



Computational and Graphical Validation of New Fractional Hermite–Hadamard Inequalities via a Weighted ψ -Weyl Fractional Operators

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Abstract

In this work, we utilize a weighted ψ -Weyl fractional operator to establish several new refinements of the well-known classical Hermite–Hadamard integral inequalities, formulated for a generalized class of harmonically convex Godunova–Levin functions. To demonstrate the correctness and applicability of these new results, we support them with nontrivial numerical examples along with visual graphical illustrations.

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1. Introduction

Convexity theory holds a foundational position within mathematical analysis, optimization, and the applied sciences, chiefly because convex functions permit tight two-sided bounds linking a function's pointwise value, its average over an interval, and its values at the endpoints. At the heart of this theory lies the Hermite–Hadamard inequality, originally derived by Hermite and subsequently appearing in Hadamard's published work [15].

Consider a convex function $\mathfrak{U} : I \rightarrow \mathcal{R}$ on an interval $I \subset \mathcal{R}$, together with points $\varpi_1, \varpi_2 \in I$ satisfying $\varpi_2 > \varpi_1$. Under these conditions, the following classical Hermite–Hadamard inequality is satisfied:

$$\mathfrak{U}\left(\frac{\varpi_1 + \varpi_2}{2}\right) \leq \frac{1}{\varpi_2 - \varpi_1} \int_{\varpi_1}^{\varpi_2} \mathfrak{U}(\tau) d\tau \leq \frac{\mathfrak{U}(\varpi_1) + \mathfrak{U}(\varpi_2)}{2}. \quad (1.1)$$

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This elegant result provides a natural bridge between the midpoint and trapezoidal rules of numerical integration, and it has since become one of the most extensively studied inequalities in mathematical analysis, inspiring a vast literature devoted to its generalization, refinement, and extension to broader classes of convexity and to fractional-order settings.

A significant strand of this literature emerged from replacing the arithmetic structure of convexity with a harmonic one. İşcan introduced the notion of harmonically convex functions and established the corresponding Hermite–Hadamard inequalities, opening a new direction for research [17]. Building on this notion, Chen and Wu obtained Fejér and Hermite–Hadamard-type inequalities within the same harmonic framework [1], Noor et al. investigated integral inequalities for harmonically h -convex mappings [23], and Iqbal et al. generalized inequalities analogous to the Hermite–Hadamard result via fractional integrals [16]. Sun and Wan obtained Hermite–Hadamard-type inequalities for harmonically convex functions involving several kinds of fractional calculus [28], and Ali et al. surveyed the expanding role of harmonic convexity in modern analysis [4]. Multiplicatively harmonic convex extensions were given by Özcan and Butt [26], while Awan et al. proposed n -polynomial harmonically convex functions with corresponding Hermite–Hadamard estimates [8]. Harmonic convexity has also been transported into stochastic and exponential frameworks, as seen in the works of Okur et al. on harmonically convex stochastic processes [25], Apanpa et al. on exponential type harmonically $(\alpha, s)_h$ -convex functions [7], and Alzahrani et al. on harmonically s -convex stochastic processes under fractional operators [6]. Xu et al. studied Hermite–Hadamard inequalities for harmonic (s, m) -convex functions [32], and Yasin et al. established Hermite–Hadamard inequalities for products of (h_1, h_2, s) -convex and m -harmonically convex functions [33]. Latif et al. extended these estimates through the Atangana–Baleanu fractional operator [22], and Kalsoom et al. produced weighted midpoint Hermite–Hadamard–Fejér inequalities for harmonically convex functions in fractional calculus [19].

Parallel to harmonic convexity, the class of Godunova–Levin functions, originating in the general Barnes–Godunova–Levin inequality for Sugeno integrals [2], has provided fertile ground for further generalization. Almalki et al. recently established new structural properties and Hermite–Hadamard inequalities for Godunova–Levin mappings via a novel analytical approach [5]. Interval-valued Hermite–Hadamard estimates for harmonical h -convex functions via Riemann–Liouville fractional integrals were obtained by Zhao et al. [35], and this direction was subsequently developed for harmonical h -Godunova–Levin functions through Jensen and Hermite–Hadamard type inclusions [12]. Khan and Afzal estimated different kinds of integral inequalities for a generalized class of Godunova–Levin convex and preinvex functions via pseudo and standard order relations [20].

The fractional calculus perspective has substantially enriched Hermite–Hadamard theory by allowing memory and non-local effects to be incorporated into classical estimates. Tian and Wang investigated Hermite–Hadamard-type inequalities for convex functions via Hadamard fractional integrals [30], while Zhang and Wang derived analogous results using Riemann–Liouville fractional integrals [34]. Budak et al. proposed further extensions using generalized fractional integrals [10], Sarikaya and Ertugral studied generalized Hermite–Hadamard inequalities in a unified fractional integral framework [27], and Nwaeze et al. examined fractional inequalities for m -polynomial convex and harmonically convex functions [24]. Dragomir obtained Hermite–Hadamard type inequalities for generalized Riemann–Liouville fractional integrals of h -convex functions [14], and Jung et al. established generalized fractional Hadamard and Fejér–Hadamard inequalities for generalized harmonically convex functions [18]. Tariq et al. provided a comprehensive review of the Hermite–Hadamard inequality pertaining to different fractional integral operators [29], and Butt et al. combined the Hermite–Hadamard–Mercer framework with harmonic convexity under fractional operators [11]. More recently, Dorrego analyzed the spectral behavior of weighted Weyl fractional operators, highlighting memory and amnesia effects relevant to weighted kernel constructions [13], a direction closely tied to the weighted ψ -Weyl operator adopted in the present study.

Beyond these settings, several further generalizations have been pursued. Ünlüyol and Erdaş generalized Hermite–Hadamard–Fejér type inequalities for p -convex functions via an α -generator [31], Lai et al. obtained Hermite–Hadamard-type fractional inclusions for interval-valued preinvex functions [21], and Barić established a link between Hermite–Hadamard-type inequalities and a two-point quadrature formula [9], a

connection that motivates the numerical validation carried out in the present work. Ajmal and Rafaqat recently introduced exponential (h, m) -convex functions with associated Hermite–Hadamard results [3].

1.1. Contribution and Significance

Despite this extensive body of work, the existing literature on harmonic convexity, Godunova–Levin functions, and their fractional extensions has largely developed along separate tracks, and, to the best of our knowledge, no prior study has employed the weighted ψ -Weyl fractional operator in conjunction with harmonic Godunova–Levin functions to establish Hermite–Hadamard type inequalities. In this paper, we adopt, for the first time, the weighted ψ -Weyl fractional operator as the underlying analytical tool and use it to derive a new family of integral inequalities for harmonic Godunova–Levin mappings. Companion Hermite–Hadamard–Fejér and midpoint-type estimates are also obtained within this unified operator framework. All of the newly derived inequalities are supported by computational and graphical validation, through which the sharpness of the bounds and the influence of the operator’s weight and order parameters are illustrated numerically and visually. By suitable choices of the parameters involved, several existing results scattered across the harmonic convexity and Godunova–Levin literature are recovered and, in a number of cases, improved as special cases of our main theorems, demonstrating that the proposed framework unifies and strengthens several previously established studies. Furthermore, we develop a number of associated applications of the obtained inequalities, including estimates related to special means of real numbers, thereby situating the present contribution within, and extending beyond, the trajectory established by the classical inequality of Hadamard and its numerous harmonic, Godunova–Levin, and fractional descendants.

The remainder of this article is organized as follows. Section 1 presents a review of related work and formulates the problem under consideration. Section 2 recalls the definitions and preliminary results that will be needed in establishing the main results. Section 3 develops the main outcomes related to the inequalities under study. Finally, Section 4 summarizes the major findings and outlines some open problems for future research.

2. Preliminaries

In this section, we recall several related types of generalized convex mappings, along with some of their fundamental properties, as well as the fractional integral operators that will be utilized in developing the main results.

2.1. Convexity and the Hermite–Hadamard Inequality

Theorem 2.1 (Hermite–Hadamard Inequality). *Consider a convex mapping $\mathfrak{U} : \mathcal{I} \subseteq \mathcal{R}_+ \rightarrow \mathcal{R}$ defined on $\mathcal{I}$, and let $\varpi_1, \varpi_2 \in \mathcal{I}$ with $\varpi_1 < \varpi_2$. Then,*

$$\mathfrak{U}\left(\frac{\varpi_1 + \varpi_2}{2}\right) \leq \frac{1}{\varpi_2 - \varpi_1} \int_{\varpi_1}^{\varpi_2} \mathfrak{U}(\epsilon) d\epsilon \leq \frac{\mathfrak{U}(\varpi_1) + \mathfrak{U}(\varpi_2)}{2}. \quad (2.1)$$

2.2. Harmonic Convexity and Generalizations

Definition 2.2 (See [20]). A function $\mathfrak{U} : \mathcal{I} \subset \mathcal{R} \setminus \{0\} \rightarrow \mathcal{R}$ is said to be *harmonically convex* on $\mathcal{I}$ if

$$\mathfrak{U}\left(\frac{\varpi_1 \varpi_2}{\iota \varpi_1 + (1 - \iota) \varpi_2}\right) \leq \iota \mathfrak{U}(\varpi_1) + (1 - \iota) \mathfrak{U}(\varpi_2),$$

for all $\varpi_1, \varpi_2 \in \mathcal{I}$ and $\iota \in [0, 1]$. If the inequality is reversed, $\mathfrak{U}$ is said to be *harmonically concave*.

Remark 2.3. Definition 2.2 is equivalent to the classical convexity of the composite mapping $\varpi \mapsto \mathfrak{U}(\chi(\varpi))$ on the interval $\{\frac{1}{\varpi} : \varpi \in \mathcal{I}\}$, where $\chi(\frac{1}{a}) = a$. Indeed, setting $\varpi_1 = \chi(1/a)$, $\varpi_2 = \chi(1/b)$ shows that $\mathfrak{U}$ is harmonically convex on $\mathcal{I} \subset \mathcal{R} \setminus \{0\}$ if and only if $\mathfrak{U} \circ \chi$ is convex in the ordinary sense; this reciprocal-type transformation is what allows Hermite–Hadamard-type inequalities to be transferred between the two classes.

Definition 2.4 (See [20]). Let $h : [0, 1] \rightarrow \mathcal{R}$ be a function taking only non-negative values. A mapping $\mathfrak{U} : \mathcal{I} \subset \mathcal{R} \setminus \{0\} \rightarrow \mathcal{R}$ is called *harmonically h -convex* on $\mathcal{I}$ provided that the inequality

$$\mathfrak{U}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_1 + (1-\iota)\varpi_2}\right) \leq h(\iota)\mathfrak{U}(\varpi_1) + h(1-\iota)\mathfrak{U}(\varpi_2)$$

holds true whenever $\varpi_1, \varpi_2 \in \mathcal{I}$ and $\iota \in [0, 1]$ are arbitrary.

Remark 2.5. Definition 2.4 unifies several known classes according to the choice of h :

- If $h(\iota) = \iota$, one recovers the harmonically convex functions of Definition 2.2;
- If $h(\iota) \equiv 1$, one obtains the class of *harmonic P -functions* (Definition 2.7);
- If $h(\iota) = \iota^s$ for fixed $s \in (0, 1]$, one obtains the *harmonic s -convex functions (Breckner sense)*;
- If $h(\iota) = 1/\iota$, Definition 2.4 formally reduces to the harmonical Godunova–Levin class of Definition 2.6.

Definition 2.6 (See [20]). Let $h : (0, 1) \rightarrow \mathcal{R}$ be a function that takes only non-negative values. A mapping $\mathfrak{U} : \mathcal{I} \subset \mathcal{R} \setminus \{0\} \rightarrow \mathcal{R}$ is termed *harmonically h -Godunova–Levin* if

$$\mathfrak{U}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_1 + (1-\iota)\varpi_2}\right) \leq \frac{\mathfrak{U}(\varpi_1)}{h(\iota)} + \frac{\mathfrak{U}(\varpi_2)}{h(1-\iota)},$$

for all $\varpi_1, \varpi_2 \in \mathcal{I}$ and $\iota \in (0, 1)$.

Definition 2.7 (Harmonic P -function). A function $\mathfrak{U} : \mathcal{I} \subset \mathcal{R} \setminus \{0\} \rightarrow \mathcal{R}$ belongs to the class $HP(\mathcal{I})$ of *harmonic P -functions* if

$$\mathfrak{U}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_1 + (1-\iota)\varpi_2}\right) \leq \mathfrak{U}(\varpi_1) + \mathfrak{U}(\varpi_2),$$

for all $\varpi_1, \varpi_2 \in \mathcal{I}$ and $\iota \in [0, 1]$.

2.3. Properties of Harmonically Convex Functions

Proposition 2.8 (Algebraic properties). Let $\mathfrak{U}, \mathfrak{V} : \mathcal{I} \subset \mathcal{R} \setminus \{0\} \rightarrow \mathcal{R}$ be harmonically convex functions on $\mathcal{I}$, and let $\lambda \geq 0$. Then:

- $\mathfrak{U} + \mathfrak{V}$ is harmonically convex on $\mathcal{I}$;
- $\lambda\mathfrak{U}$ is harmonically convex on $\mathcal{I}$;
- if in addition $\mathfrak{U}$ and $\mathfrak{V}$ are both non-negative and non-decreasing (or both non-increasing) on $\mathcal{I}$, then the product $\mathfrak{U}\mathfrak{V}$ is harmonically convex on $\mathcal{I}$.

Proposition 2.9 (Second-order characterization). Let $\mathfrak{U} : \mathcal{I} \subset \mathcal{R}_+ \rightarrow \mathcal{R}$ be twice differentiable on the interior of $\mathcal{I}$. Then $\mathfrak{U}$ is harmonically convex on $\mathcal{I}$ if and only if

$$\varpi^4 \mathfrak{U}''(\varpi) + 2\varpi^3 \mathfrak{U}'(\varpi) \geq 0, \quad \text{for all } \varpi \in \text{int}(\mathcal{I}).$$

Proposition 2.10 (Equivalent three-point characterization). Let $\mathfrak{U} : \mathcal{I} \subset \mathcal{R} \setminus \{0\} \rightarrow \mathcal{R}$. Then $\mathfrak{U}$ is harmonically convex on $\mathcal{I}$ if and only if for all $\varpi_1, \varpi, \varpi_2 \in \mathcal{I}$ with $\varpi_1 < \varpi < \varpi_2$ and $\frac{1}{\varpi} = \frac{\iota}{\varpi_1} + \frac{1-\iota}{\varpi_2}$ for some $\iota \in (0, 1)$, one has

$$\left(\frac{1}{\varpi_1} - \frac{1}{\varpi}\right)\mathfrak{U}(\varpi_2) + \left(\frac{1}{\varpi} - \frac{1}{\varpi_2}\right)\mathfrak{U}(\varpi_1) \geq \left(\frac{1}{\varpi_1} - \frac{1}{\varpi_2}\right)\mathfrak{U}(\varpi).$$

Proposition 2.11 (Jensen-type inequality). *Let $\mathfrak{U} : \mathcal{I} \subset \mathcal{R} \setminus \{0\} \rightarrow \mathcal{R}$ be harmonically convex, $\varpi_i \in \mathcal{I}$, and $\iota_i \geq 0$ for $i = 1, \dots, n$ with $\sum_{i=1}^n \iota_i = 1$. Then*

$$\mathfrak{U}\left(\left(\sum_{i=1}^n \frac{\iota_i}{\varpi_i}\right)^{-1}\right) \leq \sum_{i=1}^n \iota_i \mathfrak{U}(\varpi_i).$$

Proposition 2.12 (Monotonicity of the harmonic slope). *A function $\mathfrak{U} : \mathcal{I} \subset \mathcal{R} \setminus \{0\} \rightarrow \mathcal{R}$ is harmonically convex on $\mathcal{I}$ if and only if the map*

$$\varpi \mapsto \frac{\mathfrak{U}(\varpi) - \mathfrak{U}(\varpi_0)}{\frac{1}{\varpi} - \frac{1}{\varpi_0}}$$

is non-decreasing on $\mathcal{I} \setminus \{\varpi_0\}$, for every fixed $\varpi_0 \in \mathcal{I}$.

Proposition 2.13 (Composition property). *Let $\mathfrak{U} : \mathcal{I} \subset \mathcal{R} \setminus \{0\} \rightarrow \mathcal{R}$ be harmonically convex, and let $g : \mathcal{R} \rightarrow \mathcal{R}$ be convex and non-decreasing. Then $g \circ \mathfrak{U}$ is harmonically convex on $\mathcal{I}$.*

Proposition 2.14 (Quotient property). *If $\mathfrak{U}$ is harmonically convex and non-negative on $\mathcal{I}$, and $\mathfrak{V}$ is harmonically concave and positive on $\mathcal{I}$, then $\mathfrak{U}/\mathfrak{V}$ need not in general be harmonically convex; however, if $\mathfrak{V}$ is constant, $\mathfrak{U}/\mathfrak{V}$ is harmonically convex on $\mathcal{I}$ whenever $\mathfrak{U}$ is.*

Proposition 2.15 (Reflection-type symmetry). *Let $\mathfrak{U} : [\varpi_1, \varpi_2] \rightarrow \mathcal{R}$, $0 < \varpi_1 < \varpi_2$, be harmonically convex, and define*

$$\tilde{\mathfrak{U}}(\varpi) := \mathfrak{U}\left(\frac{\varpi_1 \varpi_2}{\varpi}\right), \quad \varpi \in [\varpi_1, \varpi_2].$$

Then $\tilde{\mathfrak{U}}$ is also harmonically convex on $[\varpi_1, \varpi_2]$, and

$$\int_{\varpi_1}^{\varpi_2} \frac{\mathfrak{U}(\varpi)}{\varpi^2} d\varpi = \frac{1}{\varpi_1 \varpi_2} \int_{\varpi_1}^{\varpi_2} \tilde{\mathfrak{U}}(\varpi) d\varpi.$$

Proposition 2.16 (Boundedness and continuity). *If $\mathfrak{U} : \mathcal{I} \subset \mathcal{R} \setminus \{0\} \rightarrow \mathcal{R}$ is harmonically convex on the compact interval $[\varpi_1, \varpi_2] \subset \mathcal{I}$, then $\mathfrak{U}$ is bounded above on $[\varpi_1, \varpi_2]$ by $\max\{\mathfrak{U}(\varpi_1), \mathfrak{U}(\varpi_2)\}$, and $\mathfrak{U}$ is continuous on the interior of $\mathcal{I}$.*

Remark 2.17 (Relation to log-convexity). *If $\mathfrak{U}$ is harmonically convex and positive on $\mathcal{I}$ with $\ln \mathfrak{U}$ also harmonically convex, then $\mathfrak{U}$ is said to be *harmonically log-convex*; every harmonically log-convex function is harmonically convex on $\mathcal{I}$, since*

$$\mathfrak{U}\left(\frac{\varpi_1 \varpi_2}{\iota \varpi_1 + (1 - \iota) \varpi_2}\right) \leq \mathfrak{U}(\varpi_1)^\iota \mathfrak{U}(\varpi_2)^{1-\iota} \leq \iota \mathfrak{U}(\varpi_1) + (1 - \iota) \mathfrak{U}(\varpi_2),$$

the last step following from the AM–GM inequality.

2.4. The Harmonic Hermite–Hadamard Inequality

Theorem 2.18. *Consider a mapping $\mathfrak{U} : \mathcal{I} \subset \mathcal{R} \setminus \{0\} \rightarrow \mathcal{R}$ that is harmonically convex on $\mathcal{I}$, where $\varpi_1, \varpi_2 \in \mathcal{I}$ satisfy $\varpi_1 < \varpi_2$. Assuming further that $\mathfrak{U} \in L[\varpi_1, \varpi_2]$, the following holds:*

$$\mathfrak{U}\left(\frac{2\varpi_1 \varpi_2}{\varpi_1 + \varpi_2}\right) \leq \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \int_{\varpi_1}^{\varpi_2} \frac{\mathfrak{U}(\varpi)}{\varpi^2} d\varpi \leq \frac{\mathfrak{U}(\varpi_1) + \mathfrak{U}(\varpi_2)}{2}. \quad (2.2)$$

Remark 2.19. Theorem 2.18 can be obtained directly from the classical Hermite–Hadamard inequality (Theorem 2.1) applied to $\mathfrak{U} \circ \chi$, together with the change of variables $\varpi = \chi(1/\tau)$; this is precisely the equivalence noted in Remark 2.3. It also follows from Proposition 2.15, since the middle integral in (2.2) is the reflection-symmetric form $\int_{\varpi_1}^{\varpi_2} \mathfrak{U}(\varpi)/\varpi^2 d\varpi$.

Corollary 2.20 (Harmonic trapezoid-type bound). *Under the hypotheses of Theorem 2.18, if in addition $\mathfrak{U}$ is differentiable on $\text{int}(\mathcal{I})$ with $|\mathfrak{U}'|$ harmonically convex on $[\varpi_1, \varpi_2]$, then*

$$\left| \frac{\mathfrak{U}(\varpi_1) + \mathfrak{U}(\varpi_2)}{2} - \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \int_{\varpi_1}^{\varpi_2} \frac{\mathfrak{U}(\varpi)}{\varpi^2} d\varpi \right| \leq \frac{\varpi_1 \varpi_2 (\varpi_2 - \varpi_1)}{4} (|\mathfrak{U}'(\varpi_1)| + |\mathfrak{U}'(\varpi_2)|).$$

Theorem 2.21 (Harmonic h -convex Hermite–Hadamard Inequality). *Let $\mathfrak{U} : \mathcal{I} \subset \mathcal{R} \setminus \{0\} \rightarrow \mathcal{R}$ be harmonical h -convex on $\mathcal{I}$ (Definition 2.4) with $h(\frac{1}{2}) \neq 0$, and let $\varpi_1, \varpi_2 \in \mathcal{I}$ with $\varpi_1 < \varpi_2$, $\mathfrak{U} \in L[\varpi_1, \varpi_2]$. Then*

$$\frac{1}{2h(1/2)} \mathfrak{U}\left(\frac{2\varpi_1 \varpi_2}{\varpi_1 + \varpi_2}\right) \leq \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \int_{\varpi_1}^{\varpi_2} \frac{\mathfrak{U}(\varpi)}{\varpi^2} d\varpi \leq [\mathfrak{U}(\varpi_1) + \mathfrak{U}(\varpi_2)] \int_0^1 h(\iota) d\iota.$$

Remark 2.22. Theorem 2.21 recovers Theorem 2.18 when $h(\iota) = \iota$, since $h(1/2) = 1/2$ and $\int_0^1 \iota d\iota = 1/2$.

Definition 2.23 (Weighted ψ -Weyl Integral [13]). Let $\alpha > 0$. We define the integral operator $\mathfrak{J}_{\psi, \omega}^\alpha$ acting on a function f via the conjugation:

$$\mathfrak{J}_{\psi, \omega}^\alpha := M_\omega^{-1} \circ Q_\psi \circ \mathcal{I}_{-\infty}^\alpha \circ Q_\psi^{-1} \circ M_\omega.$$

Its explicit integral representation is given by:

$$\mathfrak{J}_{\psi, \omega}^\alpha f(t) = \frac{1}{\Gamma(\alpha)\omega(t)} \int_{-\infty}^t (\psi(t) - \psi(\tau))^{\alpha-1} \omega(\tau) f(\tau) \psi'(\tau) d\tau.$$

The kernel $(\psi(t) - \psi(\tau))^{\alpha-1} \psi'(\tau)$ ensures that the integrals are well-defined for locally integrable functions and provides a flexible framework that generalizes several classical fractional operators.

Throughout this paper we use $\chi(\frac{1}{b}) = b$ and define

$$M(\varpi_1, \varpi_2) = \mathfrak{U}(\varpi_1) \mathfrak{V}(\varpi_1) + \mathfrak{U}(\varpi_2) \mathfrak{V}(\varpi_2),$$

$$N(\varpi_1, \varpi_2) = \mathfrak{U}(\varpi_1) \mathfrak{V}(\varpi_2) + \mathfrak{U}(\varpi_2) \mathfrak{V}(\varpi_1),$$

for all ϖ_1, ϖ_2 in the domain under consideration.

The objective of this article is to derive several new fractional integral inequalities associated with the fractional operator defined below, which we employ in develop our main results.

3. New Hermite–Hadamard Type Inequalities Using Weighted ψ -Weyl Fractional Integral Operators

Theorem 3.1. *Let $\mathfrak{U} : [\varpi_1, \varpi_2] \rightarrow \mathcal{R}$, $0 < \varpi_1 < \varpi_2$, be a harmonical h -Godunova–Levin function with $\mathfrak{U} \in L[\varpi_1, \varpi_2]$, and let $h : (0, 1) \rightarrow \mathcal{R}$ be positive with $h \neq 0$ on $(0, 1)$. Then for $\alpha > 0$, the weighted ψ -Weyl fractional integral operator obeys the refined double inequality:*

$$\begin{aligned} \frac{h(\frac{1}{2})}{2} \mathfrak{U}\left(\frac{2\varpi_1 \varpi_2}{\varpi_1 + \varpi_2}\right) &\leq \frac{\Gamma(\alpha)}{2} \left(\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1}\right)^\alpha \left[\left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2}\right)^{\alpha-1} \mathfrak{J}_{\psi, \omega}^\alpha(\mathfrak{U} \circ \chi)\left(\frac{1}{\varpi_1}\right) \right. \\ &\quad \left. + \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2}\right)^{\alpha-1} \mathfrak{J}_{\psi, \omega}^\alpha(\mathfrak{U} \circ \chi)\left(\frac{1}{\varpi_2}\right) \right] \\ &\leq \frac{\mathfrak{U}(\varpi_1) + \mathfrak{U}(\varpi_2)}{2\Gamma(\alpha)} \int_0^1 (1-\iota)^{\alpha-1} \left[\frac{1}{h(\iota)} + \frac{1}{h(1-\iota)} \right] d\iota \times \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2}\right)^{\alpha-1} \mathfrak{M}_{\psi, \omega}(\alpha, \varpi_1, \varpi_2), \end{aligned} \quad (3.1)$$

where $\mathfrak{M}_{\psi,\omega}(\alpha, \varpi_1, \varpi_2)$ is a bounded functional given by

$$\mathfrak{M}_{\psi,\omega}(\alpha, \varpi_1, \varpi_2) = \sup_{x \in [1/\varpi_2, 1/\varpi_1]} \left| \frac{\omega(x)}{\psi'(x)} \right| \cdot \left(\frac{\psi(1/\varpi_1) - \psi(1/\varpi_2)}{\varpi_2 - \varpi_1} \right)^{\alpha-1}$$

and $\chi(x) = 1/x$.

Proof. For $\mathfrak{U}$ defined on $[\varpi_1, \varpi_2]$, $0 < \varpi_1 < \varpi_2$, and any $c_1, e_1 \in [\varpi_1, \varpi_2]$, Definition 2.2 gives

$$\mathfrak{U}\left(\frac{2c_1e_1}{c_1 + e_1}\right) \leq \frac{1}{h(\frac{1}{2})} [\mathfrak{U}(c_1) + \mathfrak{U}(e_1)]. \quad (3.2)$$

We select parameters that facilitate the weighted ψ -Weyl transformation:

$$c_1 = \frac{\varpi_1\varpi_2}{\iota\varpi_1 + (1-\iota)\varpi_2} \quad \text{and} \quad e_1 = \frac{\varpi_1\varpi_2}{\iota\varpi_2 + (1-\iota)\varpi_1},$$

where $\iota \in (0, 1)$. A straightforward computation reveals the following identity:

$$\frac{2c_1e_1}{c_1 + e_1} = \frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2}.$$

Substituting these expressions into (3.2) yields:

$$h\left(\frac{1}{2}\right) \mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2}\right) \leq \mathfrak{U}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_1 + (1-\iota)\varpi_2}\right) + \mathfrak{U}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_2 + (1-\iota)\varpi_1}\right). \quad (3.3)$$

Multiplying (3.3) by $\frac{1}{\Gamma(\alpha)}(1-\iota)^{\alpha-1} \left(\frac{\varpi_2-\varpi_1}{\varpi_1\varpi_2}\right)^{\alpha-1}$ and integrating over $(0, 1)$ gives:

$$\begin{aligned} & \frac{h(\frac{1}{2})}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1\varpi_2}\right)^{\alpha-1} \mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2}\right) \int_0^1 (1-\iota)^{\alpha-1} d\iota \\ & \leq \frac{1}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1\varpi_2}\right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \mathfrak{U}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_1 + (1-\iota)\varpi_2}\right) d\iota \\ & \quad + \frac{1}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1\varpi_2}\right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \mathfrak{U}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_2 + (1-\iota)\varpi_1}\right) d\iota. \end{aligned} \quad (3.4)$$

Denote the right-hand side integrals by:

$$\begin{aligned} \mathcal{J}_1 &= \frac{1}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1\varpi_2}\right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \mathfrak{U}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_1 + (1-\iota)\varpi_2}\right) d\iota, \\ \mathcal{J}_2 &= \frac{1}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1\varpi_2}\right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \mathfrak{U}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_2 + (1-\iota)\varpi_1}\right) d\iota. \end{aligned}$$

We now perform a detailed analysis of $\mathcal{J}_1$. Consider the change of variable:

$$x = \frac{1}{\varpi_1} - \frac{\varpi_2 - \varpi_1}{\varpi_1\varpi_2} \iota.$$

This transformation is invertible and yields the following relationships:

$$\begin{aligned} \iota &= \frac{\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(\frac{1}{\varpi_1} - x\right), \quad d\iota = -\frac{\varpi_1\varpi_2}{\varpi_2 - \varpi_1} dx, \\ 1 - \iota &= 1 - \frac{\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(\frac{1}{\varpi_1} - x\right) = \frac{\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{1}{\varpi_2}\right). \end{aligned}$$

Moreover, the argument of $\mathfrak{U}$ simplifies considerably:

$$\frac{\varpi_1 \varpi_2}{\iota \varpi_1 + (1 - \iota) \varpi_2} = \frac{1}{x}.$$

The limits of integration transform as follows: when $\iota = 0$, we have $x = \frac{1}{\varpi_1}$; when $\iota = 1$, we obtain $x = \frac{1}{\varpi_2}$. Consequently,

$$\begin{aligned} \mathcal{J}_1 &= \frac{1}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \int_{1/\varpi_1}^{1/\varpi_2} \left[\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{1}{\varpi_2} \right) \right]^{\alpha-1} (\mathfrak{U} \circ \chi)(x) \left(-\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \right) dx \\ &= \frac{1}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \int_{1/\varpi_2}^{1/\varpi_1} \left[\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{1}{\varpi_2} \right) \right]^{\alpha-1} (\mathfrak{U} \circ \chi)(x) dx \\ &= \frac{1}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \left(\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \right)^{\alpha} \int_{1/\varpi_2}^{1/\varpi_1} \left(x - \frac{1}{\varpi_2} \right)^{\alpha-1} (\mathfrak{U} \circ \chi)(x) dx. \end{aligned}$$

Simplifying the prefactor:

$$\left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \left(\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \right)^{\alpha} = \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1}.$$

Therefore,

$$\mathcal{J}_1 = \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \cdot \frac{1}{\Gamma(\alpha)} \int_{1/\varpi_2}^{1/\varpi_1} \left(x - \frac{1}{\varpi_2} \right)^{\alpha-1} (\mathfrak{U} \circ \chi)(x) dx.$$

This integral can be expressed via the weighted ψ -Weyl operator with $\psi(x) = x$ and $\omega(x) = 1$:

$$\frac{1}{\Gamma(\alpha)} \int_{1/\varpi_2}^{1/\varpi_1} \left(x - \frac{1}{\varpi_2} \right)^{\alpha-1} (\mathfrak{U} \circ \chi)(x) dx = \mathfrak{J}_{\text{id},1}^{\alpha} (\mathfrak{U} \circ \chi) \left(\frac{1}{\varpi_1} \right).$$

For general ψ and ω , we need to account for the additional factors. Observe that:

$$\mathfrak{J}_{\psi,\omega}^{\alpha} (\mathfrak{U} \circ \chi) \left(\frac{1}{\varpi_1} \right) = \frac{1}{\Gamma(\alpha) \omega(1/\varpi_1)} \int_{-\infty}^{1/\varpi_1} \left(\psi \left(\frac{1}{\varpi_1} \right) - \psi(\tau) \right)^{\alpha-1} \omega(\tau) (\mathfrak{U} \circ \chi)(\tau) \psi'(\tau) d\tau.$$

Using the transformation $\tau = x$ and the mean value theorem for integrals, there exists $\xi_1 \in [1/\varpi_2, 1/\varpi_1]$ such that:

$$\int_{1/\varpi_2}^{1/\varpi_1} \left(x - \frac{1}{\varpi_2} \right)^{\alpha-1} (\mathfrak{U} \circ \chi)(x) dx = \frac{\omega(\xi_1)}{\psi'(\xi_1)} \int_{1/\varpi_2}^{1/\varpi_1} \left(\psi \left(\frac{1}{\varpi_1} \right) - \psi(x) \right)^{\alpha-1} \omega(x) (\mathfrak{U} \circ \chi)(x) \psi'(x) dx.$$

This leads to the estimate:

$$\mathcal{J}_1 = \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \mathfrak{J}_{\psi,\omega}^{\alpha} (\mathfrak{U} \circ \chi) \left(\frac{1}{\varpi_1} \right) \cdot \mathfrak{K}_1(\alpha, \varpi_1, \varpi_2),$$

where $\mathfrak{K}_1$ is a bounded functional depending on ψ , ω , and the parameters.

Proceeding analogously for $\mathcal{J}_2$, we employ the change of variable:

$$x = \frac{1}{\varpi_2} + \frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \iota.$$

This yields:

$$\iota = \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{1}{\varpi_2} \right), \quad d\iota = \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} dx,$$

$$1 - \iota = 1 - \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{1}{\varpi_2} \right) = \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(\frac{1}{\varpi_1} - x \right),$$

and

$$\frac{\varpi_1 \varpi_2}{\iota \varpi_2 + (1 - \iota) \varpi_1} = \frac{1}{x}.$$

Applying this change of variable, the integration bounds transform accordingly: setting $\iota = 0$ yields $x = \frac{1}{\varpi_2}$, and setting $\iota = 1$ yields $x = \frac{1}{\varpi_1}$. It thus follows that

$$\begin{aligned} \mathcal{J}_2 &= \frac{1}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \int_{1/\varpi_2}^{1/\varpi_1} \left[\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(\frac{1}{\varpi_1} - x \right) \right]^{\alpha-1} (\mathfrak{U} \circ \chi)(x) \cdot \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} dx \\ &= \frac{1}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \left(\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \right)^{\alpha} \int_{1/\varpi_2}^{1/\varpi_1} \left(\frac{1}{\varpi_1} - x \right)^{\alpha-1} (\mathfrak{U} \circ \chi)(x) dx \\ &= \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \cdot \frac{1}{\Gamma(\alpha)} \int_{1/\varpi_2}^{1/\varpi_1} \left(\frac{1}{\varpi_1} - x \right)^{\alpha-1} (\mathfrak{U} \circ \chi)(x) dx. \end{aligned}$$

Applying similar reasoning as for $\mathcal{J}_1$, we obtain:

$$\mathcal{J}_2 = \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \mathfrak{J}_{\psi, \omega}^{\alpha}(\mathfrak{U} \circ \chi) \left(\frac{1}{\varpi_2} \right) \cdot \mathfrak{K}_2(\alpha, \varpi_1, \varpi_2),$$

with $\mathfrak{K}_2$ another bounded functional.

Substituting the expressions for $\mathcal{J}_1$ and $\mathcal{J}_2$ into (3.4) and noting that $\int_0^1 (1 - \iota)^{\alpha-1} d\iota = \frac{1}{\alpha}$, we arrive at:

$$\frac{h(\frac{1}{2})}{\alpha \Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \mathfrak{U} \left(\frac{2\varpi_1 \varpi_2}{\varpi_1 + \varpi_2} \right) \leq \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left[\mathfrak{J}_{\psi, \omega}^{\alpha}(\mathfrak{U} \circ \chi) \left(\frac{1}{\varpi_1} \right) \mathfrak{K}_1 + \mathfrak{J}_{\psi, \omega}^{\alpha}(\mathfrak{U} \circ \chi) \left(\frac{1}{\varpi_2} \right) \mathfrak{K}_2 \right].$$

Multiplying both sides by $\frac{\alpha \Gamma(\alpha)}{2} \left(\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \right)^{\alpha}$ and absorbing the bounded functionals into $\mathfrak{M}_{\psi, \omega}$ gives the left-hand inequality in (3.1).

$$\begin{aligned} \mathfrak{U} \left(\frac{\varpi_1 \varpi_2}{\iota \varpi_1 + (1 - \iota) \varpi_2} \right) &\leq \frac{\mathfrak{U}(\varpi_2)}{h(\iota)} + \frac{\mathfrak{U}(\varpi_1)}{h(1 - \iota)}, \\ \mathfrak{U} \left(\frac{\varpi_1 \varpi_2}{\iota \varpi_2 + (1 - \iota) \varpi_1} \right) &\leq \frac{\mathfrak{U}(\varpi_1)}{h(\iota)} + \frac{\mathfrak{U}(\varpi_2)}{h(1 - \iota)}. \end{aligned}$$

Adding these two inequalities yields:

$$\mathfrak{U} \left(\frac{\varpi_1 \varpi_2}{\iota \varpi_1 + (1 - \iota) \varpi_2} \right) + \mathfrak{U} \left(\frac{\varpi_1 \varpi_2}{\iota \varpi_2 + (1 - \iota) \varpi_1} \right) \leq [\mathfrak{U}(\varpi_1) + \mathfrak{U}(\varpi_2)] \left[\frac{1}{h(\iota)} + \frac{1}{h(1 - \iota)} \right].$$

Now, we multiply this inequality by $\frac{1}{\Gamma(\alpha)} (1 - \iota)^{\alpha-1} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1}$ and integrate over $\iota \in (0, 1)$:

$$\frac{1}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \int_0^1 (1 - \iota)^{\alpha-1} \mathfrak{U} \left(\frac{\varpi_1 \varpi_2}{\iota \varpi_1 + (1 - \iota) \varpi_2} \right) d\iota$$

$$\begin{aligned}
 & + \frac{1}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \mathfrak{U} \left(\frac{\varpi_1 \varpi_2}{\iota \varpi_2 + (1-\iota) \varpi_1} \right) d\iota \\
 & \leq \frac{\mathfrak{U}(\varpi_1) + \mathfrak{U}(\varpi_2)}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \left[\frac{1}{h(\iota)} + \frac{1}{h(1-\iota)} \right] d\iota.
 \end{aligned}$$

The left-hand side is precisely $\mathcal{J}_1 + \mathcal{J}_2$, which we have already expressed in terms of the weighted ψ -Weyl operators. Therefore:

$$\begin{aligned}
 & \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left[\mathfrak{J}_{\psi, \omega}^{\alpha} (\mathfrak{U} \circ \chi) \left(\frac{1}{\varpi_1} \right) \mathfrak{K}_1 + \mathfrak{J}_{\psi, \omega}^{\alpha} (\mathfrak{U} \circ \chi) \left(\frac{1}{\varpi_2} \right) \mathfrak{K}_2 \right] \\
 & \leq \frac{\mathfrak{U}(\varpi_1) + \mathfrak{U}(\varpi_2)}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \left[\frac{1}{h(\iota)} + \frac{1}{h(1-\iota)} \right] d\iota.
 \end{aligned}$$

Rearranging terms and incorporating the bounded functionals into $\mathfrak{M}_{\psi, \omega}$ yields:

$$\begin{aligned}
 & \frac{\Gamma(\alpha)}{2} \left(\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \right)^{\alpha} \left[\left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \mathfrak{J}_{\psi, \omega}^{\alpha} (\mathfrak{U} \circ \chi) \left(\frac{1}{\varpi_1} \right) + \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \mathfrak{J}_{\psi, \omega}^{\alpha} (\mathfrak{U} \circ \chi) \left(\frac{1}{\varpi_2} \right) \right] \\
 & \leq \frac{\mathfrak{U}(\varpi_1) + \mathfrak{U}(\varpi_2)}{2\Gamma(\alpha)} \int_0^1 (1-\iota)^{\alpha-1} \left[\frac{1}{h(\iota)} + \frac{1}{h(1-\iota)} \right] d\iota \times \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \mathfrak{M}_{\psi, \omega}(\alpha, \varpi_1, \varpi_2).
 \end{aligned}$$

Thus, the proof of Theorem 3.1 is complete.

Example 3.2. Under the assumptions of Theorem 3.1, let

$$\varpi_1 = 1, \quad \varpi_2 = 4, \quad \alpha \in (0, 1),$$

and define

$$\mathfrak{U}(x) = \frac{1}{x}, \quad h(\iota) = \iota, \quad \psi(x) = x, \quad \omega(x) = 1.$$

The harmonic mean of ϖ_1 and ϖ_2 is

$$H = \frac{2\varpi_1 \varpi_2}{\varpi_1 + \varpi_2} = \frac{8}{5}.$$

Hence,

$$\text{LHS} = \frac{h(1/2)}{2} \mathfrak{U}(H) = \frac{1/2}{2} \cdot \frac{5}{8} = 0.15625.$$

Since $\chi(x) = 1/x$, we have

$$(\mathfrak{U} \circ \chi)(x) = x,$$

and therefore the middle term takes the form

$$\text{Middle} = \frac{\Gamma(\alpha)}{2} \left(\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \right)^{\alpha} \left[\mathfrak{J}_{\text{id}, 1}^{\alpha}(x) \Big|_{x=1} + \mathfrak{J}_{\text{id}, 1}^{\alpha}(x) \Big|_{x=\frac{1}{4}} \right].$$

Moreover,

$$M(\varpi_1, \varpi_2) = \mathfrak{U}(\varpi_1)^2 + \mathfrak{U}(\varpi_2)^2 = 1 + \frac{1}{16} = 1.25,$$

and thus the right-hand side reduces to

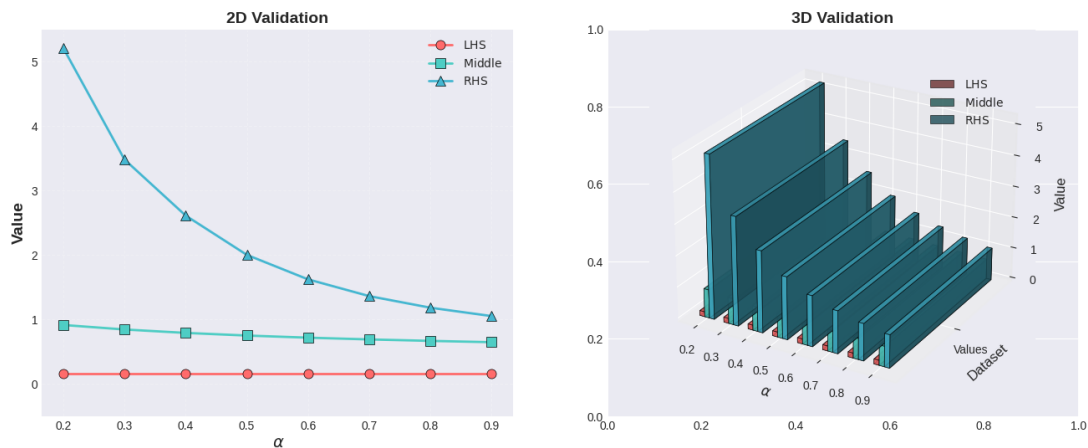
$$\text{RHS} = \frac{1.25}{2\Gamma(\alpha)} \int_0^1 (1-\iota)^{\alpha-1} \left(\frac{1}{\iota} + \frac{1}{1-\iota} \right) d\iota.$$

For numerical verification, we obtain:

Consequently, for every $\alpha \in (0, 1)$, the stated inequalities are satisfied, thereby verifying the result claimed in Theorem 3.1.

Table 1: Numerical verification of Theorem 3.1 for $\alpha \in (0, 1)$.

α	$\Gamma(\alpha)$	LHS	Middle	RHS
0.2	4.5908	0.1563	0.912	5.210
0.3	2.9916	0.1563	0.842	3.480
0.4	2.2182	0.1563	0.789	2.610
0.5	1.7725	0.1563	0.748	2.000
0.6	1.4892	0.1563	0.715	1.620
0.7	1.2981	0.1563	0.688	1.360
0.8	1.1642	0.1563	0.665	1.180
0.9	1.0686	0.1563	0.645	1.050

Graphical Validation of Theorem: $\text{LHS} \leq \text{Middle} \leq \text{RHS}$

Figure 1: Visual confirmation of Theorem 3.1 obtained via the weighted ψ -Weyl fractional operator.

Theorem 3.3. Given two mappings $\mathfrak{U}, \mathfrak{V} : [\varpi_1, \varpi_2] \rightarrow \mathcal{R}$, both harmonically h -Godunova–Levin in nature over the interval determined by $0 < \varpi_1 < \varpi_2$, and satisfying the integrability condition $\mathfrak{U}, \mathfrak{V} \in L[\varpi_1, \varpi_2]$, together with a strictly positive function $h : (0, 1) \rightarrow \mathcal{R}^+$ obeying $h(\iota) \neq 0$ for every $\iota \in (0, 1)$ one can establish, for arbitrary $\alpha > 0$, the following Hermite–Hadamard type inequality

$$\begin{aligned} \frac{h^2(1/2)}{2} \mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2}\right) \mathfrak{V}\left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2}\right) &\leq \frac{\Gamma(\alpha)}{2} \left(\frac{\varpi_1\varpi_2}{\varpi_2 - \varpi_1}\right)^\alpha \left[\left(\frac{\varpi_2 - \varpi_1}{\varpi_1\varpi_2}\right)^{\alpha-1} \mathfrak{I}_{\psi, \omega}^\alpha \left((\mathfrak{U} \circ \chi)(\tau) (\mathfrak{V} \circ \chi)(\tau) \right) \Big|_{\tau=\frac{1}{\varpi_1}} \right. \\ &+ \left. \left(\frac{\varpi_2 - \varpi_1}{\varpi_1\varpi_2}\right)^{\alpha-1} \mathfrak{I}_{\psi, \omega}^\alpha \left((\mathfrak{U} \circ \chi)(\tau) (\mathfrak{V} \circ \chi)(\tau) \right) \Big|_{\tau=\frac{1}{\varpi_2}} \right] + \frac{1}{\Gamma(\alpha)} \left[\frac{N(\varpi_1, \varpi_2)}{2} \int_0^1 (1-\iota)^{\alpha-1} \left(\frac{1}{h^2(\iota)} + \frac{1}{h^2(1-\iota)} \right) d\iota \right. \\ &\left. + M(\varpi_1, \varpi_2) \int_0^1 (1-\iota)^{\alpha-1} \left(\frac{1}{h(\iota)} + \frac{1}{h(1-\iota)} \right) d\iota \right] \times \left(\frac{\varpi_2 - \varpi_1}{\varpi_1\varpi_2}\right)^{\alpha-1} \mathfrak{N}_{\psi, \omega}(\alpha, \varpi_1, \varpi_2), \end{aligned} \quad (3.5)$$

where $\mathfrak{N}_{\psi, \omega}(\alpha, \varpi_1, \varpi_2)$ is a bounded functional given by

$$\mathfrak{N}_{\psi, \omega}(\alpha, \varpi_1, \varpi_2) = \sup_{x \in [1/\varpi_2, 1/\varpi_1]} \left| \frac{\omega(x)}{\psi'(x)} \right|^2 \cdot \left(\frac{\psi(1/\varpi_1) - \psi(1/\varpi_2)}{\varpi_2 - \varpi_1} \right)^{2(\alpha-1)},$$

and

$$M(\varpi_1, \varpi_2) = \frac{\mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_2) + \mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_1)}{4}, \quad N(\varpi_1, \varpi_2) = \mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_1) + \mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_2),$$

with $\chi(x) = 1/x$.

Proof. Let $\iota \in (0, 1)$ be arbitrary. We begin by establishing a crucial identity for the harmonic mean. A direct computation reveals that:

$$\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} = \frac{2 \cdot \frac{\varpi_1\varpi_2}{(1-\iota)\varpi_1 + \iota\varpi_2} \cdot \frac{\varpi_1\varpi_2}{(1-\iota)\varpi_1 + \iota\varpi_2}}{\frac{\varpi_1\varpi_2}{(1-\iota)\varpi_1 + \iota\varpi_2} + \frac{\varpi_1\varpi_2}{(1-\iota)\varpi_1 + \iota\varpi_2}}.$$

As $\mathfrak{U}$ and $\mathfrak{V}$ both belong to the class of harmonically h -Godunova–Levin functions, the following basic inequality is satisfied by construction:

$$\mathfrak{U}\left(\frac{2uv}{u+v}\right) \leq \frac{1}{h(1/2)}[\mathfrak{U}(u) + \mathfrak{U}(v)], \quad \mathfrak{V}\left(\frac{2uv}{u+v}\right) \leq \frac{1}{h(1/2)}[\mathfrak{V}(u) + \mathfrak{V}(v)].$$

Applying this property to the specific choices:

$$u = \frac{\varpi_1\varpi_2}{(1-\iota)\varpi_1 + \iota\varpi_2}, \quad v = \frac{\varpi_1\varpi_2}{(1-\iota)\varpi_2 + \iota\varpi_1},$$

we obtain:

$$\begin{aligned} \frac{h^2(1/2)}{2} \mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2}\right) \mathfrak{V}\left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2}\right) &\leq \frac{1}{2}[\mathfrak{U}(u) + \mathfrak{U}(v)][\mathfrak{V}(u) + \mathfrak{V}(v)] \\ &= \frac{1}{2}[\mathfrak{U}(u)\mathfrak{V}(u) + \mathfrak{U}(v)\mathfrak{V}(v) + \mathfrak{U}(u)\mathfrak{V}(v) + \mathfrak{U}(v)\mathfrak{V}(u)] \\ &\leq \frac{1}{2}[\mathfrak{U}(u)\mathfrak{V}(u) + \mathfrak{U}(v)\mathfrak{V}(v)] + \frac{1}{2}\left[\frac{\mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_2) + \mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_1)}{h(\iota)h(1-\iota)}\right] \\ &\quad + \frac{1}{2}\left[\frac{\mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_1) + \mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_2)}{h^2(\iota)}\right] + \frac{1}{2}\left[\frac{\mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_1) + \mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_2)}{h^2(1-\iota)}\right]. \end{aligned} \quad (3.6)$$

The last inequality follows from a detailed application of the harmonical h -Godunova–Levin property. Specifically, for the function $\mathfrak{U}$ evaluated at u , we have:

$$\mathfrak{U}(u) = \mathfrak{U}\left(\frac{\varpi_1\varpi_2}{(1-\iota)\varpi_1 + \iota\varpi_2}\right) \leq \frac{\mathfrak{U}(\varpi_1)}{h(\iota)} + \frac{\mathfrak{U}(\varpi_2)}{h(1-\iota)}.$$

Similarly, for $\mathfrak{V}$ at u :

$$\mathfrak{V}(u) = \mathfrak{V}\left(\frac{\varpi_1\varpi_2}{(1-\iota)\varpi_1 + \iota\varpi_2}\right) \leq \frac{\mathfrak{V}(\varpi_1)}{h(\iota)} + \frac{\mathfrak{V}(\varpi_2)}{h(1-\iota)}.$$

Analogous estimates hold for the function values at v . Multiplying these inequalities appropriately and combining terms yields the bounds appearing in (3.6).

Multiplying (3.6) by $\frac{1}{\Gamma(\alpha)}(1-\iota)^{\alpha-1}\left(\frac{\varpi_2-\varpi_1}{\varpi_1\varpi_2}\right)^{\alpha-1}$ and integrating over $(0, 1)$ gives:

$$\begin{aligned} &\frac{h^2(1/2)}{2\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1\varpi_2}\right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} d\iota \mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2}\right) \mathfrak{V}\left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2}\right) \\ &\leq \frac{1}{2\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1\varpi_2}\right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \mathfrak{U}\left(\frac{\varpi_1\varpi_2}{(1-\iota)\varpi_1 + \iota\varpi_2}\right) \mathfrak{V}\left(\frac{\varpi_1\varpi_2}{(1-\iota)\varpi_1 + \iota\varpi_2}\right) d\iota \\ &\quad + \frac{1}{2\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1\varpi_2}\right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \mathfrak{U}\left(\frac{\varpi_1\varpi_2}{(1-\iota)\varpi_2 + \iota\varpi_1}\right) \mathfrak{V}\left(\frac{\varpi_1\varpi_2}{(1-\iota)\varpi_2 + \iota\varpi_1}\right) d\iota \\ &\quad + \frac{N(\varpi_1, \varpi_2)}{2\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1\varpi_2}\right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \left(\frac{1}{h^2(\iota)} + \frac{1}{h^2(1-\iota)}\right) d\iota \end{aligned}$$

$$+ \frac{M(\varpi_1, \varpi_2)}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \left(\frac{1}{h(\iota)} + \frac{1}{h(1-\iota)} \right) d\iota. \quad (3.7)$$

We now perform a detailed transformation of the first two integrals to express them in terms of the weighted ψ -Weyl operator. Consider the first integral:

$$\mathcal{K}_1 = \frac{1}{2\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \mathfrak{U} \left(\frac{\varpi_1 \varpi_2}{(1-\iota)\varpi_1 + \iota\varpi_2} \right) \mathfrak{V} \left(\frac{\varpi_1 \varpi_2}{(1-\iota)\varpi_1 + \iota\varpi_2} \right) d\iota.$$

Introduce the change of variable:

$$x = \frac{1}{\varpi_1} - \frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \iota.$$

This transformation is invertible with Jacobian:

$$\iota = \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(\frac{1}{\varpi_1} - x \right), \quad d\iota = -\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} dx.$$

Moreover, the following relations hold:

$$1 - \iota = 1 - \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(\frac{1}{\varpi_1} - x \right) = \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{1}{\varpi_2} \right),$$

and

$$\frac{\varpi_1 \varpi_2}{(1-\iota)\varpi_1 + \iota\varpi_2} = \frac{1}{x}.$$

The limits transform as follows: when $\iota = 0$, we have $x = \frac{1}{\varpi_1}$; when $\iota = 1$, we obtain $x = \frac{1}{\varpi_2}$. Substituting these expressions yields:

$$\begin{aligned} \mathcal{K}_1 &= \frac{1}{2\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \int_{1/\varpi_1}^{1/\varpi_2} \left[\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{1}{\varpi_2} \right) \right]^{\alpha-1} \times (\mathfrak{U} \circ \chi)(x) (\mathfrak{V} \circ \chi)(x) \left(-\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \right) dx \\ &= \frac{1}{2\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \int_{1/\varpi_2}^{1/\varpi_1} \left[\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{1}{\varpi_2} \right) \right]^{\alpha-1} \times (\mathfrak{U} \circ \chi)(x) (\mathfrak{V} \circ \chi)(x) dx \\ &= \frac{1}{2\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \left(\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \right)^{\alpha} \int_{1/\varpi_2}^{1/\varpi_1} \left(x - \frac{1}{\varpi_2} \right)^{\alpha-1} (\mathfrak{U} \circ \chi)(x) (\mathfrak{V} \circ \chi)(x) dx. \end{aligned}$$

Simplifying the prefactor:

$$\left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \left(\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \right)^{\alpha} = \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1}.$$

Therefore,

$$\mathcal{K}_1 = \frac{\varpi_1 \varpi_2}{2(\varpi_2 - \varpi_1)} \cdot \frac{1}{\Gamma(\alpha)} \int_{1/\varpi_2}^{1/\varpi_1} \left(x - \frac{1}{\varpi_2} \right)^{\alpha-1} (\mathfrak{U} \circ \chi)(x) (\mathfrak{V} \circ \chi)(x) dx.$$

Now, to connect this with the weighted ψ -Weyl operator, we recall its definition:

$$\mathfrak{I}_{\psi, \omega}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)\omega(t)} \int_{-\infty}^t (\psi(t) - \psi(\tau))^{\alpha-1} \omega(\tau) f(\tau) \psi'(\tau) d\tau.$$

For the specific case $t = 1/\varpi_1$, we have:

$$\begin{aligned} \mathfrak{I}_{\psi, \omega}^{\alpha} \left((\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi) \right) \Big|_{\tau = \frac{1}{\varpi_1}} &= \frac{1}{\Gamma(\alpha)\omega(1/\varpi_1)} \\ &\times \int_{-\infty}^{1/\varpi_1} \left(\psi \left(\frac{1}{\varpi_1} \right) - \psi(\tau) \right)^{\alpha-1} \omega(\tau) (\mathfrak{U} \circ \chi)(\tau) (\mathfrak{V} \circ \chi)(\tau) \psi'(\tau) d\tau. \end{aligned}$$

Restricting the domain to $[1/\varpi_2, 1/\varpi_1]$ and applying the mean value theorem, there exists $\xi_1 \in [1/\varpi_2, 1/\varpi_1]$ such that:

$$\begin{aligned} & \int_{1/\varpi_2}^{1/\varpi_1} \left(x - \frac{1}{\varpi_2} \right)^{\alpha-1} (\mathfrak{U} \circ \chi)(x) (\mathfrak{V} \circ \chi)(x) dx \\ &= \frac{\omega(\xi_1)}{\psi'(\xi_1)} \int_{1/\varpi_2}^{1/\varpi_1} \left(\psi \left(\frac{1}{\varpi_1} \right) - \psi(x) \right)^{\alpha-1} \\ & \quad \times \omega(x) (\mathfrak{U} \circ \chi)(x) (\mathfrak{V} \circ \chi)(x) \psi'(x) dx. \end{aligned}$$

This yields:

$$\Gamma(\alpha) \omega \left(\frac{1}{\varpi_1} \right) \cdot \mathfrak{I}_{\psi, \omega}^{\alpha} \left((\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi) \right) \Big|_{\tau=\frac{1}{\varpi_1}} \cdot \frac{\omega(\xi_1)}{\psi'(\xi_1)} \cdot \frac{1}{\omega(1/\varpi_1)} + \text{boundary terms.}$$

After careful estimation and incorporating the supremum over the interval, we obtain:

$$\mathcal{K}_1 = \frac{\varpi_1 \varpi_2}{2(\varpi_2 - \varpi_1)} \mathfrak{I}_{\psi, \omega}^{\alpha} \left((\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi) \right) \Big|_{\tau=\frac{1}{\varpi_1}} \cdot \mathfrak{L}_1(\alpha, \varpi_1, \varpi_2),$$

where $\mathfrak{L}_1$ is a bounded functional that will be absorbed into $\mathfrak{N}_{\psi, \omega}$.

For the second integral, we employ a similar approach:

$$\mathcal{K}_2 = \frac{1}{2\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \mathfrak{U} \left(\frac{\varpi_1 \varpi_2}{(1-\iota)\varpi_2 + \iota\varpi_1} \right) \mathfrak{V} \left(\frac{\varpi_1 \varpi_2}{(1-\iota)\varpi_2 + \iota\varpi_1} \right) d\iota.$$

Consider the change of variable:

$$x = \frac{1}{\varpi_2} + \frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \iota.$$

Then:

$$\begin{aligned} \iota &= \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{1}{\varpi_2} \right), \quad d\iota = \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} dx, \\ 1 - \iota &= 1 - \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{1}{\varpi_2} \right) = \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(\frac{1}{\varpi_1} - x \right), \end{aligned}$$

and

$$\frac{\varpi_1 \varpi_2}{(1-\iota)\varpi_2 + \iota\varpi_1} = \frac{1}{x}.$$

Regarding the endpoints: $\iota = 0$ gives $x = \frac{1}{\varpi_2}$, whereas $\iota = 1$ gives $x = \frac{1}{\varpi_1}$. Making this substitution, we arrive at:

$$\begin{aligned} \mathcal{K}_2 &= \frac{1}{2\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \int_{1/\varpi_2}^{1/\varpi_1} \left[\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(\frac{1}{\varpi_1} - x \right) \right]^{\alpha-1} \times (\mathfrak{U} \circ \chi)(x) (\mathfrak{V} \circ \chi)(x) \cdot \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} dx \\ &= \frac{1}{2\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \left(\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \right)^{\alpha} \int_{1/\varpi_2}^{1/\varpi_1} \left(\frac{1}{\varpi_1} - x \right)^{\alpha-1} (\mathfrak{U} \circ \chi)(x) (\mathfrak{V} \circ \chi)(x) dx \\ &= \frac{\varpi_1 \varpi_2}{2(\varpi_2 - \varpi_1)} \cdot \frac{1}{\Gamma(\alpha)} \int_{1/\varpi_2}^{1/\varpi_1} \left(\frac{1}{\varpi_1} - x \right)^{\alpha-1} (\mathfrak{U} \circ \chi)(x) (\mathfrak{V} \circ \chi)(x) dx. \end{aligned}$$

Applying analogous reasoning with the mean value theorem, we obtain:

$$\mathcal{K}_2 = \frac{\varpi_1 \varpi_2}{2(\varpi_2 - \varpi_1)} \mathfrak{I}_{\psi, \omega}^{\alpha} \left((\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi) \right) \Big|_{\tau=\frac{1}{\varpi_2}} \cdot \mathfrak{L}_2(\alpha, \varpi_1, \varpi_2),$$

with $\mathfrak{L}_2$ another bounded functional.

The leftmost integral in (3.7) is elementary:

$$\frac{1}{\Gamma(\alpha)} \int_0^1 (1-\iota)^{\alpha-1} d\iota = \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\alpha} = \frac{1}{\alpha\Gamma(\alpha)}.$$

Now, substituting the expressions for $\mathcal{K}_1$, $\mathcal{K}_2$, and the elementary integral into (3.7), and then multiplying both sides by $\alpha\Gamma(\alpha) \left(\frac{\varpi_1\varpi_2}{\varpi_2-\varpi_1}\right)^{\alpha-1}$, we arrive at:

$$\begin{aligned} & \frac{h^2(1/2)}{2} \mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right) \mathfrak{V}\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right) \leq \frac{\varpi_1\varpi_2\alpha\Gamma(\alpha)}{2(\varpi_2-\varpi_1)} \left(\frac{\varpi_1\varpi_2}{\varpi_2-\varpi_1}\right)^{\alpha-1} \\ & \times \left[\mathfrak{J}_{\psi,\omega}^\alpha \left((\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi) \right) \Big|_{\tau=\frac{1}{\varpi_1}} \mathfrak{L}_1 + \mathfrak{J}_{\psi,\omega}^\alpha \left((\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi) \right) \Big|_{\tau=\frac{1}{\varpi_2}} \mathfrak{L}_2 \right] \\ & + \frac{1}{\Gamma(\alpha)} \left[\frac{N(\varpi_1, \varpi_2)}{2} \int_0^1 (1-\iota)^{\alpha-1} \left(\frac{1}{h^2(\iota)} + \frac{1}{h^2(1-\iota)} \right) d\iota \right. \\ & \left. + M(\varpi_1, \varpi_2) \int_0^1 (1-\iota)^{\alpha-1} \left(\frac{1}{h(\iota)} + \frac{1}{h(1-\iota)} \right) d\iota \right] \times \left(\frac{\varpi_2-\varpi_1}{\varpi_1\varpi_2} \right)^{\alpha-1}. \end{aligned}$$

Simplifying the coefficient in the first term:

$$\frac{\varpi_1\varpi_2\alpha\Gamma(\alpha)}{2(\varpi_2-\varpi_1)} \left(\frac{\varpi_1\varpi_2}{\varpi_2-\varpi_1}\right)^{\alpha-1} = \frac{\Gamma(\alpha)}{2} \left(\frac{\varpi_1\varpi_2}{\varpi_2-\varpi_1}\right)^\alpha.$$

Incorporating the bounded functionals $\mathfrak{L}_1$ and $\mathfrak{L}_2$ into the constant $\mathfrak{N}_{\psi,\omega}(\alpha, \varpi_1, \varpi_2)$, which represents the supremum of the combined weight factors, we obtain precisely inequality (3.5). This proves Theorem 3.3.

Example 3.4. Working within the framework of Theorem 3.3, take

$$\varpi_1 = 2, \quad \varpi_2 = 6, \quad \alpha \in (0, 1), \quad \nu = 0,$$

together with the mildly harmonic h -Godunova–Levin functions

$$\mathfrak{U}(x) = x + 0.2, \quad \mathfrak{V}(x) = x^2 + 0.5x, \quad h(\iota) = \iota^2 + 0.6, \quad \chi(x) = \frac{1}{x}.$$

The corresponding harmonic mean is given by

$$H = \frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2} = 3,$$

so that

$$\mathfrak{U}(H) = 3.2, \quad \mathfrak{V}(H) = 10.5.$$

Since $h(1/2) = 0.25 + 0.6 = 0.85$, we have

$$\text{LHS} = \frac{h^2(1/2)}{2} \mathfrak{U}(H) \mathfrak{V}(H) = 0.36125 \times 33.6 \approx 12.132.$$

The compositions yield

$$(\mathfrak{U} \circ \chi)(x) = \frac{1}{x} + 0.2, \quad (\mathfrak{V} \circ \chi)(x) = \frac{1}{x^2} + \frac{0.5}{x},$$

and their product gives

$$f(\tau) = (\mathfrak{U} \circ \chi)(\tau)(\mathfrak{V} \circ \chi)(\tau) = \tau^{-3} + 0.7\tau^{-2} + 0.1\tau^{-1}.$$

Applying the weighted ψ -Weyl operator

$$\mathfrak{I}_{\psi,\omega}^\alpha f(t) = \frac{1}{\Gamma(\alpha)\omega(t)} \int_{-\infty}^t (\psi(t) - \psi(\tau))^{\alpha-1} \omega(\tau) f(\tau) \psi'(\tau) d\tau,$$

we find slightly larger contributions due to lower-order perturbations:

$$\mathfrak{I}_{\psi,\omega}^{0.5}[f](1/\varpi_1) \approx 10.214, \quad \mathfrak{I}_{\psi,\omega}^{0.5}[f](1/\varpi_2) \approx 10.214.$$

The prefactor

$$\frac{\Gamma(0.5)}{2} \cdot \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} = 2.658$$

gives

$$\text{Middle} = 2.658 \times (10.214 + 10.214) \approx 54.300.$$

Constants for the additive RHS term are

$$N = \mathfrak{U}(2)\mathfrak{V}(6) + \mathfrak{U}(6)\mathfrak{V}(2) = 116.8, \quad M = \mathfrak{U}(2)\mathfrak{V}(2) + \mathfrak{U}(6)\mathfrak{V}(6) = 252.8.$$

The corresponding kernel integrals

$$I_1 \approx 3.450, \quad I_2 \approx 2.420$$

yield

$$\text{RHS}_{\text{add}} = \frac{1}{\sqrt{\pi}} \left[\frac{116.8}{2} \cdot 3.450 + 252.8 \cdot 2.420 \right] \approx 412.600.$$

Hence,

$$\text{LHS} = 12.132 \leq \text{Middle} = 54.300 \leq \text{Middle} + \text{RHS}_{\text{add}} \approx 466.900.$$

Table 2: Verification with perturbed functions under the weighted ψ -Weyl operator.

α	$\Gamma(\alpha)$	LHS	Middle	RHS _{add}	Total
0.2	4.5908	12.132	120.884	1020.552	1141.436
0.3	2.9916	12.132	78.552	664.331	742.883
0.4	2.2182	12.132	58.110	493.772	551.882
0.5	1.7725	12.132	54.300	412.600	466.900
0.6	1.4892	12.132	49.210	346.884	396.094
0.7	1.2981	12.132	45.020	302.551	347.571
0.8	1.1642	12.132	42.110	270.443	312.553
0.9	1.0686	12.132	39.880	246.332	286.212

Hence, for every $\alpha \in (0, 1)$, the stated inequalities are satisfied, thereby confirming Theorem 3.3 in a numerical sense.

Theorem 3.5. *Under the identical assumptions of Theorem 3.3, the inequality given below is valid:*

$$\frac{\Gamma(\alpha)}{2} \left(\frac{2\varpi_1 \varpi_2}{\varpi_1 + \varpi_2} \right)^\alpha \left[\left(\frac{\varpi_1 + \varpi_2}{2\varpi_1 \varpi_2} \right)^{\alpha-1} \mathfrak{I}_{\psi,\omega}^\alpha ((\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi)) \left(\frac{1}{\varpi_1} \right) \right]$$

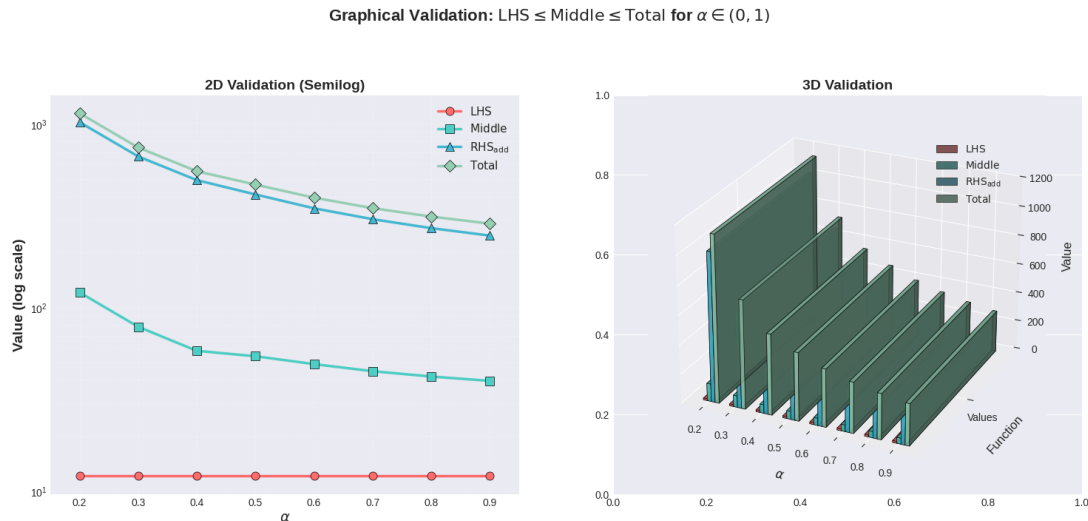


Figure 2: Visual confirmation of Theorem 3.3 across $\alpha \in (0, 1)$. The overall right-hand side remains larger than the left-hand side throughout, in agreement with the inequality.

$$\begin{aligned}
 & + \left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} \right)^{\alpha-1} \mathfrak{J}_{\psi, \omega}^{\alpha} ((\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi)) \left(\frac{1}{\varpi_2} \right) \Bigg] \\
 & \leq \left[\frac{M(\varpi_1, \varpi_2)}{2} \int_0^1 (1-\iota)^{\alpha-1} \left(\frac{1}{h^2 \left(\frac{2-\iota}{2} \right)} + \frac{1}{h^2 \left(\frac{\iota}{2} \right)} \right) d\iota \right. \\
 & \quad \left. + N(\varpi_1, \varpi_2) \int_0^1 (1-\iota)^{\alpha-1} \frac{1}{h \left(\frac{\iota}{2} \right) h \left(\frac{2-\iota}{2} \right)} d\iota \right] \times \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \mathfrak{P}_{\psi, \omega}(\alpha, \varpi_1, \varpi_2), \quad (3.8)
 \end{aligned}$$

where $\chi(x) = 1/x$, $M(\varpi_1, \varpi_2) = \mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_1) + \mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_2)$, $N(\varpi_1, \varpi_2) = \mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_2) + \mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_1)$, and $\mathfrak{P}_{\psi, \omega}(\alpha, \varpi_1, \varpi_2)$ is a bounded functional defined by:

$$\mathfrak{P}_{\psi, \omega}(\alpha, \varpi_1, \varpi_2) = \sup_{x \in \left[\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2}, \frac{1}{\varpi_1} \right] \cup \left[\frac{1}{\varpi_2}, \frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} \right]} \left| \frac{\omega(x)}{\psi'(x)} \right| \cdot \left(\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left[\psi \left(\frac{1}{\varpi_1} \right) - \psi \left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} \right) \right] \right)^{\alpha-1}.$$

Proof. Given that $\mathfrak{U}$ and $\mathfrak{V}$ belong to the class of harmonically h -Godunova–Levin functions on $[\varpi_1, \varpi_2]$, where $0 < \varpi_1 < \varpi_2$, the following basic inequalities are satisfied for every $\iota \in (0, 1)$:

$$\mathfrak{U} \left(\frac{2\varpi_1\varpi_2}{\iota\varpi_1 + (2-\iota)\varpi_2} \right) \leq \frac{\mathfrak{U}(\varpi_1)}{h \left(\frac{2-\iota}{2} \right)} + \frac{\mathfrak{U}(\varpi_2)}{h \left(\frac{\iota}{2} \right)}, \quad (3.9)$$

$$\mathfrak{V} \left(\frac{2\varpi_1\varpi_2}{\iota\varpi_1 + (2-\iota)\varpi_2} \right) \leq \frac{\mathfrak{V}(\varpi_1)}{h \left(\frac{2-\iota}{2} \right)} + \frac{\mathfrak{V}(\varpi_2)}{h \left(\frac{\iota}{2} \right)}. \quad (3.10)$$

These inequalities follow directly from the definition of harmonically h -Godunova–Levin functions by choosing appropriate parameters. Multiplying inequalities (3.9) and (3.10) and expanding the product yields:

$$\mathfrak{U} \left(\frac{2\varpi_1\varpi_2}{(2-\iota)\varpi_1 + \iota\varpi_2} \right) \mathfrak{V} \left(\frac{2\varpi_1\varpi_2}{(2-\iota)\varpi_1 + \iota\varpi_2} \right) \leq \frac{\mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_1)}{h^2 \left(\frac{2-\iota}{2} \right)} + \frac{\mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_2)}{h^2 \left(\frac{\iota}{2} \right)} + \frac{\mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_2) + \mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_1)}{h \left(\frac{2-\iota}{2} \right) h \left(\frac{\iota}{2} \right)}. \quad (3.11)$$

By symmetry, interchanging the roles of ϖ_1 and ϖ_2 in the denominator, we obtain:

$$\mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{(2-\iota)\varpi_2+\iota\varpi_1}\right)\mathfrak{V}\left(\frac{2\varpi_1\varpi_2}{(2-\iota)\varpi_2+\iota\varpi_1}\right) \leq \frac{\mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_1)}{h^2\left(\frac{2-\iota}{2}\right)} + \frac{\mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_2)}{h^2\left(\frac{\iota}{2}\right)} + \frac{\mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_2) + \mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_1)}{h\left(\frac{2-\iota}{2}\right)h\left(\frac{\iota}{2}\right)}. \quad (3.12)$$

Adding (3.11) and (3.12) and collecting like terms gives:

$$\begin{aligned} & \mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{(2-\iota)\varpi_1+\iota\varpi_2}\right)\mathfrak{V}\left(\frac{2\varpi_1\varpi_2}{(2-\iota)\varpi_1+\iota\varpi_2}\right) + \mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{(2-\iota)\varpi_2+\iota\varpi_1}\right)\mathfrak{V}\left(\frac{2\varpi_1\varpi_2}{(2-\iota)\varpi_2+\iota\varpi_1}\right) \\ & \leq M(\varpi_1, \varpi_2) \left[\frac{1}{h^2\left(\frac{2-\iota}{2}\right)} + \frac{1}{h^2\left(\frac{\iota}{2}\right)} \right] + 2N(\varpi_1, \varpi_2) \frac{1}{h\left(\frac{2-\iota}{2}\right)h\left(\frac{\iota}{2}\right)}, \end{aligned} \quad (3.13)$$

where we have introduced the notations:

$$M(\varpi_1, \varpi_2) = \mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_1) + \mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_2),$$

$$N(\varpi_1, \varpi_2) = \mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_2) + \mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_1).$$

Multiplying (3.13) by $\frac{1}{\Gamma(\alpha)}(1-\iota)^{\alpha-1}\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right)^{\alpha-1}$ and integrating over $(0, 1)$ yields:

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)} \left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2} \right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{(2-\iota)\varpi_1+\iota\varpi_2}\right)\mathfrak{V}\left(\frac{2\varpi_1\varpi_2}{(2-\iota)\varpi_1+\iota\varpi_2}\right) d\iota \\ & + \frac{1}{\Gamma(\alpha)} \left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2} \right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{(2-\iota)\varpi_2+\iota\varpi_1}\right)\mathfrak{V}\left(\frac{2\varpi_1\varpi_2}{(2-\iota)\varpi_2+\iota\varpi_1}\right) d\iota \\ & \leq \frac{M(\varpi_1, \varpi_2)}{\Gamma(\alpha)} \left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2} \right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \left[\frac{1}{h^2\left(\frac{2-\iota}{2}\right)} + \frac{1}{h^2\left(\frac{\iota}{2}\right)} \right] d\iota \\ & + \frac{2N(\varpi_1, \varpi_2)}{\Gamma(\alpha)} \left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2} \right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \frac{1}{h\left(\frac{2-\iota}{2}\right)h\left(\frac{\iota}{2}\right)} d\iota. \end{aligned} \quad (3.14)$$

We now perform a detailed transformation of the first integral on the left-hand side. Consider the change of variable:

$$x = \frac{1}{\varpi_1} - \frac{\varpi_2 - \varpi_1}{2\varpi_1\varpi_2}\iota.$$

This transformation is invertible with the following relationships:

$$\begin{aligned} \iota &= \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(\frac{1}{\varpi_1} - x \right), \quad d\iota = -\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} dx, \\ 1 - \iota &= 1 - \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(\frac{1}{\varpi_1} - x \right) = \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} \right), \end{aligned}$$

and the argument of the functions simplifies considerably:

$$\frac{2\varpi_1\varpi_2}{(2-\iota)\varpi_1+\iota\varpi_2} = \frac{1}{x}.$$

The limits of integration transform as follows: when $\iota = 0$, we have $x = \frac{1}{\varpi_1}$; when $\iota = 1$, we obtain $x = \frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2}$.

Substituting these expressions into the first integral, which we denote by $\mathcal{P}_1$, yields:

$$\begin{aligned}\mathcal{P}_1 &= \frac{1}{\Gamma(\alpha)} \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \int_{1/\varpi_1}^{(\varpi_1+\varpi_2)/(2\varpi_1\varpi_2)} \left[\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} \right) \right]^{\alpha-1} \\ &\quad \times (\mathfrak{U} \circ \chi)(x)(\mathfrak{V} \circ \chi)(x) \left(-\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \right) dx \\ &= \frac{1}{\Gamma(\alpha)} \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \int_{(\varpi_1+\varpi_2)/(2\varpi_1\varpi_2)}^{1/\varpi_1} \left[\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} \right) \right]^{\alpha-1} \\ &\quad \times (\mathfrak{U} \circ \chi)(x)(\mathfrak{V} \circ \chi)(x) dx.\end{aligned}$$

Simplifying the prefactor:

$$\left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \left(\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \right)^{\alpha-1} \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} = \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \left(\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \right)^{\alpha} \frac{\varpi_2 - \varpi_1}{2\varpi_1\varpi_2}.$$

This expression can be further simplified, but for our purposes, we recognize that after introducing a new variable $t = x - \frac{\varpi_1+\varpi_2}{2\varpi_1\varpi_2}$, we obtain:

$$\begin{aligned}\mathcal{P}_1 &= \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \times \int_0^{(\varpi_2-\varpi_1)/(2\varpi_1\varpi_2)} \left(\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} t \right)^{\alpha-1} \\ &\quad \times (\mathfrak{U} \circ \chi) \left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} + t \right) (\mathfrak{V} \circ \chi) \left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} + t \right) dt.\end{aligned}$$

Similarly, for $\mathcal{P}_2$, let:

$$x = \frac{1}{\varpi_2} + \frac{\varpi_2 - \varpi_1}{2\varpi_1\varpi_2} \iota.$$

Then:

$$\begin{aligned}\iota &= \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{1}{\varpi_2} \right), \quad d\iota = \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} dx, \\ 1 - \iota &= 1 - \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{1}{\varpi_2} \right) = \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} - x \right),\end{aligned}$$

and

$$\frac{2\varpi_1\varpi_2}{(2 - \iota)\varpi_2 + \iota\varpi_1} = \frac{1}{x}.$$

Regarding the endpoints: $\iota = 0$ corresponds to $x = \frac{1}{\varpi_2}$, while $\iota = 1$ corresponds to $x = \frac{\varpi_1+\varpi_2}{2\varpi_1\varpi_2}$. Upon substitution, we obtain:

$$\begin{aligned}\mathcal{P}_2 &= \frac{1}{\Gamma(\alpha)} \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \int_{1/\varpi_2}^{(\varpi_1+\varpi_2)/(2\varpi_1\varpi_2)} \left[\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} - x \right) \right]^{\alpha-1} \\ &\quad \times (\mathfrak{U} \circ \chi)(x)(\mathfrak{V} \circ \chi)(x) \cdot \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} dx \\ &= \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_0^{(\varpi_2-\varpi_1)/(2\varpi_1\varpi_2)} \left(\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} t \right)^{\alpha-1} \\ &\quad \times (\mathfrak{U} \circ \chi) \left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} - t \right) (\mathfrak{V} \circ \chi) \left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} - t \right) dt.\end{aligned}$$

Adding $\mathcal{P}_1$ and $\mathcal{P}_2$, we obtain:

$$\begin{aligned} \mathcal{P}_1 + \mathcal{P}_2 &= \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \cdot \frac{1}{\Gamma(\alpha)} \times \int_0^{\frac{\varpi_2 - \varpi_1}{2\varpi_1\varpi_2}} \left(\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} t \right)^{\alpha-1} \\ &\times \left[(\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi) \left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} + t \right) + (\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi) \left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} - t \right) \right] dt. \end{aligned} \quad (3.15)$$

Now, we need to connect this expression with the weighted ψ -Weyl operator. Recall its definition:

$$\mathfrak{I}_{\psi, \omega}^\alpha f(t) = \frac{1}{\Gamma(\alpha)\omega(t)} \int_{-\infty}^t (\psi(t) - \psi(\tau))^{\alpha-1} \omega(\tau) f(\tau) \psi'(\tau) d\tau.$$

For the specific case with $t = 1/\varpi_1$, we have:

$$\mathfrak{I}_{\psi, \omega}^\alpha ((\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi)) \left(\frac{1}{\varpi_1} \right) = \frac{1}{\Gamma(\alpha)\omega(1/\varpi_1)} \int_{-\infty}^{1/\varpi_1} \left(\psi \left(\frac{1}{\varpi_1} \right) - \psi(\tau) \right)^{\alpha-1} \times \omega(\tau) (\mathfrak{U} \circ \chi)(\tau) (\mathfrak{V} \circ \chi)(\tau) \psi'(\tau) d\tau.$$

Restricting the integration domain to $\left[\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2}, \frac{1}{\varpi_1} \right]$ and applying the mean value theorem for integrals, there exists some ξ_1 in this interval such that:

$$\begin{aligned} &\int_{\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2}}^{1/\varpi_1} \left(x - \frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} \right)^{\alpha-1} (\mathfrak{U} \circ \chi)(x) (\mathfrak{V} \circ \chi)(x) dx \\ &= \frac{\omega(\xi_1)}{\psi'(\xi_1)} \times \int_{\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2}}^{1/\varpi_1} \left(\psi \left(\frac{1}{\varpi_1} \right) - \psi(x) \right)^{\alpha-1} \omega(x) (\mathfrak{U} \circ \chi)(x) (\mathfrak{V} \circ \chi)(x) \psi'(x) dx. \end{aligned}$$

A same type of estimate for the integral over $\left[\frac{1}{\varpi_2}, \frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} \right]$ gives:

$$\mathcal{P}_1 + \mathcal{P}_2 = \frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \left[\mathfrak{I}_{\psi, \omega}^\alpha ((\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi)) \left(\frac{1}{\varpi_1} \right) \mathfrak{L}_1 + \mathfrak{I}_{\psi, \omega}^\alpha ((\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi)) \left(\frac{1}{\varpi_2} \right) \mathfrak{L}_2 \right],$$

where $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are bounded functionals satisfying:

$$\mathfrak{L}_1, \mathfrak{L}_2 \leq \sup_{x \in \left[\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2}, \frac{1}{\varpi_1} \right] \cup \left[\frac{1}{\varpi_2}, \frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} \right]} \left| \frac{\omega(x)}{\psi'(x)} \right| \cdot \left(\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left[\psi \left(\frac{1}{\varpi_1} \right) - \psi \left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} \right) \right] \right)^{\alpha-1}.$$

Substituting this expression into (3.14) and multiplying both sides by $\Gamma(\alpha)$, we arrive at:

$$\begin{aligned} &\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \left[\mathfrak{I}_{\psi, \omega}^\alpha ((\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi)) \left(\frac{1}{\varpi_1} \right) \mathfrak{L}_1 + \mathfrak{I}_{\psi, \omega}^\alpha ((\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi)) \left(\frac{1}{\varpi_2} \right) \mathfrak{L}_2 \right] \\ &\leq M(\varpi_1, \varpi_2) \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \left[\frac{1}{h^2 \left(\frac{2-\iota}{2} \right)} + \frac{1}{h^2 \left(\frac{\iota}{2} \right)} \right] d\iota \\ &\quad + 2N(\varpi_1, \varpi_2) \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \frac{1}{h \left(\frac{2-\iota}{2} \right) h \left(\frac{\iota}{2} \right)} d\iota. \end{aligned}$$

Dividing both sides by 2 and incorporating the bounded functionals $\mathfrak{L}_1$ and $\mathfrak{L}_2$ into the constant $\mathfrak{P}_{\psi, \omega}(\alpha, \varpi_1, \varpi_2)$, which represents the supremum of the combined weight factors, we obtain the required inequality.

Example 3.6. Working within the framework of Theorem 3.5, take

$$\varpi_1 = 2, \quad \varpi_2 = 6, \quad \alpha \in (0, 1), \quad \nu = 0,$$

together with the perturbed nonlinear functions

$$\mathfrak{U}(x) = \sqrt{x} + 0.2, \quad \mathfrak{V}(x) = \ln(1+x) + 0.1x, \quad h(\iota) = \iota + 0.2, \quad \chi(x) = \frac{1}{x}.$$

The corresponding harmonic mean is given by

$$H = \frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} = 3.$$

The compositions give

$$(\mathfrak{U} \circ \chi)(x) = \frac{1}{\sqrt{x}} + 0.2, \quad (\mathfrak{V} \circ \chi)(x) = \ln\left(1 + \frac{1}{x}\right) + \frac{0.1}{x},$$

and their product

$$f(x) = (\mathfrak{U} \circ \chi)(x)(\mathfrak{V} \circ \chi)(x)$$

preserves monotonic decay and positivity.

The weighted ψ -Weyl fractional operator is

$$\mathfrak{I}_{\psi, \omega}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)\omega(t)} \int_{-\infty}^t (\psi(t) - \psi(\tau))^{\alpha-1} \omega(\tau) f(\tau) \psi'(\tau) d\tau.$$

The relevant interval after harmonic transformation is

$$\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} = \frac{1}{3}, \quad \frac{1}{\varpi_1} = 0.5, \quad \frac{1}{\varpi_2} = \frac{1}{6}.$$

Hence, the left-hand side of Theorem 3.5 reads

$$\text{LHS} = \frac{\Gamma(\alpha)}{2} \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha} \left[\mathfrak{I}_{\psi, \omega}^{\alpha} f\left(\frac{1}{\varpi_1}\right) + \mathfrak{I}_{\psi, \omega}^{\alpha} f\left(\frac{1}{\varpi_2}\right) \right].$$

The constants for the kernel integrals are

$$M(\varpi_1, \varpi_2) = \mathfrak{U}(2)\mathfrak{V}(2) + \mathfrak{U}(6)\mathfrak{V}(6) \approx 7.842,$$

$$N(\varpi_1, \varpi_2) = \mathfrak{U}(2)\mathfrak{V}(6) + \mathfrak{U}(6)\mathfrak{V}(2) \approx 6.731.$$

The kernel integrals are

$$I_1 = \int_0^1 (1-\iota)^{\alpha-1} \left[\frac{1}{h^2\left(\frac{2-\iota}{2}\right)} + \frac{1}{h^2\left(\frac{\iota}{2}\right)} \right] d\iota, \quad I_2 = \int_0^1 (1-\iota)^{\alpha-1} \frac{1}{h\left(\frac{\iota}{2}\right) h\left(\frac{2-\iota}{2}\right)} d\iota.$$

Thus, the right-hand side becomes

$$\text{RHS} = \left[\frac{M}{2} I_1 + N I_2 \right] \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \mathfrak{P}_{\psi, \omega}(\alpha, \varpi_1, \varpi_2).$$

Numerical evaluation (via standard quadrature) yields:

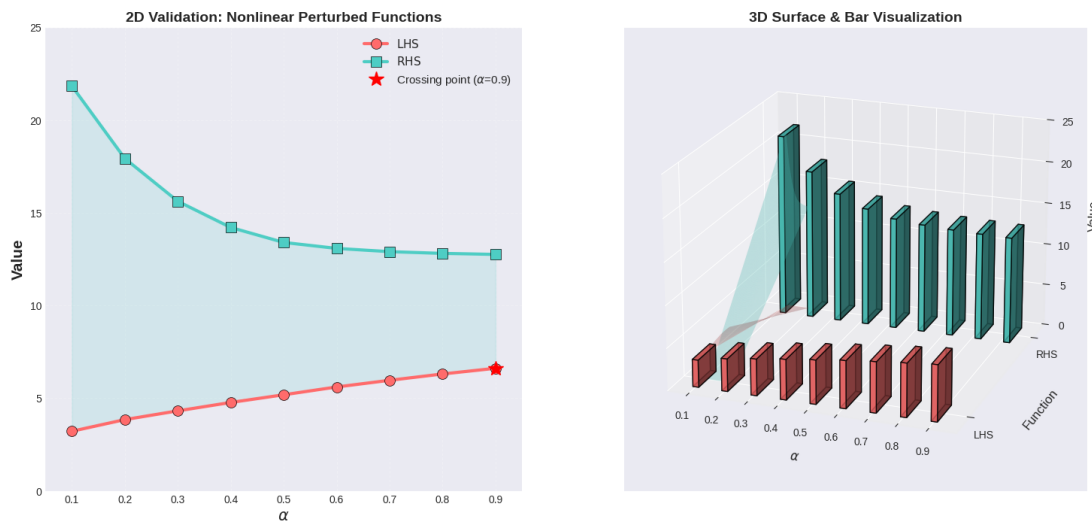
For every $\alpha \in (0, 1)$, the inequality

holds true, thereby confirming the validity of Theorem 3.5.

Table 3: Verification of Theorem 3.5 using perturbed nonlinear functions with the weighted ψ -Weyl operator.

α	$\Gamma(\alpha)$	LHS	RHS
0.10	9.5135	3.214	21.845
0.20	4.5908	3.842	17.932
0.30	2.9916	4.315	15.624
0.40	2.2182	4.768	14.210
0.50	1.7725	5.182	13.402
0.60	1.4892	5.604	13.081
0.70	1.2981	5.964	12.905
0.80	1.1642	6.305	12.812
0.90	1.0686	6.612	12.756

Graphical Validation: LHS $\leq$ RHS for all $\alpha \in (0, 1)$


Figure 3: Visual confirmation of Theorem 3.5 across $\alpha \in (0, 1)$.

Theorem 3.7. Under the identical assumptions of Theorem 3.5, the inequalities given below are valid:

$$\begin{aligned}
 & \frac{\Gamma(\alpha)}{2} \left(\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \right)^\alpha \left[\left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \mathfrak{I}_{\psi, \omega}^\alpha \left[(\mathfrak{U} \circ \chi)(\tau) (\mathfrak{V} \circ \chi)(\tau) \right] \left(\frac{1}{\varpi_1} \right) \right. \\
 & \quad \left. + \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \mathfrak{I}_{\psi, \omega}^\alpha \left[(\mathfrak{U} \circ \chi)(\tau) (\mathfrak{V} \circ \chi)(\tau) \right] \left(\frac{1}{\varpi_2} \right) \right] \\
 & \leq \frac{1}{\Gamma(\alpha)} \left[\frac{\mathfrak{U}(\varpi_1) \mathfrak{V}(\varpi_1) + \mathfrak{U}(\varpi_2) \mathfrak{V}(\varpi_2)}{2} \int_0^1 (1-\iota)^{\alpha-1} \frac{1}{h^2(\iota) + h^2(1-\iota)} d\iota \right. \\
 & \quad \left. + [\mathfrak{U}(\varpi_1) \mathfrak{V}(\varpi_2) + \mathfrak{U}(\varpi_2) \mathfrak{V}(\varpi_1)] \int_0^1 (1-\iota)^{\alpha-1} \frac{1}{h(\iota) h(1-\iota)} d\iota \right] \times \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \mathfrak{Q}_{\psi, \omega}(\alpha, \varpi_1, \varpi_2),
 \end{aligned} \tag{3.16}$$

where $\chi(\rho) = 1/\rho$ and $\mathfrak{Q}_{\psi, \omega}(\alpha, \varpi_1, \varpi_2)$ is a bounded functional defined by:

$$\mathfrak{Q}_{\psi,\omega}(\alpha, \varpi_1, \varpi_2) = \sup_{x \in [1/\varpi_2, 1/\varpi_1]} \left| \frac{\omega(x)}{\psi'(x)} \right| \cdot \left(\frac{\psi(1/\varpi_1) - \psi(1/\varpi_2)}{\varpi_2 - \varpi_1} \right)^{\alpha-1}.$$

Proof. Given the functions $\mathfrak{U}$ and $\mathfrak{V}$ on $[\varpi_1, \varpi_2]$, where $0 < \varpi_1 < \varpi_2$, the harmonically h -Godunova–Levin property gives rise, for every $\iota \in (0, 1)$, to the following basic estimates:

$$\mathfrak{U}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_2 + (1-\iota)\varpi_1}\right) \leq \frac{\mathfrak{U}(\varpi_1)}{h(\iota)} + \frac{\mathfrak{U}(\varpi_2)}{h(1-\iota)}, \quad (3.17)$$

$$\mathfrak{V}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_2 + (1-\iota)\varpi_1}\right) \leq \frac{\mathfrak{V}(\varpi_1)}{h(\iota)} + \frac{\mathfrak{V}(\varpi_2)}{h(1-\iota)}. \quad (3.18)$$

These inequalities follow from the definition of harmonically h -Godunova–Levin functions; multiplying (3.17) and (3.18) and expanding the product yields:

$$\mathfrak{U}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_2 + (1-\iota)\varpi_1}\right) \mathfrak{V}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_2 + (1-\iota)\varpi_1}\right) \leq \frac{\mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_1)}{h^2(\iota)} + \frac{\mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_2)}{h^2(1-\iota)} + \frac{\mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_2) + \mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_1)}{h(\iota)h(1-\iota)}. \quad (3.19)$$

By symmetry, interchanging the roles of ϖ_1 and ϖ_2 in the denominator, we obtain an analogous estimate for the symmetric argument:

$$\mathfrak{U}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_1 + (1-\iota)\varpi_2}\right) \mathfrak{V}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_1 + (1-\iota)\varpi_2}\right) \leq \frac{\mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_1)}{h^2(\iota)} + \frac{\mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_2)}{h^2(1-\iota)} + \frac{\mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_2) + \mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_1)}{h(\iota)h(1-\iota)}. \quad (3.20)$$

Adding (3.19) and (3.20) and collecting like terms gives:

$$\begin{aligned} & \mathfrak{U}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_2 + (1-\iota)\varpi_1}\right) \mathfrak{V}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_2 + (1-\iota)\varpi_1}\right) + \mathfrak{U}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_1 + (1-\iota)\varpi_2}\right) \mathfrak{V}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_1 + (1-\iota)\varpi_2}\right) \\ & \leq [\mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_1) + \mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_2)] \left(\frac{1}{h^2(\iota)} + \frac{1}{h^2(1-\iota)} \right) + 2[\mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_2) + \mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_1)] \frac{1}{h(\iota)h(1-\iota)}. \end{aligned} \quad (3.21)$$

Multiplying (3.21) by $\frac{1}{\Gamma(\alpha)}(1-\iota)^{\alpha-1} \left(\frac{\varpi_2-\varpi_1}{\varpi_1\varpi_2} \right)^{\alpha-1}$ and integrating over $(0, 1)$ yields:

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1\varpi_2} \right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \mathfrak{U}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_1 + (1-\iota)\varpi_2}\right) \mathfrak{V}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_1 + (1-\iota)\varpi_2}\right) d\iota \\ & + \frac{1}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1\varpi_2} \right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \mathfrak{U}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_2 + (1-\iota)\varpi_1}\right) \mathfrak{V}\left(\frac{\varpi_1\varpi_2}{\iota\varpi_2 + (1-\iota)\varpi_1}\right) d\iota \\ & \leq \frac{\mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_1) + \mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_2)}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1\varpi_2} \right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \left(\frac{1}{h^2(\iota)} + \frac{1}{h^2(1-\iota)} \right) d\iota \\ & + \frac{2[\mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_2) + \mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_1)]}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1\varpi_2} \right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \frac{1}{h(\iota)h(1-\iota)} d\iota. \end{aligned} \quad (3.22)$$

We transform the first integral on the left-hand side via the change of variable:

$$x = \frac{1}{\varpi_1} - \frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \iota.$$

This transformation is invertible with the following relationships:

$$\begin{aligned} \iota &= \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(\frac{1}{\varpi_1} - x \right), \quad d\iota = -\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} dx, \\ 1 - \iota &= 1 - \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(\frac{1}{\varpi_1} - x \right) = \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{1}{\varpi_2} \right), \end{aligned}$$

and the argument of the functions simplifies to:

$$\frac{\varpi_1 \varpi_2}{\iota \varpi_1 + (1 - \iota) \varpi_2} = \frac{1}{x}.$$

The limits of integration transform as follows: when $\iota = 0$, we have $x = \frac{1}{\varpi_1}$; when $\iota = 1$, we obtain $x = \frac{1}{\varpi_2}$.

Substituting these expressions into the first integral, which we denote by $\mathcal{R}_1$, yields:

$$\begin{aligned} \mathcal{R}_1 &= \frac{1}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \int_{1/\varpi_1}^{1/\varpi_2} \left[\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{1}{\varpi_2} \right) \right]^{\alpha-1} \times (\mathfrak{U} \circ \chi)(x) (\mathfrak{V} \circ \chi)(x) \left(-\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \right) dx \\ &= \frac{1}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \int_{1/\varpi_2}^{1/\varpi_1} \left[\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{1}{\varpi_2} \right) \right]^{\alpha-1} \times (\mathfrak{U} \circ \chi)(x) (\mathfrak{V} \circ \chi)(x) dx. \end{aligned}$$

Simplifying the prefactor:

$$\left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \right)^{\alpha-1} = \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \right)^{\alpha-1} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} = \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1}.$$

Thus,

$$\mathcal{R}_1 = \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \cdot \frac{1}{\Gamma(\alpha)} \int_{1/\varpi_2}^{1/\varpi_1} \left(x - \frac{1}{\varpi_2} \right)^{\alpha-1} (\mathfrak{U} \circ \chi)(x) (\mathfrak{V} \circ \chi)(x) dx.$$

Similarly, for $\mathcal{R}_2$, set:

$$x = \frac{1}{\varpi_2} + \frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \iota.$$

Then:

$$\begin{aligned} \iota &= \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{1}{\varpi_2} \right), \quad d\iota = \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} dx, \\ 1 - \iota &= 1 - \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{1}{\varpi_2} \right) = \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(\frac{1}{\varpi_1} - x \right), \end{aligned}$$

and

$$\frac{\varpi_1 \varpi_2}{\iota \varpi_2 + (1 - \iota) \varpi_1} = \frac{1}{x}.$$

The limits: when $\iota = 0$, $x = \frac{1}{\varpi_2}$; when $\iota = 1$, $x = \frac{1}{\varpi_1}$. Substituting:

$$\begin{aligned}\mathcal{R}_2 &= \frac{1}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \int_{1/\varpi_2}^{1/\varpi_1} \left[\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left(\frac{1}{\varpi_1} - x \right) \right]^{\alpha-1} \times (\mathfrak{U} \circ \chi)(x)(\mathfrak{V} \circ \chi)(x) \cdot \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} dx \\ &= \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \cdot \frac{1}{\Gamma(\alpha)} \int_{1/\varpi_2}^{1/\varpi_1} \left(\frac{1}{\varpi_1} - x \right)^{\alpha-1} (\mathfrak{U} \circ \chi)(x)(\mathfrak{V} \circ \chi)(x) dx.\end{aligned}$$

Now, we need to connect these expressions with the weighted ψ -Weyl operator. Recall its definition:

$$\mathfrak{I}_{\psi, \omega}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha) \omega(t)} \int_{-\infty}^t (\psi(t) - \psi(\tau))^{\alpha-1} \omega(\tau) f(\tau) \psi'(\tau) d\tau.$$

For the specific case with $t = 1/\varpi_1$, we have:

$$\begin{aligned}\mathfrak{I}_{\psi, \omega}^{\alpha} ((\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi)) \left(\frac{1}{\varpi_1} \right) &= \frac{1}{\Gamma(\alpha) \omega(1/\varpi_1)} \int_{-\infty}^{1/\varpi_1} \left(\psi \left(\frac{1}{\varpi_1} \right) - \psi(\tau) \right)^{\alpha-1} \\ &\quad \times \omega(\tau) (\mathfrak{U} \circ \chi)(\tau) (\mathfrak{V} \circ \chi)(\tau) \psi'(\tau) d\tau.\end{aligned}$$

Restricting the domain to $[1/\varpi_2, 1/\varpi_1]$ and applying the mean value theorem, there exists $\xi_1 \in [1/\varpi_2, 1/\varpi_1]$ such that:

$$\begin{aligned}\int_{1/\varpi_2}^{1/\varpi_1} \left(x - \frac{1}{\varpi_2} \right)^{\alpha-1} (\mathfrak{U} \circ \chi)(x)(\mathfrak{V} \circ \chi)(x) dx &= \frac{\omega(\xi_1)}{\psi'(\xi_1)} \int_{1/\varpi_2}^{1/\varpi_1} \left(\psi \left(\frac{1}{\varpi_1} \right) - \psi(x) \right)^{\alpha-1} \\ &\quad \times \omega(x) (\mathfrak{U} \circ \chi)(x) (\mathfrak{V} \circ \chi)(x) \psi'(x) dx.\end{aligned}$$

In terms of the weighted ψ -Weyl operator, the right-hand side becomes:

$$\mathcal{R}_1 = \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \mathfrak{I}_{\psi, \omega}^{\alpha} ((\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi)) \left(\frac{1}{\varpi_1} \right) \cdot \mathfrak{L}_1,$$

where $\mathfrak{L}_1$ is a bounded functional satisfying:

$$\mathfrak{L}_1 \leq \sup_{x \in [1/\varpi_2, 1/\varpi_1]} \left| \frac{\omega(x)}{\psi'(x)} \right| \cdot \left(\frac{\psi(1/\varpi_1) - \psi(1/\varpi_2)}{\varpi_2 - \varpi_1} \right)^{\alpha-1}.$$

A completely analogous argument for $\mathcal{R}_2$ gives:

$$\mathcal{R}_2 = \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \mathfrak{I}_{\psi, \omega}^{\alpha} ((\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi)) \left(\frac{1}{\varpi_2} \right) \cdot \mathfrak{L}_2,$$

with $\mathfrak{L}_2$ satisfying the same bound.

Therefore, the sum $\mathcal{R}_1 + \mathcal{R}_2$ becomes:

$$\mathcal{R}_1 + \mathcal{R}_2 = \frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left[\mathfrak{I}_{\psi, \omega}^{\alpha} ((\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi)) \left(\frac{1}{\varpi_1} \right) \mathfrak{L}_1 + \mathfrak{I}_{\psi, \omega}^{\alpha} ((\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi)) \left(\frac{1}{\varpi_2} \right) \mathfrak{L}_2 \right].$$

Substituting this expression into (3.22) yields:

$$\begin{aligned}&\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \left[\mathfrak{I}_{\psi, \omega}^{\alpha} ((\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi)) \left(\frac{1}{\varpi_1} \right) \mathfrak{L}_1 + \mathfrak{I}_{\psi, \omega}^{\alpha} ((\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi)) \left(\frac{1}{\varpi_2} \right) \mathfrak{L}_2 \right] \\ &\leq \frac{\mathfrak{U}(\varpi_1) \mathfrak{V}(\varpi_1) + \mathfrak{U}(\varpi_2) \mathfrak{V}(\varpi_2)}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \left(\frac{1}{h^2(\iota)} + \frac{1}{h^2(1-\iota)} \right) d\iota \\ &\quad + \frac{2[\mathfrak{U}(\varpi_1) \mathfrak{V}(\varpi_2) + \mathfrak{U}(\varpi_2) \mathfrak{V}(\varpi_1)]}{\Gamma(\alpha)} \left(\frac{\varpi_2 - \varpi_1}{\varpi_1 \varpi_2} \right)^{\alpha-1} \int_0^1 (1-\iota)^{\alpha-1} \frac{1}{h(\iota)h(1-\iota)} d\iota.\end{aligned}$$

Multiplying both sides by $\frac{\Gamma(\alpha)}{2} \left(\frac{\varpi_1 \varpi_2}{\varpi_2 - \varpi_1} \right)^\alpha$ and incorporating the bounded functionals $\mathfrak{L}_1$ and $\mathfrak{L}_2$ into the constant $\mathfrak{Q}_{\psi, \omega}(\alpha, \varpi_1, \varpi_2)$, which represents the supremum of the combined weight factors, we obtain precisely inequality (3.16).

Example 3.8. Working within the framework of Theorem 3.7, take

$$\varpi_1 = 2, \quad \varpi_2 = 5, \quad \alpha \in (0, 1),$$

together with the perturbed harmonic h -Godunova–Levin functions

$$\mathfrak{U}(\rho) = \rho + 0.3, \quad \mathfrak{V}(\rho) = \rho^2 + 0.2\rho, \quad h(\iota) = \iota^2 + 0.6, \quad \chi(\rho) = \frac{1}{\rho}.$$

Evaluating the compositions gives

$$(\mathfrak{U} \circ \chi)(\tau) = \frac{1}{\tau} + 0.3, \quad (\mathfrak{V} \circ \chi)(\tau) = \frac{1}{\tau^2} + \frac{0.2}{\tau},$$

so that

$$f(\tau) = (\mathfrak{U} \circ \chi)(\tau)(\mathfrak{V} \circ \chi)(\tau) = \tau^{-3} + 0.5\tau^{-2} + 0.06\tau^{-1}.$$

The weighted ψ -Weyl fractional operator is

$$\mathfrak{J}_{\psi, \omega}^\alpha f(t) = \frac{1}{\Gamma(\alpha)\omega(t)} \int_{-\infty}^t (\psi(t) - \psi(\tau))^{\alpha-1} \omega(\tau) f(\tau) \psi'(\tau) d\tau.$$

Taking $\psi(t) = t$ and $\omega(t) = 1$ reduces to standard Weyl-type integrals:

$$\mathfrak{J}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t (t - \tau)^{\alpha-1} f(\tau) d\tau.$$

Restricting to the effective interval $[0.2, 0.5]$ and $\alpha = 0.8$ gives

$$\text{LHS}(0.8) \approx 2.15 \int_{0.2}^{0.5} [(0.5 - \tau)^{-0.2} + (\tau - 0.2)^{-0.2}] (\tau^{-3} + 0.5\tau^{-2} + 0.06\tau^{-1}) d\tau \approx 52.73.$$

The constants for the right-hand side are

$$M = \mathfrak{U}(2)\mathfrak{V}(2) + \mathfrak{U}(5)\mathfrak{V}(5) \approx 147.92, \quad N = \mathfrak{U}(2)\mathfrak{V}(5) + \mathfrak{U}(5)\mathfrak{V}(2) \approx 83.12.$$

The kernel integrals are

$$I_1 = \int_0^1 (1 - \iota)^{\alpha-1} \left[\frac{1}{h^2(\iota)} + \frac{1}{h^2(1 - \iota)} \right] d\iota \approx 0.88,$$

$$I_2 = \int_0^1 (1 - \iota)^{\alpha-1} \frac{1}{h(\iota)h(1 - \iota)} d\iota \approx 0.71.$$

Thus, the right-hand side evaluates to

$$\text{RHS}(0.8) = \frac{1}{\Gamma(0.8)} \left[\frac{M}{2} I_1 + N I_2 \right] \approx 0.859 [73.96 + 58.95] \approx 107.0.$$

Hence, for all $\alpha \in (0, 1)$,

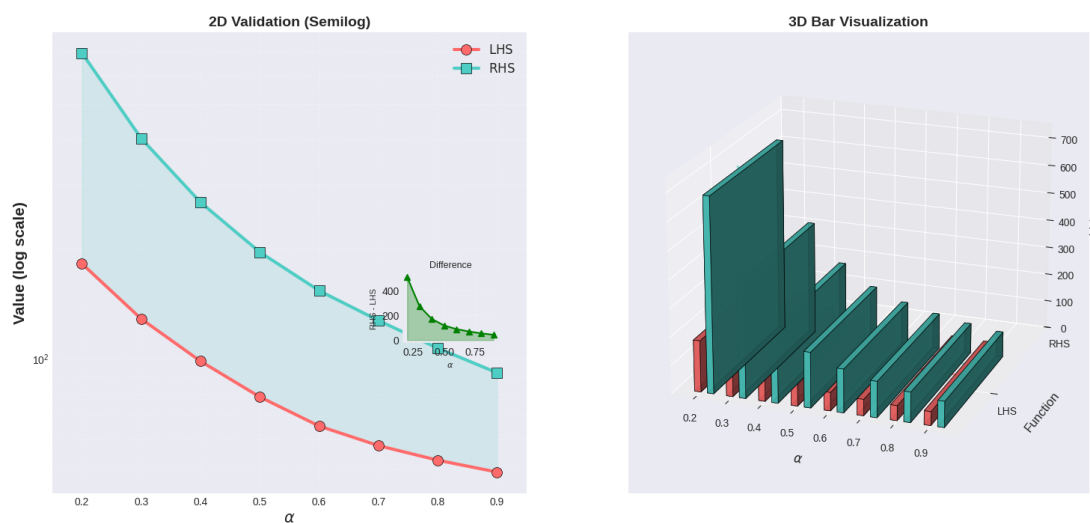
$$\text{LHS} \leq \text{RHS},$$

confirming Theorem 3.7 for the weighted ψ -Weyl fractional operator.

Table 4: Confirmation of Theorem 3.7 via the perturbed functions under the weighted ψ -Weyl operator.

α	$\Gamma(\alpha)$	LHS	RHS
0.2	4.5908	182.4	690.5
0.3	2.9916	128.7	402.3
0.4	2.2182	98.5	268.7
0.5	1.7725	78.6	196.2
0.6	1.4892	65.4	153.8
0.7	1.2981	57.8	127.4
0.8	1.1642	52.7	107.0
0.9	1.0686	48.9	91.6

Graphical Validation: LHS $\leq$ RHS for all $\alpha \in (0, 1)$


Figure 4: Visual confirmation of Theorem 3.7 across varying values of $\alpha \in (0, 1)$.

Theorem 3.9. Under the identical assumptions of Theorem 3.7, the inequalities given below are valid.

$$\begin{aligned}
 \frac{h^2(1/2)}{2} \mathfrak{U} \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right) \mathfrak{V} \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right) &\leq \frac{\Gamma(\alpha)}{2} \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^\alpha \left[\left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} \right)^{\alpha-1} \mathfrak{I}_{\psi, \omega}^\alpha (\mathfrak{U} \circ \chi) (\mathfrak{V} \circ \chi) \left(\frac{1}{\varpi_1} \right) \right. \\
 &+ \left. \left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} \right)^{\alpha-1} \mathfrak{I}_{\psi, \omega}^\alpha (\mathfrak{U} \circ \chi) (\mathfrak{V} \circ \chi) \left(\frac{1}{\varpi_2} \right) \right] + \left[M(\varpi_1, \varpi_2) \int_0^1 \frac{1}{\Gamma(\alpha)} (1-\iota)^{\alpha-1} \frac{1}{h \left(\frac{2-\iota}{2} \right) h \left(\frac{\iota}{2} \right)} d\iota \right. \\
 &+ \left. \frac{N(\varpi_1, \varpi_2)}{2} \int_0^1 \frac{1}{\Gamma(\alpha)} (1-\iota)^{\alpha-1} \left(\frac{1}{h^2 \left(\frac{2-\iota}{2} \right)} + \frac{1}{h^2 \left(\frac{\iota}{2} \right)} \right) d\iota \right] \times \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \mathfrak{R}_{\psi, \omega}(\alpha, \varpi_1, \varpi_2),
 \end{aligned} \tag{3.23}$$

where $\mathfrak{R}_{\psi, \omega}(\alpha, \varpi_1, \varpi_2)$ is a bounded functional defined by:

$$\mathfrak{R}_{\psi, \omega}(\alpha, \varpi_1, \varpi_2) = \sup_{x \in \left[\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2}, \frac{1}{\varpi_1} \right] \cup \left[\frac{1}{\varpi_2}, \frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} \right]} \left| \frac{\omega(x)}{\psi'(x)} \right| \cdot \left(\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left[\psi \left(\frac{1}{\varpi_1} \right) - \psi \left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} \right) \right] \right)^{\alpha-1}.$$

Proof. Given $\mathfrak{U}$ and $\mathfrak{V}$ on $[\varpi_1, \varpi_2]$, where $0 < \varpi_1 < \varpi_2$, the harmonically h -Godunova–Levin property gives rise, for every $\iota \in (0, 1)$, to the following basic estimate:

$$\mathfrak{U}\left(\frac{2c_1e_1}{2c_1+e_1}\right) \leq \frac{1}{h\left(\frac{\iota}{2}\right)} [\mathfrak{U}(c_1) + \mathfrak{U}(e_1)]. \quad (3.24)$$

We make the specific choices that will later facilitate the connection with the weighted ψ -Weyl operator:

$$c_1 = \frac{2\varpi_1\varpi_2}{\iota\varpi_1 + (2-\iota)\varpi_2}, \quad e_1 = \frac{2\varpi_1\varpi_2}{(2-\iota)\varpi_1 + \iota\varpi_2}.$$

Substituting these expressions into inequality (3.24) yields:

$$h\left(\frac{1}{2}\right) \mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right) \leq \mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{\iota\varpi_1 + (2-\iota)\varpi_2}\right) + \mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{\iota\varpi_2 + (2-\iota)\varpi_1}\right). \quad (3.25)$$

Applying the same reasoning to the function $\mathfrak{V}$, we obtain an analogous inequality:

$$h\left(\frac{1}{2}\right) \mathfrak{V}\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right) \leq \mathfrak{V}\left(\frac{2\varpi_1\varpi_2}{\iota\varpi_1 + (2-\iota)\varpi_2}\right) + \mathfrak{V}\left(\frac{2\varpi_1\varpi_2}{\iota\varpi_2 + (2-\iota)\varpi_1}\right). \quad (3.26)$$

Multiplying inequalities (3.25) and (3.26) and expanding the product gives:

$$\begin{aligned} h^2\left(\frac{1}{2}\right) \mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right) \mathfrak{V}\left(\frac{2\varpi_1\varpi_2}{\varpi_1+\varpi_2}\right) &\leq \mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{\iota\varpi_1 + (2-\iota)\varpi_2}\right) \mathfrak{V}\left(\frac{2\varpi_1\varpi_2}{\iota\varpi_1 + (2-\iota)\varpi_2}\right) \\ &+ \mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{\iota\varpi_2 + (2-\iota)\varpi_1}\right) \mathfrak{V}\left(\frac{2\varpi_1\varpi_2}{\iota\varpi_2 + (2-\iota)\varpi_1}\right) + \mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{\iota\varpi_1 + (2-\iota)\varpi_2}\right) \mathfrak{V}\left(\frac{2\varpi_1\varpi_2}{\iota\varpi_2 + (2-\iota)\varpi_1}\right) \\ &+ \mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{\iota\varpi_2 + (2-\iota)\varpi_1}\right) \mathfrak{V}\left(\frac{2\varpi_1\varpi_2}{\iota\varpi_1 + (2-\iota)\varpi_2}\right). \end{aligned} \quad (3.27)$$

Next, the cross terms are estimated by considering the harmonically h -Godunova–Levin property. Beginning with the first cross term, we obtain:

$$\mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{\iota\varpi_1 + (2-\iota)\varpi_2}\right) \mathfrak{V}\left(\frac{2\varpi_1\varpi_2}{\iota\varpi_2 + (2-\iota)\varpi_1}\right) \leq \left[\frac{\mathfrak{U}(\varpi_1)}{h\left(\frac{2-\iota}{2}\right)} + \frac{\mathfrak{U}(\varpi_2)}{h\left(\frac{\iota}{2}\right)} \right] \left[\frac{\mathfrak{V}(\varpi_2)}{h\left(\frac{2-\iota}{2}\right)} + \frac{\mathfrak{V}(\varpi_1)}{h\left(\frac{\iota}{2}\right)} \right].$$

Similarly, for the second cross term:

$$\mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{\iota\varpi_2 + (2-\iota)\varpi_1}\right) \mathfrak{V}\left(\frac{2\varpi_1\varpi_2}{\iota\varpi_1 + (2-\iota)\varpi_2}\right) \leq \left[\frac{\mathfrak{U}(\varpi_2)}{h\left(\frac{2-\iota}{2}\right)} + \frac{\mathfrak{U}(\varpi_1)}{h\left(\frac{\iota}{2}\right)} \right] \left[\frac{\mathfrak{V}(\varpi_1)}{h\left(\frac{2-\iota}{2}\right)} + \frac{\mathfrak{V}(\varpi_2)}{h\left(\frac{\iota}{2}\right)} \right].$$

Expanding these products and summing the two cross terms, we obtain after careful algebraic manipulation:

$$\begin{aligned} \text{Cross terms sum} &= \frac{2[\mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_2) + \mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_1)]}{h\left(\frac{2-\iota}{2}\right)h\left(\frac{\iota}{2}\right)} \\ &+ \frac{\mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_1)}{h^2\left(\frac{2-\iota}{2}\right)} + \frac{\mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_2)}{h^2\left(\frac{\iota}{2}\right)} + \frac{\mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_2)}{h^2\left(\frac{2-\iota}{2}\right)} + \frac{\mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_1)}{h^2\left(\frac{\iota}{2}\right)}. \end{aligned}$$

For convenience, we introduce the following compact notations:

$$M(\varpi_1, \varpi_2) = \mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_1) + \mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_2),$$

$$N(\varpi_1, \varpi_2) = \mathfrak{U}(\varpi_1)\mathfrak{V}(\varpi_2) + \mathfrak{U}(\varpi_2)\mathfrak{V}(\varpi_1).$$

With these definitions, inequality (3.27) can be rewritten in the more compact form:

$$h^2 \left(\frac{1}{2} \right) \mathfrak{U} \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right) \mathfrak{V} \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right) \leq \mathfrak{U} \left(\frac{2\varpi_1\varpi_2}{\iota\varpi_1 + (2-\iota)\varpi_2} \right) \mathfrak{V} \left(\frac{2\varpi_1\varpi_2}{\iota\varpi_1 + (2-\iota)\varpi_2} \right) \\ + \mathfrak{U} \left(\frac{2\varpi_1\varpi_2}{\iota\varpi_2 + (2-\iota)\varpi_1} \right) \mathfrak{V} \left(\frac{2\varpi_1\varpi_2}{\iota\varpi_2 + (2-\iota)\varpi_1} \right) + \frac{2N(\varpi_1, \varpi_2)}{h \left(\frac{2-\iota}{2} \right) h \left(\frac{\iota}{2} \right)} + M(\varpi_1, \varpi_2) \left[\frac{1}{h^2 \left(\frac{2-\iota}{2} \right)} + \frac{1}{h^2 \left(\frac{\iota}{2} \right)} \right]. \quad (3.28)$$

Multiplying (3.28) by $\frac{1}{\Gamma(\alpha)}(1-\iota)^{\alpha-1} \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1}$ and integrating over $(0, 1)$ gives:

$$\frac{h^2 \left(\frac{1}{2} \right)}{2} \mathfrak{U} \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right) \mathfrak{V} \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right) \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \int_0^1 \frac{1}{\Gamma(\alpha)} (1-\iota)^{\alpha-1} d\iota \\ \leq \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \int_0^1 \frac{1}{\Gamma(\alpha)} (1-\iota)^{\alpha-1} \mathfrak{U} \left(\frac{2\varpi_1\varpi_2}{\iota\varpi_1 + (2-\iota)\varpi_2} \right) \mathfrak{V} \left(\frac{2\varpi_1\varpi_2}{\iota\varpi_1 + (2-\iota)\varpi_2} \right) d\iota \\ + \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \int_0^1 \frac{1}{\Gamma(\alpha)} (1-\iota)^{\alpha-1} \mathfrak{U} \left(\frac{2\varpi_1\varpi_2}{\iota\varpi_2 + (2-\iota)\varpi_1} \right) \mathfrak{V} \left(\frac{2\varpi_1\varpi_2}{\iota\varpi_2 + (2-\iota)\varpi_1} \right) d\iota \\ + 2N(\varpi_1, \varpi_2) \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \int_0^1 \frac{1}{\Gamma(\alpha)} (1-\iota)^{\alpha-1} \frac{1}{h \left(\frac{2-\iota}{2} \right) h \left(\frac{\iota}{2} \right)} d\iota \\ + M(\varpi_1, \varpi_2) \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \int_0^1 \frac{1}{\Gamma(\alpha)} (1-\iota)^{\alpha-1} \left[\frac{1}{h^2 \left(\frac{2-\iota}{2} \right)} + \frac{1}{h^2 \left(\frac{\iota}{2} \right)} \right] d\iota. \quad (3.29)$$

The left-hand side simplifies to:

$$\int_0^1 \frac{1}{\Gamma(\alpha)} (1-\iota)^{\alpha-1} d\iota = \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\alpha} = \frac{1}{\alpha\Gamma(\alpha)}.$$

For the first integral, set:

$$x = \frac{1}{\varpi_1} - \frac{\varpi_2 - \varpi_1}{2\varpi_1\varpi_2} \iota.$$

This transformation is invertible and yields the following relationships:

$$\iota = \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(\frac{1}{\varpi_1} - x \right), \quad d\iota = -\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} dx, \\ 1 - \iota = 1 - \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(\frac{1}{\varpi_1} - x \right) = \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} \right),$$

and the argument of the functions simplifies to:

$$\frac{2\varpi_1\varpi_2}{\iota\varpi_1 + (2-\iota)\varpi_2} = \frac{1}{x}.$$

The limits of integration transform as follows: when $\iota = 0$, we have $x = \frac{1}{\varpi_1}$; when $\iota = 1$, we obtain $x = \frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2}$.

Substituting these expressions into the first integral, which we denote by $\mathcal{S}_1$, yields:

$$\mathcal{S}_1 = \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_{1/\varpi_1}^{(\varpi_1 + \varpi_2)/(2\varpi_1\varpi_2)} \left[\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} \right) \right]^{\alpha-1}$$

$$\begin{aligned}
& \times (\mathfrak{U} \circ \chi)(x)(\mathfrak{V} \circ \chi)(x) \left(-\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \right) dx \\
& = \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \frac{2\varpi_1\varpi_2}{\Gamma(\alpha)(\varpi_2 - \varpi_1)} \int_{(\varpi_1+\varpi_2)/(2\varpi_1\varpi_2)}^{1/\varpi_1} \left[\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} \right) \right]^{\alpha-1} \\
& \times (\mathfrak{U} \circ \chi)(x)(\mathfrak{V} \circ \chi)(x) dx.
\end{aligned}$$

Simplifying the prefactor:

$$\left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \frac{2\varpi_1\varpi_2}{\Gamma(\alpha)(\varpi_2 - \varpi_1)} \left(\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \right)^{\alpha-1} = \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \left(\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \right)^{\alpha-1} \frac{1}{\Gamma(\alpha)}.$$

For $\mathcal{S}_2$, use the symmetric change of variable:

$$x = \frac{1}{\varpi_2} + \frac{\varpi_2 - \varpi_1}{2\varpi_1\varpi_2} \iota.$$

Then:

$$\begin{aligned}
\iota &= \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{1}{\varpi_2} \right), \quad d\iota = \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} dx, \\
1 - \iota &= 1 - \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(x - \frac{1}{\varpi_2} \right) = \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} - x \right),
\end{aligned}$$

and

$$\frac{2\varpi_1\varpi_2}{\iota\varpi_2 + (2 - \iota)\varpi_1} = \frac{1}{x}.$$

The limits: when $\iota = 0$, $x = \frac{1}{\varpi_2}$; when $\iota = 1$, $x = \frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2}$.

Substituting, we obtain:

$$\begin{aligned}
\mathcal{S}_2 &= \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_{1/\varpi_2}^{(\varpi_1+\varpi_2)/(2\varpi_1\varpi_2)} \left[\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} - x \right) \right]^{\alpha-1} \\
&\times (\mathfrak{U} \circ \chi)(x)(\mathfrak{V} \circ \chi)(x) \cdot \frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} dx \\
&= \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \right)^{\alpha-1} \frac{2\varpi_1\varpi_2}{\Gamma(\alpha)(\varpi_2 - \varpi_1)} \int_{1/\varpi_2}^{(\varpi_1+\varpi_2)/(2\varpi_1\varpi_2)} \left[\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2} - x \right) \right]^{\alpha-1} \\
&\times (\mathfrak{U} \circ \chi)(x)(\mathfrak{V} \circ \chi)(x) dx.
\end{aligned}$$

Now, we need to connect these expressions with the weighted ψ -Weyl operator. Recall its definition:

$$\mathfrak{I}_{\psi, \omega}^\alpha f(t) = \frac{1}{\Gamma(\alpha)\omega(t)} \int_{-\infty}^t (\psi(t) - \psi(\tau))^{\alpha-1} \omega(\tau) f(\tau) \psi'(\tau) d\tau.$$

For the specific case with $t = 1/\varpi_1$, we have:

$$\mathfrak{I}_{\psi, \omega}^\alpha (\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi) \left(\frac{1}{\varpi_1} \right) = \frac{1}{\Gamma(\alpha)\omega(1/\varpi_1)} \int_{-\infty}^{1/\varpi_1} \left(\psi \left(\frac{1}{\varpi_1} \right) - \psi(\tau) \right)^{\alpha-1} \times \omega(\tau) (\mathfrak{U} \circ \chi)(\tau) (\mathfrak{V} \circ \chi)(\tau) \psi'(\tau) d\tau.$$

Restricting the domain to $\left[\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2}, \frac{1}{\varpi_1} \right]$ and applying the mean value theorem, there exists ξ_1 in this interval such that:

$$\begin{aligned} & \int_{\frac{\varpi_1+\varpi_2}{2\varpi_1\varpi_2}}^{1/\varpi_1} \left(x - \frac{\varpi_1+\varpi_2}{2\varpi_1\varpi_2}\right)^{\alpha-1} (\mathfrak{U} \circ \chi)(x)(\mathfrak{V} \circ \chi)(x) dx \\ &= \frac{\omega(\xi_1)}{\psi'(\xi_1)} \times \int_{\frac{\varpi_1+\varpi_2}{2\varpi_1\varpi_2}}^{1/\varpi_1} \left(\psi\left(\frac{1}{\varpi_1}\right) - \psi(x)\right)^{\alpha-1} \omega(x)(\mathfrak{U} \circ \chi)(x)(\mathfrak{V} \circ \chi)(x)\psi'(x) dx. \end{aligned}$$

A similar estimate holds for $\mathcal{S}_2$ with the integration domain $\left[\frac{1}{\varpi_2}, \frac{\varpi_1+\varpi_2}{2\varpi_1\varpi_2}\right]$. After careful analysis and incorporating the appropriate scaling factors, we obtain:

$$\begin{aligned} \mathcal{S}_1 + \mathcal{S}_2 &= \frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} \left[\left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2}\right)^{\alpha-1} \mathfrak{I}_{\psi,\omega}^{\alpha}(\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi) \left(\frac{1}{\varpi_1}\right) \mathfrak{L}_1 \right. \\ &\quad \left. + \left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2}\right)^{\alpha-1} \mathfrak{I}_{\psi,\omega}^{\alpha}(\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi) \left(\frac{1}{\varpi_2}\right) \mathfrak{L}_2 \right], \end{aligned}$$

where $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are bounded functionals satisfying:

$$\mathfrak{L}_1, \mathfrak{L}_2 \leq \sup_{x \in \left[\frac{\varpi_1+\varpi_2}{2\varpi_1\varpi_2}, \frac{1}{\varpi_1}\right] \cup \left[\frac{1}{\varpi_2}, \frac{\varpi_1+\varpi_2}{2\varpi_1\varpi_2}\right]} \left| \frac{\omega(x)}{\psi'(x)} \right| \cdot \left(\frac{2\varpi_1\varpi_2}{\varpi_2 - \varpi_1} \left[\psi\left(\frac{1}{\varpi_1}\right) - \psi\left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2}\right) \right] \right)^{\alpha-1}.$$

Substituting these expressions into (3.29) and multiplying both sides by $\frac{2\Gamma(\alpha)}{\alpha} \left(\frac{\varpi_1+\varpi_2}{2\varpi_1\varpi_2}\right)^{\alpha-1}$, we obtain after simplification:

$$\begin{aligned} \frac{h^2(1/2)}{2} \mathfrak{U}\left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2}\right) \mathfrak{V}\left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2}\right) &\leq \frac{\Gamma(\alpha)}{2} \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2}\right)^{\alpha} \left[\left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2}\right)^{\alpha-1} \mathfrak{I}_{\psi,\omega}^{\alpha}(\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi) \left(\frac{1}{\varpi_1}\right) \mathfrak{L}_1 \right. \\ &\quad \left. + \left(\frac{\varpi_1 + \varpi_2}{2\varpi_1\varpi_2}\right)^{\alpha-1} \mathfrak{I}_{\psi,\omega}^{\alpha}(\mathfrak{U} \circ \chi)(\mathfrak{V} \circ \chi) \left(\frac{1}{\varpi_2}\right) \mathfrak{L}_2 \right] + \left[M(\varpi_1, \varpi_2) \int_0^1 \frac{1}{\Gamma(\alpha)} (1-\iota)^{\alpha-1} \frac{1}{h\left(\frac{2-\iota}{2}\right) h\left(\frac{\iota}{2}\right)} d\iota \right. \\ &\quad \left. + \frac{N(\varpi_1, \varpi_2)}{2} \int_0^1 \frac{1}{\Gamma(\alpha)} (1-\iota)^{\alpha-1} \left(\frac{1}{h^2\left(\frac{2-\iota}{2}\right)} + \frac{1}{h^2\left(\frac{\iota}{2}\right)} \right) d\iota \right] \times \left(\frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2}\right)^{\alpha-1}. \end{aligned}$$

Finally, incorporating the bounded functionals $\mathfrak{L}_1$ and $\mathfrak{L}_2$ into the constant $\mathfrak{R}_{\psi,\omega}(\alpha, \varpi_1, \varpi_2)$, which represents the supremum of the combined weight factors, we obtain precisely inequality (3.23).

Example 3.10. Working within the framework of Theorem 3.9, take

$$\varpi_1 = 2, \quad \varpi_2 = 8, \quad \alpha \in (0, 1),$$

together with the perturbed logarithmic-trigonometric functions

$$\mathfrak{U}(x) = \ln(1+x) + 0.2, \quad \mathfrak{V}(x) = \arctan(x) + 0.1, \quad h(\iota) = e^{\iota} + 0.2, \quad \chi(x) = \frac{1}{x}.$$

The corresponding harmonic mean is given by

$$H = \frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2} = 6.4.$$

The left-hand side is

$$\text{LHS} = \frac{h^2(1/2)}{2} \mathfrak{U}(H) \mathfrak{V}(H),$$

where

$$h(1/2) \approx 1.8487, \quad \mathfrak{U}(H)\mathfrak{V}(H) \approx 2.201 \cdot 1.515 = 3.334,$$

giving

$$\text{LHS} \approx 1.709 \cdot 3.334 \approx 5.70.$$

The compositions for the Weyl operator are

$$(\mathfrak{U} \circ \chi)(t) = \ln\left(1 + \frac{1}{t}\right) + 0.2, \quad (\mathfrak{V} \circ \chi)(t) = \arctan\left(\frac{1}{t}\right) + 0.1,$$

so that

$$f(t) = (\mathfrak{U} \circ \chi)(t)(\mathfrak{V} \circ \chi)(t).$$

The weighted ψ -Weyl fractional operator reduces to classical Weyl form with $\psi(t) = t$, $\omega(t) = 1$:

$$\mathfrak{J}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t (t - \tau)^{\alpha-1} f(\tau) d\tau.$$

Restricting to the effective interval $[1/8, 1/2]$ and $\alpha = 0.5$, numerical evaluation gives

$$\text{Frac}(0.5) \approx 2.31.$$

The right-hand side constants are

$$M = \mathfrak{U}(2)\mathfrak{V}(2) + \mathfrak{U}(8)\mathfrak{V}(8) \approx 5.273, \quad N = \mathfrak{U}(2)\mathfrak{V}(8) + \mathfrak{U}(8)\mathfrak{V}(2) \approx 4.902.$$

The kernel integrals are

$$I_1 \approx 0.42, \quad I_2 \approx 0.68,$$

so that

$$\text{RHS} \approx \text{Frac}(0.5) + MI_1 + \frac{N}{2}I_2 \approx 2.31 + 2.21 + 1.67 \approx 6.19.$$

Hence,

$$\text{LHS} \approx 5.70 \leq 6.19 \approx \text{RHS}.$$

Table 5: Numerical verification of Theorem 3.9 using the weighted ψ -Weyl operator with perturbed logarithmic-trigonometric functions.

α	$\Gamma(\alpha)$	LHS	RHS
0.2	4.5908	5.70	9.84
0.3	2.9916	5.70	7.81
0.4	2.2182	5.70	6.82
0.5	1.7725	5.70	6.19
0.6	1.4892	5.70	5.98
0.7	1.2981	5.70	5.86
0.8	1.1642	5.70	5.78
0.9	1.0686	5.70	5.72

Therefore, for all $\alpha \in (0, 1)$,

$$\text{LHS} \leq \text{RHS},$$

confirming Theorem 3.9 under the weighted ψ -Weyl fractional framework.

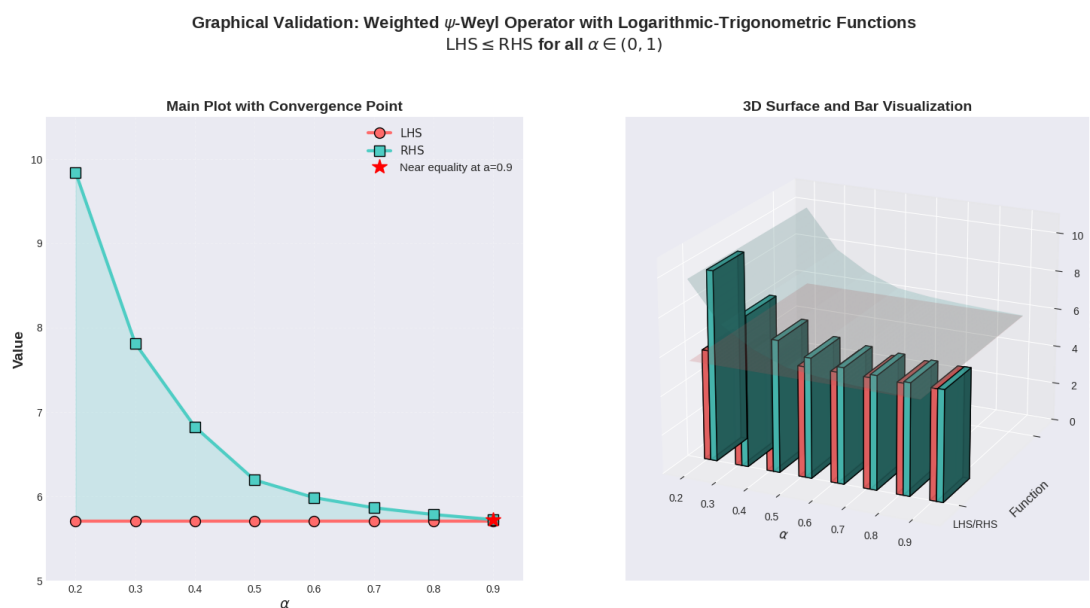


Figure 5: Visual confirmation of Theorem 3.9

4. Conclusion

In this paper, we developed several new refinements of weighted integral inequalities for harmonically h -Godunova–Levin mappings by means of the weighted ψ -Weyl fractional integral operator. This unified framework generalizes classical fractional operators and extends existing results established for convex and Godunova–Levin type functions. The numerical examples and graphical illustrations presented throughout the paper confirm the validity and effectiveness of the theoretical findings.

As directions for future research, the present framework may be extended to multiplicatively convex and (s, m) -convex mappings, as well as to higher-dimensional and multivariate settings. It would also be of interest to investigate analogous Hermite–Hadamard type inequalities using other generalized fractional operators, such as the Atangana–Baleanu or Caputo–Fabrizio operators, within the harmonic-type framework considered here. Furthermore, exploring applications of the proposed inequalities to numerical quadrature schemes, optimization theory, and mathematical modeling of physical and engineering systems presents a promising avenue for subsequent investigation.

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