ uniformly on $\mathbb{R}^n$. On the other hand we get $\lim_{k\to\infty} f(\eta_j + x_{p_k}) = g_1(\eta_j) = g_j = \lim_{k\to\infty} f(\eta_j + x_{q_k}) = g_2(\eta_j)$, $\forall\ j = 1, 2, 3, ...$ From the continuity of g_1 and g_2 and the density of $\{\eta_j\}_{j=1}^{\infty}$ in $\mathbb{R}^n$ we obtain $g_1(x) = g_2(x)$, $\forall\ x \in \mathbb{R}^n$ this is a contradiction. This completes the proof. $\square$

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