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Transcendental continued β-fraction with formal power series over finite fields

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Abstract

In the framework of formal power series defined over a finite field, this study provides a new theoretical result that facilitates the derivation of a transcendence criterion. Our approach specifically leverages the structural properties of continued β-fraction expansions associated with quadratic Pisot series, where $deg(\beta) = m$. By analysing these expansions, we establish conditions under which an element of $\mathbb{F}_q((x^{-1}))$ is transcendental.

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1. Introduction

Let $\mathbb{F}_q$ denote a finite field with q elements and characteristic p. In this mathematical setting, we consider the ring of polynomials $\mathbb{F}_q[x]$ and its associated field of fractions $\mathbb{F}_q(x)$. The completion of $\mathbb{F}_q(x)$ relative to the ultrametric absolute value is represented as $\mathbb{F}_q((x^{-1}))$. This absolute value is defined such that $|0| = 0$ and $|P/Q| = q^{\deg P - \deg Q}$, $\forall P, Q \in (\mathbb{F}_q[x])^*$. Every non-zero element $\alpha \in \mathbb{F}_q((x^{-1}))$ can be expressed via its Laurent series expansion:

$$\alpha = \sum_{i \leq i_0} a_i x^i$$

where $i_0 \in \mathbb{Z}$, $a_i \in \mathbb{F}_q$, and the leading coefficient a_{i_0} is non-null.

From this, we describe the polynomial part $[\alpha] = \sum_{0 \leq i \leq i_0} a_i x^i$ and the fractional part $\{\alpha\} = \alpha - [\alpha]$, the latter being in the open unit disk $D(0, 1)$.

We define $\mathbb{F}_q(x, \beta)$ as the smallest field extension containing both x and an element $\beta \in \mathbb{F}_q((x^{-1}))$, with $\mathbb{F}_q[x, \beta]$ serving as the corresponding minimal ring. α is defined as a β-polynomial if the condition $\{\alpha\} = 0$ is satisfied. The collection of all such elements is denoted by $\mathbb{F}_q[x]_\beta$. It is important to note that while this set contains these specific polynomials, $\mathbb{F}_q[x]_\beta$ is not stable under standard multiplication.
A key focus of this work involves Pisot elements. An element $\beta \in \mathbb{F}_q((x^{-1}))$ is identified as a Pisot element if it is an algebraic integer over $\mathbb{F}_q[x]$ with $|\beta| > 1$, while all its conjugates satisfy $|\beta_i| < 1$.

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Consider $\beta \in \mathbb{F}_q((x^{-1}))$ such that $|\beta| > 1$.
We define a β-representation of α as any infinite sequence $(\mu_i)_{i\geq 1}$ of polynomials in $\mathbb{F}_q[x]$ satisfying the series expansion $\alpha = \sum_{i=1}^{\infty} \mu_i \beta^{-i}$. The β-expansion of α, denoted $d_\beta(\alpha)$, is the specific representation generated by the map $T_\beta : D(0,1) \to D(0,1)$, defined by $T_\beta(\alpha) = \beta\alpha - [\beta\alpha]$. In this case, the digits are uniquely determined by $\mu_i = [\beta T_\beta^{i-1}(\alpha)]$. We compute the β-expansion via the recursion $z_0 = \alpha$, with

$$\mu_i = [\beta z_{i-1}] \quad \text{and} \quad z_i = \beta z_{i-1} - \mu_i$$

for $i \geq 1$. We say that the β-expansion $d_\beta(\alpha)$ is finite if the sequence of digits eventually terminates, that is, if there exists an index $k \geq 0$ such that $\mu_i = 0$ for all $i > k$. Based on this definition, let:

$$\mathrm{Fin}(\beta) = \{\alpha \in \mathbb{F}_q((x^{-1})) \mid \exists k \geq 0, \mu_i = 0 \text{ for all } i > k\}.$$

Furthermore, we define $L_\odot$ as the smallest natural number n such that for any two β-polynomials P_1 and P_2, the condition $P_1 \cdot P_2 \in \mathrm{Fin}(\beta)$ implies that $\beta^n(P_1 \cdot P_2)$ is itself a β-polynomial. Specifically:

$$L_\odot = min\{n \in \mathbb{N} : \forall P_1, P_2 \in \mathbb{F}_q[x]_\beta; P_1.P_2 \in Fin(\beta) \Rightarrow \beta^n(P_1 \cdot P_2) \in \mathbb{F}_q[x]_\beta\}.$$

Previous studies have determined $L_\odot$ for various Pisot series over finite fields. For further details and a general reference on this subject, see, for instance, [6].

Theorem 1.1. *[6] Let $f \in \mathbb{F}_q((x^{-1}))$ be algebraic over $\mathbb{F}_q[x]$ with $\deg(f) \geq 2$. If there is a conjugate f_i of f such that $|f_i| \cdot |f| = 1$, then $L_\odot \in \{0,1\}$.*

As a result of these findings, the following corollary provides a description of the continued fraction expansion for quadratic Pisot series satisfying $\deg(\beta) \geq 2$.

Corollary 1.2. *[6] Let β be a quadratic Pisot series satisfying $\deg(\beta) \geq 2$. Then, $L_\odot = 1$.*

The relationship between transcendence criteria and the continued fraction expansions of elements in $\mathbb{F}_q((x^{-1}))$ has been a subject of extensive investigation. This connection has been explored in depth across several key studies, notably those by [1, 2, 3, 4, 8].
The primary objective of this paper is to establish a new transcendence criterion by leveraging the properties of continued β-fraction expansions for quadratic Pisot series with $\deg(\beta) = m$. By investigating the structural irregularities and patterns within these expansions, we provide a framework for identifying transcendental elements in the field of formal Laurent series. To facilitate this discussion, we have organised the work into several key sections. Section 2 establishes the necessary mathematical preliminaries, specifically focusing on the construction of continued β-fractions and the essential algebraic properties of the underlying Pisot series. In Section 3, we move to the core of our research, where we rigorously prove our main outcomes and discuss their implications within the field.

2. THE CONTINUED β-FRACTION EXPANSION ALGORITHM

For an element $\beta \in \mathbb{F}_q((x^{-1}))$ satisfying $|\beta| > 1$, consider the mapping $T'_\beta : D(0,1) \to D(0,1)$ given by:

$$T'_\beta(\alpha) = \left\{\frac{1}{\alpha}\right\}_\beta = \frac{1}{\alpha} - \left[\frac{1}{\alpha}\right]_\beta,$$

where $[\cdot]_\beta$ denotes the β-integer part. For any α in the unit disk $D(0,1)$, we can express α as a continued fraction of the form:

$$\alpha = \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{\ddots}}} = [0, A_1, A_2, \dots]_\beta$$

where the sequence $(A_i)_{i\geq 1}$ consists of β-polynomials. These coefficients, referred to as the partial β-quotients, are generated iteratively for any positive integer i by:

$$A_i = \left[\frac{1}{T_\beta'^{(i-1)}(\alpha)}\right]_\beta$$

More generally, for any $\alpha \in \mathbb{F}_q((x^{-1}))$, we let $A_0 = [\alpha]_\beta$ denote the initial β-integer part. The full expansion is then given by:

$$\alpha = A_0 + \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{\ddots}}} = [A_0, A_1, A_2, \dots]_\beta$$

This expression is termed the continued β-fraction expansion of α. To analyse the convergence and properties of this expansion, we define the n^{th} β-complete quotient as α_n, which satisfies $\alpha = [A_0, A_1, \dots, A_{n-1}, \alpha_n]_\beta$. It is important to note that, unlike standard continued fractions, the partial quotients A_k for $k \geq 1$ are non-constant β-polynomials and do not belong to the constant field $\mathbb{F}_q$.

In a manner analogous to the classical theory of continued fractions, we introduce the sequences $(P_n)_{n\in\mathbb{N}}$ and $(Q_n)_{n\in\mathbb{N}}$ within the ring $\mathbb{F}_q[x, \beta^{-1}]$. These sequences are defined recursively by the following initial conditions and relations:

$$\begin{cases} P_0 = A_0, & P_1 = A_0A_1 + 1 \\ Q_0 = 1, & Q_1 = A_1 \end{cases}$$

and for all $n \geq 2$:

$$\begin{cases} P_n = A_nP_{n-1} + P_{n-2} \\ Q_n = A_nQ_{n-1} + Q_{n-2} \end{cases}$$

For each $n \geq 0$, the pair (P_n, Q_n) corresponds to reduced β-fractionary expansion of α. Now, let β be a quadratic Pisot series of degree m. In the specific case where β is a quadratic Pisot series of degree m, we define the mapping $\gamma : \mathbb{F}_q[x]_\beta \to \mathbb{N}$. For any β-polynomial $P = a_s\beta^s + \cdots + a_0$, we set:

$$\gamma(P) = m \cdot s + \deg(a_s).$$

Below, we recall several fundamental properties of these sequences and the map γ that will be essential for the proofs presented in Section 3. For the sake of self-containedness, we explicitly restate the key underlying assertions from reference [7] and [5] as follows:

Proposition 2.1. *[7] Let β be a quadratic Pisot unit in the field of formal power series with* $\deg(\beta) = m$*.Then for all $P_1, \dots, P_m \in \mathbb{F}_q[x]_\beta$, we have:*

$$\beta^{m-1}(P_1P_2 \cdots P_m) \in \mathbb{F}_q[x]_\beta.$$

Corollary 2.2. *[7] Let β be a quadratic Pisot unit in the field of formal power series with* $\deg(\beta) = m$*. For the n-th β-convergent P_n/Q_n of $\alpha \in \mathbb{F}_q((x^{-1}))$, we have:*

$$\beta^{\frac{(m-1)n}{m}} P_n \in \mathbb{F}_q[x]_\beta \quad \textit{and} \quad \beta^{\frac{(m-1)n}{m}} Q_n \in \mathbb{F}_q[x]_\beta.$$

Lemma 2.3. *[5] Let $A, B \in \mathbb{F}_q[x]_\beta$ satisfy $\gamma(A) > \gamma(B)$. One can find C, A_1, and B_1 in $\mathbb{F}_q[x]_\beta$ that allow for the expansion:*

$$\frac{A}{B} = C + \left(\frac{A_1}{B_1}\right)^{-1}$$

such that the inequality $\gamma(A_1) > \gamma(B_1)$ holds.

Remark 2.4. By combining the results of Corollary 2.2 and Lemma 2.3, it follows that the quantity $\beta^{\frac{(m-1)(2n+1)}{m}}(P_{n+1}Q_n - P_nQ_{n+1})$ is an element of $\mathbb{F}_q[x]_\beta$.

3. RESULTS

Theorem 3.1. *Let $\alpha = [A_0, A_1, \dots]_\beta$ be the continued β-fraction expansion of $\alpha \in \mathbb{F}_q((x^{-1}))$, where β is a quadratic Pisot series. α is transcendental whenever the following condition holds:*

$$\limsup_{n\to\infty} \frac{1}{n} \log\left(\sum_{i=1}^{n} \gamma(A_i)\right) = \infty.$$

To establish the necessary framework for proving Theorem 3.1, we first present a fundamental corollary.

Corollary 3.2. *Let β be a quadratic Pisot series of degree m, and let α be an algebraic series of degree ξ over $\mathbb{F}_q(x)$. Then, there exists a constant $w = w(\alpha) > 0$ such that for any pair of β-polynomials $P, Q \in \mathbb{F}_q[x]_\beta$ with $Q \neq 0$, the following inequality holds:*

$$\left|\alpha - \frac{P}{Q}\right| > w \cdot |\beta|^{-\frac{(\xi+1)(m-1)}{m}} |Q|^{-\xi}.$$

Proof. Let $\psi(y) = B_\xi y^\xi + \cdots + B_0 \in \mathbb{F}_q[x, y]$ be the minimal irreducible polynomial of α, such that $\psi(\alpha) = 0$. By applying Corollary 2.2, it follows that the expression $\beta^{\frac{(\xi+1)(m-1)}{m}} Q^\xi \psi(P/Q)$ belongs to the set of β-polynomials $\mathbb{F}_q[x]_\beta$ for any $P, Q \in \mathbb{F}_q[x]_\beta$. Assuming P/Q is not a root of ψ, we have the lower bound:

$$\left|\psi\left(\frac{P}{Q}\right)\right| \geq \frac{1}{|\beta|^{\frac{(\xi+1)(m-1)}{m}} |Q|^\xi}.$$

Using the Mean Value Theorem in this ultrametric context and the fact that $\psi(\alpha) = 0$, we obtain:

$$\left|\psi\left(\frac{P}{Q}\right) - \psi(\alpha)\right| \leq \left|\frac{P}{Q} - \alpha\right| \cdot \max_{1\leq i\leq \xi} |B_i \alpha^{i-1}|.$$

Let $w_1 = \max_{1\leq i\leq \xi} |B_i\alpha^{i-1}|$. Combining the above inequalities, it follows that:

$$\left|\frac{P}{Q} - \alpha\right| \geq \frac{1}{w_1 |\beta|^{\frac{(\xi+1)(m-1)}{m}} |Q|^\xi}.$$

Because $\psi(P/Q) \neq 0$ and the absolute value in $\mathbb{F}_q((x^{-1}))$ takes non-zero discrete values, the non-strict lower bound can be strengthened. Specifically, by picking a constant w strictly smaller than $\min(1, \frac{1}{w_1})$, the strict inequality is preserved over the valuation grid, yielding:

$$\left|\alpha - \frac{P}{Q}\right| > \frac{w}{|\beta|^{\frac{(\xi+1)(m-1)}{m}} |Q|^\xi}.$$

□

Proof of Theorem 3.1

Let P_n/Q_n be the n-th β-convergent of the formal power series α. In a manner analogous to the classical theory of continued fractions, the approximation error is given by the determinant-like formula:

$$\left|\alpha - \frac{P_n}{Q_n}\right| = \left|\frac{P_{n+1}Q_n - P_nQ_{n+1}}{Q_nQ_{n+1}}\right|.$$

Applying the result from Remark 2.4, we obtain an upper bound for the shifted error:

$$\left|\beta^{\frac{(m-1)(2n+1)}{m}}\left(\alpha - \frac{P_n}{Q_n}\right)\right| \leq \frac{|\beta|^{\frac{(m-1)(2n+1)}{m}}}{|Q_nQ_{n+1}|}.$$

Suppose, for the sake of contradiction, that α is an algebraic series of degree $\xi > 1$. By Corollary 3.2, there exists a constant $w > 0$ such that:

$$\frac{w}{|\beta|^{\frac{(\xi+1)(m-1)}{m}}|Q_n|^{\xi}} < \left|\beta^{\frac{(m-1)(2n+1)}{m}}\left(\alpha - \frac{P_n}{Q_n}\right)\right| \leq \frac{|\beta|^{\frac{(m-1)(2n+1)}{m}}}{|Q_nQ_{n+1}|}.$$

By isolating $|Q_{n+1}|$, we derive the following recurrence inequality:

$$|Q_{n+1}| < \frac{1}{w}|Q_n|^{\xi-1}|\beta|^{\frac{(m-1)(n(\xi+2)+2)}{m}}.$$

Let $\theta = 1/w$. By iterating this relation, it follows that the growth of $|Q_n|$ is bounded by:

$$|Q_n| < \theta^{(\xi-1)^n}|Q_1|^{(\xi-1)^n}|\beta|^{\frac{(m-1)(n(\xi+2)+2)(\xi-1)^n}{m}}.$$

We recall from [7] that under the ultrametric absolute value in $\mathbb{F}_q((x^{-1}))$, the continuants satisfy the metric identity $|Q_n| = q^{\gamma(Q_n)}$, which structurally yields the cumulative degree-sum relationship:

$$\gamma(Q_n) = \sum_{i=1}^{n}\gamma(A_i).$$

Furthermore,

$$\sum_{i=1}^{n}\gamma(A_i) < (\xi-1)^n\left[\log\theta + \log|Q_1| + \frac{(m-1)(n(\xi+2)+2)}{m}\log|\beta|\right].$$

Taking the logarithm again yields:

$$\log\left(\sum_{i=1}^{n}\gamma(A_i)\right) < n\log(\xi-1) + \log\left[\log\theta + \log|Q_1| + \frac{(m-1)(n(\xi+2)+2)}{m}\log|\beta|\right].$$

Upon dividing by n and taking the limit superior as $n \to +\infty$, the right-hand side converges to $\log(\xi-1)$, which is finite. This directly contradicts the hypothesis:

$$\limsup_{n\to+\infty}\frac{\log\left(\sum_{i=1}^{n}\gamma(A_i)\right)}{n} = +\infty.$$

We therefore conclude that α must be transcendental.

Example 3.3. Let α be defined by the continued β-fraction expansion with $A_n = \beta^{n^n}$. Since $\gamma(A_n) = n^n$, the partial sums of the degrees are dominated by the leading term, yielding

$$\sum_{i=1}^{n}\gamma(A_i) \geq n^n.$$

The condition for transcendence is satisfied as:

$$\frac{\log\left(\sum_{i=1}^{n}\gamma(A_i)\right)}{n} \geq \log n \xrightarrow[n\to\infty]{} +\infty.$$

Hence, α is a transcendental element in $\mathbb{F}_q((x^{-1}))$.

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