



Advancing Oscillation Theory for Neutral Delay Differential Equations

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Abstract

In applied mathematics, oscillation is widely recognized, as is its role in fields such as physics, engineering, and medicine. This paper proposes novel criteria for characterizing the oscillatory nature of delay differential equations that contain a mixture of neutral terms. The outcomes of this research expand and simplify existing conditions previously established in the literature. A practical illustration and a numerical analysis are provided as an example to demonstrate the practical significance of the primary findings.

Keywords: Oscillation Theory, Neutral Differential Equations, Several Deviating Arguments, Mixed Neutral Term.

2010 MSC: 35B05, 34C15, 34K40

1. Introduction

Neutral differential equations have become an increasingly popular area of study in recent years, with diverse applications in fields such as fluid dynamics, physics, chemistry, and biology. Researchers have invested considerable effort in investigating various phenomena in these areas, such as the impact of vibrating systems attached to elastic bars. Examples of such problems can be seen in the Euler and Taylor–Goldstein equations in fluid dynamics, as well as in the perturbed vertical velocity in stratified flow affected by viscosity.

Moreover, the oscillatory properties of neutral differential equations play a crucial role in application-oriented research, mechanical engineering, and civil engineering, which could potentially support the development of industries such as shipbuilding, airplane, and rocket manufacturing, as well as the creation of microelectromechanical systems (MEMS) and gyroscopes [1, 5].

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The focus of the paper is to examine the oscillatory properties of a specific category of second-order differential equations

$$\left(r(\vartheta) (z'(\vartheta))^\beta \right)' + \sum_{i=1}^m q_i(\vartheta) v^\beta(\lambda_i(\vartheta)) = 0, \tag{1.1}$$

where $\vartheta \geq \vartheta_0$, β is a quotient of odd positive integers and

$$z(\vartheta) = v(\vartheta) + p_1(\vartheta) v(\tau_1(\vartheta)) + p_2(\vartheta) v(\tau_2(\vartheta)),$$

and the conditions listed below are always presumed to hold

- (I₁) $p_1, p_2, q_i \in C([\vartheta_0, \infty), [0, \infty))$ and $q_i(\vartheta)$ is not identically zero for large ϑ ;
- (I₂) $r \in C([\vartheta_0, \infty), (0, \infty))$ and $\eta(\vartheta_0) = \int_{\vartheta_0}^\infty r^{-1/\beta}(s) ds < \infty$;
- (I₃) $\tau_1, \tau_2, \lambda_i \in C([\vartheta_0, \infty), \mathbb{R})$, $\tau_1(\vartheta) \leq \vartheta$, $\tau_2(\vartheta) \geq \vartheta$, $\lambda_i(\vartheta) \leq \vartheta$ and $\lim_{\vartheta \rightarrow \infty} \tau_1(\vartheta) = \lim_{\vartheta \rightarrow \infty} \tau_2(\vartheta) = \lim_{\vartheta \rightarrow \infty} \lambda_i(\vartheta) = \infty$.

If $v \in C^1[\vartheta_v, \infty)$, $\vartheta_v \geq \vartheta_0$, $r(\vartheta) (z'(\vartheta))^\beta \in C^1[\vartheta_v, \infty)$ for all $\vartheta_v \geq \vartheta_0$, and v satisfies (1.1) on $[\vartheta_v, \infty)$, then v is called a solution of (1.1). We consider only those solutions v of (1.1) which satisfy $\sup \{|v(\vartheta)| : \vartheta_c \leq \vartheta < \infty\} > 0$ for any $\vartheta_c \geq \vartheta_v$.

The topic of oscillation in neutral equations has been widely explored in various studies, as can be seen in the works cited in [7, 8, 3]. However, there are fewer studies that have explored oscillation in mixed neutral differential equations, as noted in [2, 6, 12]. In particular, Graef et al. in [4] investigated the damped second-order linear neutral differential equation of mixed type

$$\left(r(\vartheta) (v(\vartheta) + p_1(\vartheta) v(\tau_1(\vartheta)) + p_2(\vartheta) v(\tau_2(\vartheta)))' \right)' + p(\vartheta) z'(\vartheta) + q(\vartheta) v(\lambda(\vartheta)) = 0,$$

where $\vartheta \geq \vartheta_0 > 0$ and

$$\int_{\vartheta_0}^\infty \frac{1}{r(\vartheta)} \exp\left(-\int_{\vartheta_0}^\vartheta \frac{p(s)}{r(s)} ds\right) d\vartheta = \infty.$$

Li et al. [9] obtained the oscillatory conditions of the equation

$$\left(r(\vartheta) |z'(\vartheta)|^{\beta-1} z'(\vartheta) \right)' + q(\vartheta) |v(\lambda(\vartheta))|^{\beta-1} v(\lambda(\vartheta)) = 0,$$

where

$$z(\vartheta) = v(\vartheta) + \sum_{i=1}^m p_i(\vartheta) v(\tau_i(\vartheta)).$$

The oscillatory behavior of solutions to neutral differential equations was studied by Thandapani et al. in [11] and by Thandapani and Rama in [10]. The studied equation is represented as

$$\left(r(\vartheta) (z'(\vartheta))^\beta \right)' + q_1(\vartheta) v^\beta(\vartheta - \tau_1) + q_2(\vartheta) v^\gamma(\vartheta + \tau_2) = 0,$$

where $z(\vartheta) = v(\vartheta) + p_1(\vartheta) v(\vartheta - \lambda_1) + p_2(\vartheta) v(\vartheta + \lambda_2)$.

The aim of the current paper is to broaden the findings of previous studies and introduce new, sufficient conditions that guarantee that every solution of (1.1) oscillates. To derive these conditions, various methods, such as Riccati transformations, were used. The applicability of the main results is demonstrated through a specific example.

2. Oscillation results

Throughout this paper, we assume that:

$$\begin{aligned} \pi(u, s) &= \int_u^s r^{-1/\beta}(\varrho) \, d\varrho, \\ \Omega_1(\vartheta) &:= 1 - p_1(\vartheta) \frac{\pi(\tau_1(\vartheta), \infty)}{\pi(\vartheta, \infty)} - p_2(\vartheta), \\ \Omega_2(\vartheta) &:= 1 - p_1(\vartheta) - p_2(\vartheta) \frac{\pi(\vartheta_1, \tau_2(\vartheta))}{\pi(\vartheta_1, \vartheta)} \end{aligned}$$

and

$$\lambda(\vartheta) = \max \{ \lambda_i(\vartheta) : i = 1, 2, \dots, m \}.$$

Lemma 2.1. *Suppose that $\eta(s) := As - B(s - C)^{(\beta+1)/\beta}$, where A, B and C are real constants, $B > 0$ and β is a quotient of odd positive integers. Then,*

$$\eta(s^*) \leq \max_{s \in \mathbb{R}} \eta(s) = AC + \frac{\beta^\beta}{(\beta + 1)^{\beta+1}} A^{\beta+1} B^{-\beta}.$$

Lemma 2.2. *Suppose that v is a positive solution of (1.1). Then*

$$\left(\frac{z(\vartheta)}{\pi(\vartheta, \infty)} \right)' \geq 0, \tag{2.1}$$

and there exists a $\vartheta_1 \geq \vartheta_0$ such that

$$\left(\frac{z(\vartheta)}{\pi(\vartheta_1, \vartheta)} \right)' \leq 0, \tag{2.2}$$

for all $\vartheta \geq \vartheta_1$.

Proof. Suppose that there exists a $\vartheta_1 \geq \vartheta_0$ such that equation (1.1) has a positive solution v on $[\vartheta_0, \infty)$. Obviously, for all $\vartheta \geq \vartheta_1$, $z(\vartheta) \geq v(\vartheta) > 0$ and $\left(r(\vartheta) (z'(\vartheta))^\beta \right)' \leq 0$. Since

$$\left(r(\vartheta) (z'(\vartheta))^\beta \right)' = - \sum_{i=1}^m q_i(\vartheta) v^\beta(\lambda_i(\vartheta)) \leq 0,$$

therefore, $z' < 0$ or $z' > 0$.

(1) Assume that $z' < 0$ on $[\vartheta, \infty)$, we obtain

$$z(\vartheta) \geq - \int_\vartheta^\infty r^{-1/\beta}(s) r^{1/\beta}(s) z'(s) \, ds \geq -\pi(\vartheta, \infty) r^{1/\beta}(\vartheta) z'(\vartheta) \tag{2.3}$$

and so

$$\left(\frac{z(\vartheta)}{\pi(\vartheta, \infty)} \right)' = \frac{\pi(\vartheta, \infty) z'(\vartheta) + r^{-1/\beta}(\vartheta) z(\vartheta)}{(\pi(\vartheta, \infty))^2} \geq 0.$$

(2) Assume that $z' > 0$ on $[\vartheta_1, \vartheta)$, we obtain

$$z(\vartheta) \geq \int_{\vartheta_1}^\vartheta r^{-1/\beta}(s) r^{1/\beta}(s) z'(s) \, ds \geq \pi(\vartheta_1, \vartheta) r^{1/\beta}(\vartheta) z'(\vartheta),$$

it follows that

$$\left(\frac{z(\vartheta)}{\pi(\vartheta_1, \vartheta)} \right)' = \frac{\pi(\vartheta_1, \vartheta) z'(\vartheta) - r^{-1/\beta}(\vartheta) z(\vartheta)}{\pi^2(\vartheta_1, \vartheta)} \leq 0.$$

Therefore, the proof has been completed. □

Theorem 2.3. Assume that $\Omega_2(\vartheta) \geq \Omega_1(\vartheta) > 0$. If

$$\limsup_{\vartheta \rightarrow \infty} \int_{\vartheta_1}^{\vartheta} \frac{1}{r^{1/\beta}(u)} \left(\int_{\vartheta_1}^u \pi^\beta(s, \infty) \sum_{i=1}^m q_i(s) \Omega_1^\beta(\lambda_i(s)) ds \right)^{1/\beta} du = \infty, \tag{2.4}$$

then, (1.1) would be oscillatory.

Proof. Suppose, for the sake of contradiction, that there exists an eventually positive solution v for equation (1.1). Then there exists $\vartheta_1 \geq \vartheta_0$ such that $v(\tau_1(\vartheta))$, $v(\tau_2(\vartheta))$ and $v(\lambda_i(\vartheta))$ are positive for each $\vartheta \geq \vartheta_1$. Therefore, by utilizing (1.1) and the definition of z , we can deduce that $z(\vartheta) \geq v(\vartheta) > 0$ and $r(\vartheta)(z'(\vartheta))^\beta \leq 0$. Therefore, $z' < 0$ or $z' > 0$.

(1) Assume that $z'(\vartheta) < 0$ on $[\vartheta_1, \infty)$. By using Lemma 2.2, we have

$$z(\tau_1(\vartheta)) \leq \frac{\pi(\tau_1(\vartheta), \infty)}{\pi(\vartheta, \infty)} z(\vartheta),$$

based on the fact that $\tau_1(\vartheta) \leq \vartheta$. Therefore,

$$\begin{aligned} v(\vartheta) &= z(\vartheta) - p_1(\vartheta)v(\tau_1(\vartheta)) - p_2(\vartheta)v(\tau_2(\vartheta)) \\ &\geq z(\vartheta) - p_1(\vartheta)z(\tau_1(\vartheta)) - p_2(\vartheta)z(\tau_2(\vartheta)) \\ &\geq \left(1 - p_1(\vartheta) \frac{\pi(\tau_1(\vartheta), \infty)}{\pi(\vartheta, \infty)} - p_2(\vartheta) \right) z(\vartheta) \\ &= \Omega_1(\vartheta) z(\vartheta). \end{aligned}$$

Hence, (1.1) becomes

$$\begin{aligned} \left(r(\vartheta)(z'(\vartheta))^\beta \right)' &\leq - \sum_{i=1}^m q_i(\vartheta) v^\beta(\lambda_i(\vartheta)) \\ &\leq - \sum_{i=1}^m q_i(\vartheta) \Omega_1^\beta(\lambda_i(\vartheta)) z^\beta(\lambda_i(\vartheta)) \\ &\leq -z^\beta(\vartheta) \sum_{i=1}^m q_i(\vartheta) \Omega_1^\beta(\lambda_i(\vartheta)). \end{aligned} \tag{2.5}$$

Since $\left(r(\vartheta)(z'(\vartheta))^\beta \right)' \leq 0$, we have

$$r(\vartheta)(z'(\vartheta))^\beta \leq r(\vartheta_1)(z'(\vartheta_1))^\beta = -N < 0, \tag{2.6}$$

for all $\vartheta \geq \vartheta_1$, from (2.3) and (2.6), we have

$$z^\beta(\vartheta) \geq N\pi^\beta(\vartheta, \infty) \text{ for all } \vartheta \geq \vartheta_1. \tag{2.7}$$

Combining (2.5) with (2.7) yields

$$\left(r(\vartheta)(z'(\vartheta))^\beta \right)' \leq -N\pi^\beta(\vartheta, \infty) \sum_{i=1}^m q_i(\vartheta) \Omega_1^\beta(\lambda_i(\vartheta)), \tag{2.8}$$

for all $\vartheta \geq \vartheta_1$. Integrating (2.8) from ϑ_1 to ϑ , we obtain

$$\begin{aligned} r(\vartheta)(z'(\vartheta))^\beta &\leq r(\vartheta_1)(z'(\vartheta_1))^\beta - N \int_{\vartheta_1}^{\vartheta} \pi^\beta(s, \infty) \sum_{i=1}^m q_i(s) \Omega_1^\beta(\lambda_i(s)) ds \\ &\leq -N \int_{\vartheta_1}^{\vartheta} \pi^\beta(s, \infty) \sum_{i=1}^m q_i(s) \Omega_1^\beta(\lambda_i(s)) ds. \end{aligned}$$

By integrating the previous inequality from ϑ_1 to ϑ , we get

$$z(\vartheta) \leq z(\vartheta_1) - N^{1/\beta} \int_{\vartheta_1}^{\vartheta} \frac{1}{r^{1/\beta}(u)} \left(\int_{\vartheta_1}^u \pi^\beta(s, \infty) \sum_{i=1}^m q_i(s) \Omega_1^\beta(\lambda_i(s)) ds \right)^{1/\beta} du.$$

which contradicts condition (2.4) at $\vartheta \rightarrow \infty$.

(2) Assume that $z'(\vartheta) > 0$ on $[\vartheta_1, \infty)$. From Lemma 2.2, we arrive at

$$z(\tau_2(\vartheta)) \leq \frac{\pi(\vartheta_1, \tau_2(\vartheta))}{\pi(\vartheta_1, \vartheta)} z(\vartheta). \tag{2.9}$$

From the definition of z , we obtain

$$\begin{aligned} v(\vartheta) &= z(\vartheta) - p_1(\vartheta)v(\tau_1(\vartheta)) - p_2(\vartheta)v(\tau_2(\vartheta)) \\ &\geq z(\vartheta) - p_1(\vartheta)z(\tau_1(\vartheta)) - p_2(\vartheta)z(\tau_2(\vartheta)). \end{aligned} \tag{2.10}$$

Using (2.9) and $z(\tau_1(\vartheta)) \leq z(\vartheta)$ where $\tau_1(\vartheta) < \vartheta$ in (2.10), we obtain

$$\begin{aligned} v(\vartheta) &\geq z(\vartheta) \left(1 - p_1(\vartheta) - p_2(\vartheta) \frac{\pi(\vartheta_1, \tau_2(\vartheta))}{\pi(\vartheta_1, \vartheta)} \right) \\ &\geq \Omega_2(\vartheta) z(\vartheta). \end{aligned} \tag{2.11}$$

Thus, (1.1) becomes

$$\begin{aligned} \left(r(\vartheta) (z'(\vartheta))^\beta \right)' &\leq - \sum_{i=1}^m q_i(\vartheta) v^\beta(\lambda_i(\vartheta)) \\ &\leq - \sum_{i=1}^m q_i(\vartheta) \Omega_2^\beta(\lambda_i(\vartheta)) z^\beta(\lambda_i(\vartheta)) \\ &\leq -z^\beta(\lambda(\vartheta)) \sum_{i=1}^m q_i(\vartheta) \Omega_2^\beta(\lambda_i(\vartheta)). \end{aligned} \tag{2.12}$$

On the other hand, it follows from (2.4) and (I₂) that

$$\int_{\vartheta_1}^{\vartheta} \pi^\beta(s, \infty) \sum_{i=1}^m q_i(s) \Omega_1^\beta(\lambda_i(s)) ds$$

must be unbounded. Further, since $\pi'(\vartheta, \infty) < 0$, it is easy to see that

$$\int_{\vartheta_1}^{\vartheta} \sum_{i=1}^m q_i(s) \Omega_1^\beta(\lambda_i(s)) ds \rightarrow \infty \text{ as } \vartheta \rightarrow \infty. \tag{2.13}$$

An integration of (2.12) from ϑ_2 to ϑ yields

$$\begin{aligned} r(\vartheta) (z'(\vartheta))^\beta &\leq r(\vartheta_2) (z'(\vartheta_2))^\beta - \int_{\vartheta_2}^{\vartheta} z^\beta(\lambda(s)) \sum_{i=1}^m q_i(s) \Omega_2^\beta(\lambda_i(s)) ds \\ &\leq r(\vartheta_2) (z'(\vartheta_2))^\beta - z^\beta(\lambda(\vartheta_2)) \int_{\vartheta_2}^{\vartheta} \sum_{i=1}^m q_i(s) \Omega_2^\beta(\lambda_i(s)) ds. \end{aligned}$$

Since $\Omega_2(\vartheta) > \Omega_1(\vartheta)$, we get

$$r(\vartheta) (z'(\vartheta))^\beta \leq r(\vartheta_2) (z'(\vartheta_2))^\beta - z^\beta(\lambda(\vartheta_2)) \int_{\vartheta_2}^{\vartheta} \sum_{i=1}^m q_i(s) \Omega_1^\beta(\lambda_i(s)) ds, \tag{2.14}$$

and from (2.13) and (2.14) we get a contradiction to the positivity of $z'(\vartheta)$ as $\vartheta \rightarrow \infty$. Thus, the proof is complete. □

Theorem 2.4. *Suppose that $\Omega_2(\vartheta) \geq \Omega_1(\vartheta) > 0$. If*

$$\limsup_{\vartheta \rightarrow \infty} \pi^\beta(\vartheta, \infty) \int_{\vartheta_1}^{\vartheta} \sum_{i=1}^m q_i(s) \Omega_1^\beta(\lambda_i(s)) ds > 1, \tag{2.15}$$

then, (1.1) is oscillatory.

Proof. Suppose, to the contrary, that there exists an eventually positive solution v for (1.1). Then there exists $\vartheta_1 \geq \vartheta_0$ such that $v(\tau_1(\vartheta)) > 0$, $v(\tau_2(\vartheta)) > 0$ and $v(\lambda_i(\vartheta)) > 0$ for all $\vartheta \geq \vartheta_1$.

(1) Assume that $z' < 0$ on $[\vartheta_1, \infty)$. Integrating (2.5) from ϑ_1 to ϑ , we have

$$\begin{aligned} r(\vartheta) (z'(\vartheta))^\beta &\leq r(\vartheta_1) (z'(\vartheta_1))^\beta - \int_{\vartheta_1}^{\vartheta} z^\beta(s) \sum_{i=1}^m q_i(s) \Omega_1^\beta(\lambda_i(s)) ds \\ &\leq -z^\beta(\vartheta) \int_{\vartheta_1}^{\vartheta} \sum_{i=1}^m q_i(s) \Omega_1^\beta(\lambda_i(s)) ds. \end{aligned} \tag{2.16}$$

Using (2.3) in (2.16), we obtain

$$-r(\vartheta) (z'(\vartheta))^\beta \geq -\pi^\beta(\vartheta, \infty) r(\vartheta) (z'(\vartheta))^\beta \int_{\vartheta_1}^{\vartheta} \sum_{i=1}^m q_i(s) \Omega_1^\beta(\lambda_i(s)) ds. \tag{2.17}$$

By dividing both sides of inequality (2.17) by $-r(\vartheta) (z'(\vartheta))^\beta$ and taking the limit superior, we obtain

$$\limsup_{\vartheta \rightarrow \infty} \pi^\beta(\vartheta, \infty) \int_{\vartheta_1}^{\vartheta} \sum_{i=1}^m q_i(s) \Omega_1^\beta(\lambda_i(s)) ds \leq 1,$$

which contradicts condition (2.15).

(2) Assume that $z' > 0$ on $[\vartheta_1, \infty)$. As $\pi(\vartheta, \infty) < \infty$ and (2.15) holds, (2.13) must also hold. The remainder of the proof then follows similarly to the proof of Theorem 2.3; thus, the proof is complete. \square

Theorem 2.5. *Suppose that $\Omega_2(\vartheta) > 0$, $\Omega_1(\vartheta) > 0$ and $r' > 0$. If there exist functions $\rho, \varrho \in C^1([\vartheta_0, \infty), (0, \infty))$ and $\vartheta \in [\vartheta_0, \infty)$ such that*

$$\limsup_{\vartheta \rightarrow \infty} \frac{\pi^\beta(\vartheta, \infty)}{\varrho(\vartheta)} \int_{\vartheta_1}^{\vartheta} \left(\varrho(s) \sum_{i=1}^m q_i(s) \Omega_1^\beta(\lambda_i(s)) - \frac{r(s)}{(\beta + 1)^{\beta+1}} \frac{(\varrho'(s))^{\beta+1}}{(\varrho(s))^\beta} \right) ds > 1 \tag{2.18}$$

and

$$\limsup_{\vartheta \rightarrow \infty} \int_{\vartheta_2}^{\vartheta} \left(\rho(s) \sum_{i=1}^m q_i(s) \Omega_2^\beta(\lambda_i(s)) - \frac{1}{(\beta + 1)^{\beta+1}} \frac{r(s) (\rho'(s))^{\beta+1}}{\rho^\beta(s) (\lambda'(s))^\beta} \right) ds = \infty, \tag{2.19}$$

then, (1.1) is oscillatory.

Proof. Suppose, to the contrary, that there exists an eventually positive solution v for (1.1). Then there exists $\vartheta_1 \geq \vartheta_0$ such that $v(\tau_1(\vartheta)) > 0$, $v(\tau_2(\vartheta)) > 0$ and $v(\lambda_i(\vartheta)) > 0$ for all $\vartheta \geq \vartheta_1$.

(1) Assume that $z' < 0$ on $[\vartheta_1, \infty)$. By following the same steps as in the proof of Theorem 2.3, it can be deduced that z satisfies the equation (2.5). Now, we define the Riccati function

$$\varpi(\vartheta) = \varrho(\vartheta) \left(\frac{r(\vartheta) (z'(\vartheta))^\beta}{z^\beta(\vartheta)} + \frac{1}{\pi^\beta(\vartheta, \infty)} \right) \text{ on } [\vartheta_1, \infty). \tag{2.20}$$

from (2.3), we have $\varpi \geq 0$ on $[\vartheta_1, \infty)$. Differentiating (2.20), we get

$$\begin{aligned} \varpi'(\vartheta) &= \frac{\varrho'(\vartheta)}{\varrho(\vartheta)} \varpi(\vartheta) + \varrho(\vartheta) \frac{\left(r(\vartheta) (z'(\vartheta))^\beta\right)'}{z^\beta(\vartheta)} - \beta \varrho(\vartheta) r(\vartheta) \left(\frac{z'(\vartheta)}{z(\vartheta)}\right)^{\beta+1} \\ &\quad + \frac{\beta \varrho(\vartheta)}{r^{1/\beta}(\vartheta) \pi^{\beta+1}(\vartheta, \infty)} \\ &\leq \frac{\varrho'(\vartheta)}{\varrho(\vartheta)} \varpi(\vartheta) + \varrho(\vartheta) \frac{\left(r(\vartheta) (z'(\vartheta))^\beta\right)'}{z^\beta(\vartheta)} \\ &\quad - \frac{\beta}{(\varrho(\vartheta) r(\vartheta))^{1/\beta}} \left(\varpi(\vartheta) - \frac{\varrho(\vartheta)}{\pi^\beta(\vartheta, \infty)}\right)^{(\beta+1)/\beta} \\ &\quad + \frac{\beta \varrho(\vartheta)}{r^{1/\beta}(\vartheta) \pi^{\beta+1}(\vartheta, \infty)}. \end{aligned} \tag{2.21}$$

Combining (2.5) and (2.21), we have

$$\begin{aligned} \varpi'(\vartheta) &\leq -\frac{\beta}{(\varrho(\vartheta) r(\vartheta))^{1/\beta}} \left(\varpi(\vartheta) - \frac{\varrho(\vartheta)}{\pi^\beta(\vartheta, \infty)}\right)^{(\beta+1)/\beta} \\ &\quad - \varrho(\vartheta) \sum_{i=1}^m q_i(\vartheta) \Omega_1^\beta(\lambda_i(\vartheta)) \\ &\quad + \frac{\beta \varrho(\vartheta)}{r^{1/\beta}(\vartheta) \pi^{\beta+1}(\vartheta, \infty)} + \frac{\varrho'(\vartheta)}{\varrho(\vartheta)} \varpi(\vartheta). \end{aligned} \tag{2.22}$$

Using Lemma 2.1 with $A := \varrho'(\vartheta) / \varrho(\vartheta)$, $B := \beta (\varrho(\vartheta) r(\vartheta))^{-1/\beta}$, $C := \varrho(\vartheta) / \pi^\beta(\vartheta, \infty)$ and $s := \varpi$, we get

$$\begin{aligned} \frac{\varrho'(\vartheta)}{\varrho(\vartheta)} \varpi(\vartheta) - \frac{\beta}{(\varrho(\vartheta) r(\vartheta))^{1/\beta}} \left(\varpi(\vartheta) - \frac{\varrho(\vartheta)}{\pi^\beta(\vartheta, \infty)}\right)^{(\beta+1)/\beta} \\ \leq \frac{1}{(\beta+1)^{\beta+1}} r(\vartheta) \frac{(\varrho'(\vartheta))^{\beta+1}}{(\varrho(\vartheta))^\beta} + \frac{\varrho'(\vartheta)}{\pi^\beta(\vartheta, \infty)}. \end{aligned}$$

Now, in view of (2.22), we have

$$\begin{aligned} \varpi'(\vartheta) &\leq \frac{\varrho'(\vartheta)}{\pi^\beta(\vartheta, \infty)} + \frac{1}{(\beta+1)^{\beta+1}} r(\vartheta) \frac{(\varrho'(\vartheta))^{\beta+1}}{(\varrho(\vartheta))^\beta} - \varrho(\vartheta) \sum_{i=1}^m q_i(\vartheta) \Omega_1^\beta(\lambda_i(\vartheta)) \\ &\quad + \frac{\beta \varrho(\vartheta)}{r^{1/\beta}(\vartheta) \pi^{\beta+1}(\vartheta, \infty)} \\ &\leq -\varrho(\vartheta) \sum_{i=1}^m q_i(\vartheta) \Omega_1^\beta(\lambda_i(\vartheta)) + \left(\frac{\varrho(\vartheta)}{\pi^\beta(\vartheta, \infty)}\right)' \\ &\quad + \frac{r(\vartheta)}{(\beta+1)^{\beta+1}} \frac{(\varrho'(\vartheta))^{\beta+1}}{(\varrho(\vartheta))^\beta}. \end{aligned} \tag{2.23}$$

Integrating (2.23) from ϑ_2 to ϑ , we arrive at

$$\begin{aligned} \int_{\vartheta_2}^{\vartheta} \left(\varrho(s) \sum_{i=1}^m q_i(s) \Omega_1^\beta(\lambda_i(s)) - \frac{r(s)}{(\beta+1)^{\beta+1}} \frac{(\varrho'(s))^{\beta+1}}{(\varrho(s))^\beta}\right) ds \\ \leq \left(\frac{\varrho(\vartheta)}{\pi^\beta(\vartheta, \infty)} - \varpi(\vartheta)\right) \Big|_{\vartheta_2}^{\vartheta} \\ \leq -\left(\varrho(\vartheta) \frac{r(\vartheta) (z'(\vartheta))^\beta}{z^\beta(\vartheta)}\right) \Big|_{\vartheta_2}^{\vartheta}. \end{aligned} \tag{2.24}$$

From (2.3), we have

$$-\frac{r^{1/\beta}(\vartheta) z'(\vartheta)}{z(\vartheta)} \leq \frac{1}{\pi(\vartheta, \infty)}.$$

Now, in view of (2.24), we get

$$\frac{\pi^\beta(\vartheta, \infty)}{\varrho(\vartheta)} \int_{\vartheta_2}^{\vartheta} \left(\varrho(s) \sum_{i=1}^m q_i(s) \Omega_1^\beta(\lambda_i(s)) - \frac{r(s)}{(\beta+1)^{\beta+1}} \frac{(\varrho'(s))^{\beta+1}}{(\varrho(s))^\beta} \right) ds \leq 1. \tag{2.25}$$

Taking the limit superior of (2.25), we are led to a contradiction with (2.18).

(2) Assume that $z'(\vartheta) > 0$ on $[\vartheta_1, \infty)$. Introducing the concept of the Riccati function,

$$\varphi(\vartheta) = \rho(\vartheta) \frac{r(\vartheta) (z'(\vartheta))^\beta}{z^\beta(\lambda(\vartheta))}, \quad \text{on } [\vartheta_1, \infty), \tag{2.26}$$

we observe that $\varphi \geq 0$ on $[\vartheta_1, \infty)$. Differentiating (2.26), we obtain

$$\begin{aligned} \varphi'(\vartheta) &= \frac{\rho'(\vartheta)}{\rho(\vartheta)} \varphi(\vartheta) + \rho(\vartheta) \frac{(r(\vartheta)(z'(\vartheta))^\beta)'}{z^\beta(\lambda(\vartheta))} \\ &\quad - \beta \rho(\vartheta) r(\vartheta) \frac{(z'(\vartheta))^\beta z'(\lambda(\vartheta)) \lambda'(\vartheta)}{z^{\beta+1}(\lambda(\vartheta))}. \end{aligned} \tag{2.27}$$

Combining (2.12) and (2.27), we have

$$\varphi'(\vartheta) \leq \frac{\rho'(\vartheta)}{\rho(\vartheta)} \varphi(\vartheta) - \rho(\vartheta) \sum_{i=1}^m q_i(\vartheta) \Omega_2^\beta(\lambda_i(\vartheta)) - \beta \rho(\vartheta) r(\vartheta) \frac{(z'(\vartheta))^\beta z'(\lambda(\vartheta)) \lambda'(\vartheta)}{z^{\beta+1}(\lambda(\vartheta))}.$$

Since $(r(\vartheta) (z'(\vartheta))^\beta)' < 0$ and $\lambda(\vartheta) \leq \vartheta$, we obtain

$$\varphi'(\vartheta) \leq \frac{\rho'(\vartheta)}{\rho(\vartheta)} \varphi(\vartheta) - \rho(\vartheta) \sum_{i=1}^m q_i(\vartheta) \Omega_2^\beta(\lambda_i(\vartheta)) - \beta \rho(\vartheta) r(\vartheta) \lambda'(\vartheta) \frac{(z'(\vartheta))^{\beta+1}}{z^{\beta+1}(\lambda(\vartheta))},$$

from (2.26), we have

$$\varphi'(\vartheta) \leq \frac{\rho'(\vartheta)}{\rho(\vartheta)} \varphi(\vartheta) - \rho(\vartheta) \sum_{i=1}^m q_i(\vartheta) \Omega_2^\beta(\lambda_i(\vartheta)) - \frac{\beta \lambda'(\vartheta)}{\rho^{1/\beta}(\vartheta) r^{1/\beta}(\vartheta)} \varphi^{(\beta+1)/\beta}(\vartheta).$$

Using the inequality

$$KG - NG^{(\beta+1)/\beta} \leq \frac{\beta^\beta}{(\beta+1)^{\beta+1}} \frac{K^{\beta+1}}{N^\beta}, \quad N > 0, \tag{2.28}$$

with $K = \rho'(\vartheta) / \rho(\vartheta)$, $N = \beta \lambda'(\vartheta) / (\rho^{1/\beta}(\vartheta) r^{1/\beta}(\vartheta))$ and $G = \varphi$, we have

$$\varphi'(\vartheta) \leq -\rho(\vartheta) \sum_{i=1}^m q_i(\vartheta) \Omega_2^\beta(\lambda_i(\vartheta)) + \frac{1}{(\beta+1)^{\beta+1}} \frac{r(\vartheta) (\rho'(\vartheta))^{\beta+1}}{\rho^\beta(\vartheta) (\lambda'(\vartheta))^\beta}. \tag{2.29}$$

Integrating (2.29) from ϑ_2 to ϑ , we have

$$\int_{\vartheta_2}^{\vartheta} \left(\rho(s) \sum_{i=1}^m q_i(s) \Omega_2^\beta(\lambda_i(s)) - \frac{1}{(\beta+1)^{\beta+1}} \frac{r(s) (\rho'(s))^{\beta+1}}{\rho^\beta(s) (\lambda'(s))^\beta} \right) ds \leq \varphi(\vartheta_2). \tag{2.30}$$

By taking the limit superior of (2.30), a contradiction with (2.19) is produced, thus completing the proof. \square

2.1. Practical results

In this section, we provide and discuss an example to examine the applicability of our results.

Example 2.6. Consider the neutral equation

$$\left(\vartheta^2 \left(v(\vartheta) + \frac{1}{7}v\left(\frac{\vartheta}{3}\right) + \frac{1}{2}v(3\vartheta) \right) \right)' + a_0\vartheta v\left(\frac{\vartheta}{3}\right) + a_0\vartheta v\left(\frac{\vartheta}{2}\right) = 0, \tag{2.31}$$

where $\beta = 1$ and $\vartheta \geq \frac{13}{6}$. Here, $r(\vartheta) = \vartheta^2$, $p_1(\vartheta) = 1/7$, $p_2(\vartheta) = 1/2$, $\tau_1(\vartheta) = \vartheta/3$, $\tau_2(\vartheta) = 3\vartheta$, $\lambda_1(\vartheta) = \vartheta/3$, $\lambda_2(\vartheta) = \vartheta/2$ and $q_1(\vartheta) = q_2(\vartheta) = a_0\vartheta$. Thus, we have

$$\eta(\vartheta) = \int_{\vartheta}^{\infty} r^{-1/\beta}(s) ds = \frac{1}{\vartheta}.$$

It is easy to verify the conditions $(I_1) - (I_3)$. Moreover, we find

$$\Omega_1(\vartheta) = 1 - p_1(\vartheta) \frac{\pi(\tau_1(\vartheta), \infty)}{\pi(\vartheta, \infty)} - p_2(\vartheta) = 1 - \frac{3}{7} - \frac{1}{2} > 0$$

and

$$\Omega_2(\vartheta) = 1 - p_1(\vartheta) - p_2(\vartheta) \frac{\pi(\vartheta_1, \tau_2(\vartheta))}{\pi(\vartheta_1, \vartheta)} = 1 - \frac{1}{7} - \frac{1}{2} \left(\frac{\vartheta - \frac{1}{3}}{\vartheta - 1} \right)$$

and so $\Omega_2(\vartheta) \geq \Omega_1(\vartheta) > 0$.

Now, from Theorem 2.3, we have

$$\begin{aligned} \limsup_{\vartheta \rightarrow \infty} \int_{\vartheta_1}^{\vartheta} \frac{1}{r^{1/\beta}(u)} \left(\int_{\vartheta_1}^u \pi^{\beta}(s, \infty) \sum_{i=1}^m q_i(s) \Omega_1^{\beta}(\lambda_i(s)) ds \right)^{1/\beta} du \\ = \limsup_{\vartheta \rightarrow \infty} \int_{\vartheta_1}^{\vartheta} \frac{1}{u^2} \left(\int_{\vartheta_1}^u \frac{2a_0}{14} ds \right) du = \infty, \end{aligned}$$

Thus, equation (2.31) is oscillatory.

Conclusion

In this study, various techniques and criteria were employed to analyze the oscillatory behavior of second-order noncanonical differential equations with a mixed neutral term. The results obtained extend and complement the previous findings in the related literature. It would be interesting to further the results of this paper to higher-order differential equations of the form:

$$\left(r(\vartheta) \left(z^{(n)}(\vartheta) \right)^{\beta} \right)' + \sum_{i=1}^m q_i(\vartheta) v^{\beta}(\lambda_i(\vartheta)) = 0,$$

where $z(\vartheta) = v(\vartheta) + p_1(\vartheta) v(\tau_1(\vartheta)) + p_2(\vartheta) v(\tau_2(\vartheta))$.

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