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ON EIGENVALUES AND SURJECTIVITY USING FIXED POINTS

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Abstract. Here first we will prove the existence of fixed points for a weakly continuous, strictly quasi bounded operator on a reflexive Banach space and a completely continuous, strictly quasi bounded operator on any normed linear space. Using these results we can deduce the existence of eigen values and surjectivity of quasi bounded operator in similar situations.

Key words: reflexive banach space, weakly continuous functions, completely continuous functions, quasi bounded functions, strictly quasi bounded functions.
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1. Introduction

In [2] G. Isac and S.Z. Nemeth and in [4] In- Sook Kim have proved some interesting results on fixed points, eigen values and surjectivity. Also in [1] we can find a result on surjectivity of a continuous, strictly quasi bounded function which maps each bounded subset of a normed space X into a compact subset of X. Here we will prove that any strictly quasi bounded, weakly continuous operator on a reflexive Banach space has a fixed point. Also we will prove that a strictly quasi bounded, completely continuous operator on any normed linear space has a fixed point. Using these fixed point theorems we will prove the existence of eigen values and surjectivity of some quasi bounded mappings. Before proving these results let us recall some important definitions and theorems

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Definition 1. *Let X and Y be normed spaces and $F : X \to Y$. $F$is weakly continuous at $x_0 \in X$if, for any sequence $\{x_n\}$ which converges weakly to x_0,the sequence $\{Fx_n\}$ converges weakly to Fx_0.*

Definition 2. *Let X and Y be metric spaces and $F : X \to Y$. $F$is completely continuous if the image of each bounded set in X is contained in a compact subset of Y.*

Definition 3. *Let X be a normed space and $F : X \to X$. F is quasi bounded if $\limsup_{\|x\| \to \infty} \frac{\|Fx\|}{\|x\|} < \infty$ and it is strictly quasi bounded if $\limsup_{\|x\| \to \infty} \frac{\|Fx\|}{\|x\|} < 1$.*

Theorem 1. *Let X be a reflexive Banach space,K a closed convex subset of X and Fa weakly continuous mapping of K into a bounded subset of K. Then F has a fixed point in K.[8]*

Theorem 2. *Let X be a normed linear space, $C \subset E$ be a convex set and let U be open in C such that $0 \in U$. Then each compact map $F : \overline{U} \to C$ has at least one of the following two properties*

(1) *F has a fixed point*
(2) *There exists $x \in \partial U$ and $\lambda \in (0, 1)$ such that $x = \lambda\, Fx$ where ∂U denote boundary of U.[3]*

Theorem 3. *A normed space X is reflexive iff every bounded sequence in X has a weak convergent subsequence.[1]*

Theorem 4. *If a sequence $\{x_n\}$ in a normed space X is weak convergent then it is bounded.[1]*

2. Main Results

Using the above mentioned definitions and theorems now we will prove our main results.First let us prove a fixed point theorem for a strictly quasi bounded operator on a reflexive Banach space.

Theorem 5. *Let X be reflexive Banach space. $F : X \to X$ be a weakly continuous and strictly quasi bounded operator on X. Then F has a fixed point in X.*

Proof. For $n = 1, 2, 3, ...$define $S_n = \{x \in X : \|x\| \leq n\}$. We will prove that $F(S_n) \subseteq S_n$ for some n.

If possible assume that $F(S_n)$ is not a subset of S_n, $\forall n$. Then for each $n = 1, 2, 3, ...$ $\exists x_n \in S_n$ such that

$$\|Fx_n\| > n \tag{1}$$

Now if $\{x_n\}$ is a bounded sequence in X, since X is reflexive by theorem(1.6) it has a weak convergent subsequence $\{x_{n_i}\}$. Then as F is weakly continuous $\{Fx_{n_i}\}$ is a weakly convergent sequence in X. Therefore by theorem 4, $\{Fx_{n_i}\}$ is a bounded sequence in X, which is a contradiction to (1).

Hence $\|x_n\| \to \infty$ as $n \to \infty$.

We have $\|Fx_n\| > n \geq \|x_n\| \quad \forall n.$

Therefore $\limsup_{\|x\| \to \infty} \frac{\|Fx\|}{\|x\|} \geq \limsup_{\|x_n\| \to \infty} \frac{\|Fx_n\|}{\|x_n\|} \geq 1$ which is a contradiction to the fact that $F$is strictly quasi bounded. Hence $F(S_n) \subseteq S_n$ for some n. Then using theorem 1, F has a fixed point in S_n and hence in X. □

Corollary 6. *Let X be reflexive Banach space. $F : X \to X$ be a weakly continuous and quasi bounded operator on X. Let $l = \limsup_{\|x\| \to \infty} \frac{\|Fx\|}{\|x\|}$. Then for each $\lambda > l$, λ is an eigen value of F provided $F(0) \neq 0$.*

Proof. For $x \in X$, define $G(x) = \frac{1}{\lambda}F(x)$ where $\lambda > l$.

Let $m = \limsup_{\|x\| \to \infty} \frac{\|Gx\|}{\|x\|}$. Then $m = \frac{l}{\lambda} < 1$. Hence G is a strictly quasi bounded, weakly continuous function from $X$to X. Then by theorem 5, G has a fixed point $x_0 \in X$. Further as $F(0) \neq 0$ we have $G(0) \neq 0$. Hence $x_0 \neq 0$.

ie., there exists a non zero element $x_0 \in X$ such that $Fx_0 = \lambda x_0$.

Therefore λ is an eigen value of F. □

Corollary 7. *Let X be reflexive Banach space. $F : X \to X$ be a weakly continuous and quasi bounded operator on X. Then $I - \frac{1}{\lambda}F$ is surjective for all $\lambda > l$ where $l = \limsup_{\|x\| \to \infty} \frac{\|Fx\|}{\|x\|}$.*

Proof. Let $y \in X$. Define $G(x) = y + \frac{1}{\lambda}F(x)$.

Let $m = \limsup_{\|x\| \to \infty} \frac{\|Gx\|}{\|x\|}$. Then $m \leq \frac{l}{\lambda} < 1$. Hence G is a strictly quasi bounded, weakly continuous function from $X$to $X$and hence by theorem 5 G has a fixed point $x_0 \in X$.

Then $y + \frac{1}{\lambda}Fx_0 = x_0$. ie., $\left(I - \frac{1}{\lambda}F\right) x_0 = y$.

Therefore $I - \frac{1}{\lambda}F$ is surjective. □

Before proving the next theorem let us discuss a very simple example which verifies our arguments.

Example 1. *Define $F : R \to R$ by $Fx = \frac{2}{2x^2+1}$.*

Then one can easily see that F is not a contraction, even it is not non expansive. But it is strictly quasi bounded and weakly continuous operator on R(Remember that since R is finite dimensional both weak convergence and *strong convergence coincides in R). Clearly Fhas a fixed point which lies in* $(0,1)$.

Further $\limsup_{\|x\|\to\infty} \frac{\|Fx\|}{\|x\|} = 0$ *and any real number* $\lambda > 0$ *is an eigen value of* F.

Theorem 8. *Let* X *be normed linear space,* $F : X \to X$ *be a completely continuous and strictly quasi bounded operator on* X. *Then* F *has a fixed point in* X.

Proof. Since F is strictly quasi bounded , $l = \limsup_{\|x\|\to\infty} \frac{\|Fx\|}{\|x\|} < 1$. Therefore there exists $r > 0$such that $\frac{\|Fx\|}{\|x\|} < 1 \quad \forall x \in X$ with $\|x\| \geq r$.

If possible suppose there exists $x \in X$ with $\|x\| = r$ and $\lambda \in (0,1)$ such that $x = \lambda Fx$. Then $\|x\| = |\lambda| \, \|Fx\| < \|Fx\|$ which is a contradiction.

Hence by non linear alternative of Fon $\{x : \|x\| \leq r\}$ (theorem 2) F has a fixed point. □

Corollary 9. *Let* X *be normed linear space,* $F : X \to X$ *be a completely continuous and quasi bounded operator on* X. *Let* $l = \limsup_{\|x\|\to\infty} \frac{\|Fx\|}{\|x\|}$. *Then for each* $\lambda > l$, λ *is an eigen value of* F *provided* $F(0) \neq 0$.

Corollary 10. *Let* X *be normed linear space,* $F : X \to X$ *be a completely continuous and quasi bounded operator on* X. *Then for all* $\lambda > l$, $I - \frac{1}{\lambda}F$ *is surjective ,where* $l = \limsup_{\|x\|\to\infty} \frac{\|Fx\|}{\|x\|}$.

Proofs of the above two corollaries are similar to the proofs of corollary 6 and corollary 7.

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