Available online at http://jprm.sms.edu.pk/
Journal of Prime Research in Mathematics, 22(2) (2026), 151–161

Journal of Prime Research in Mathematics

The Game Chromatic Number of Some Families of Direct Graph Products of Graphs

Mohammad Reza Salmanpour[a], Zahid Raza[a,*]

[a] *Department of Mathematics, University of Sharjah, Sharjah-27272, UAE.*

Abstract

Given a graph G, the graph coloring game is a non-cooperative two player game such that the two players take turns properly coloring the graph. Alice wins if all vertices receive a proper coloring, otherwise Bob wins. A proper coloring is a type of coloring such that no two adjacent vertices are labelled with the same color. Our object of study is the game chromatic number $\mathcal{X}_g(G)$ which is the least number of colors needed for Alice to have a winning strategy for some graph G. We will mainly consider the game chromatic number of the direct products of stars, cycles, complete graphs, friendship graphs and paths in this paper.

Keywords: Graph coloring; chromatic number; direct graph product.
2010 Mathematics Subject Classification: 05C10, 05C35, 05C57

1. Background

A graph G is an ordered pair (V, E) comprised of two sets, the set of vertices V and the set of edges E. Since we will be dealing with simple, finite undirected graphs only, we can define an edge as an unordered pair (v_1, v_2) (or written as v_1v_2) comprised of two vertices, and in this case, we call v_1 and v_2 the end points of said edge. Two vertices v_1 and v_2 are adjacent if they share an edge, or in other words, there exists an edge defined as (v_1, v_2). To properly colour such a graph, we label the vertices using the elements of a colour set C such that no two adjacent vertices share the same colour. The chromatic number of G, defined as $\chi(G)$, is the least amount of colours needed to fully colour a graph G.

We now turn our attention to the main subject of this research paper, that being the graph coloring game. This is a non-cooperative game involving a graph G and a colour set C. There are two players, Alice and Bob. For Alice to win, she needs to properly colour all the vertices of the graph G with all the elements of C. Otherwise, Bob wins. The game chromatic number of G is the lowest amount of colors that Alice needs to win, and it is denoted by $\chi_g(G)$. We can also see that $\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$. To study the parameter $\chi_g(G)$, readers are referred to [11, 5, 8, 10, 4].

In this paper, the graphs that we will be dealing with are direct graph products. They are defined as follows:

*Corresponding author
Email addresses: `msalman222001@outlook.com` (Mohammad Reza Salmanpour), `zraza@sharjah.ac.ae` (Zahid Raza)

Received : 01 June 2026*; Accepted:* 07 July 2026*; Published Online:* 29 August 2026

Consider two graphs G and H. We denote the direct product of G and H as $G \times H$, whose vertex set is defined as $V(G \times H) = V(G) \times V(H)$. Any two vertices $(u_1, v_1) \in V(G \times H)$ and $(u_2, v_2) \in V(G \times H)$ form an edge if and only if

$$u_1u_2 \in E(G) \quad \text{and} \quad v_1v_2 \in E(H).$$

Note that the direct product is commutative up to isomorphism, that is,

$$G \times H \cong H \times G.$$

Often times when we want to find such bounds, there are strategies for Alice to win that revolve around coloring a handful of 'dangerous' vertices depending on the value of the game chromatic number. We also would like to introduce an object that can define this, which we called the danger zone. The danger zone of a graph G and a given game chromatic number $\chi_g(G)$, denoted $DZ(G, \chi_g(G))$, is a set of vertices such that for all $v \in DZ(G, \chi_g(G))$, $deg(v) \geq \chi_g(G)$.

A graph G is an ordered pair such that $G = (V(G), E(G))$. The set $V(G)$ is called the Vertex Set of G, and is often written as V. The set $E(G)$ is called the Edge Set of G, often denoted as E. The set $E(G)$ is composed of ordered pairs of vertices. Take for example $(v, w) \in E(G)$: v and w are said to share an edge, where v and w are endpoints. Any two vertices that share an edge are said to be adjacent or neighbours. The neighbourhood of a vertex v is the set of all vertices that neighbour v, and it is denoted as Neighbourhood(v). The degree of a vertex v, denoted as $deg(v)$, is the number of vertices that neighbour v.

Consider a graph G. We define a color set of G to be

$$C(G) = \{c_1, c_2, \ldots, c_k\};$$

such that any element of $C(G)$ is called a color. By properly coloring G, we mean that we are assigning a color to a vertex of G such that none of its neighbours have been assigned with the same color. A colored vertex is a vertex of G that has been assigned with a color, whilst an uncolored vertex is a vertex that has not been assigned any color. By a fully colored graph G or a fully colored set of vertices, we mean that no two vertices that share an edge have the same assigned color. We will refer to any instance of proper coloring as coloring instead.

We will now define the two-player graph coloring game. Consider a two-player, non-cooperative, turn-based game, with Player 1 starting first. Let G be a simple, undirected graph and let $C(G) = c_1, c_2, \ldots, c_k$ be the color set of G. Let Player 1 be named Alice and let Player 2 be named Bob. On their turns, the only action available to Alice and Bob is properly coloring some uncolored vertex of G. The winning condition for Alice and Bob are the following:

- Alice: Graph G must be fully colored with $|C(G)| = k$ colors
- Bob: At least one vertex must be uncolorable, meaning that there are already k colors assigned to the neighbours of said vertex and there are no legal moves left.

With that, we can say that a winning strategy of some player is a series of actions which leads to the outcome of a player winning the game.

The game chromatic number $\chi_g(G)$ is the least number of colors which Alice needs to win. We can see that for $\chi_g(G) = k$, the following necessary and sufficient conditions apply: Alice has a winning strategy with k colors, thus defining $\chi_g(G) \leq k$ Bob has a winning strategy with $k-1$ colors or less, thus defining $\chi_g(G) \geq k$

Let us now consider the Danger Zone of a graph G with a color set $C(G)$.

Let G be a graph and let $C(G) = c_1, c_2, \ldots, c_k$ be a color set, and let the graph coloring game be played. The Danger Zone of some graph G with a color set $C(G)$, denoted as $DG(G, C(G))$, is the set of all vertices such that $deg(v) \geq |C(G)|$.

The Danger Zone proves useful to finding winning strategies for either Alice or Bob with a given number of colors $|C(G)|$, since those are the vertices that can be surrounded by $|C(G)|$ colors, and thus strategies could be built with those vertices in mind. However, sometimes the Danger Zone doesn't prove as useful, like when all the vertices of a graph G are part of the Danger Zone.

With regards to recent studies surrounding the game chromatic number, one common topic one might see is the game chromatic number of different classes of graph products. Some examples being [6] for Cartesian graph product classes, [6], [3], [7] for Corona graph product classes, [9] for strong graph product classes, [1] for lexicographic graph product classes and [2] for direct graph product classes. With this in mind however, there are some graph products who still have largely unexplored classes. Thus, we decided to consider some direct graph product classes, namely those involving stars, complete graphs, cycles, paths and friendship graphs.

Let us thus remind ourselves of the definition of the direct graph product. Let G and H be graphs and let $V(G \times H) = V(G) \times V(H)$. The direct product $G \times H$ is the graph whose vertex set is $V(G) \times V(H)$, where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if

$$u_1u_2 \in E(G) \quad \text{and} \quad v_1v_2 \in E(H).$$

2. Results

Throughout this section, the main results of this paper will be proved by using the necessary and sufficient conditions for some game chromatic number $\chi_g(G) = k$ and creating bounds. However, we will also sometimes use the inequality $\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$, where Δ denotes the highest degree count of G.

Theorem 2.1. *The game chromatic number of products of two cycles is given as:*

$$\chi_g(C_n \times C_m) = \begin{cases} 1 & \text{if } m = 1 \\ 2 & \text{if } n = 2 \text{ and } m = 2 \\ 3 & \text{if } n \geq 2 \text{ and } m = 3 \\ 4 & \text{if } n \geq 3 \text{ and } m \geq 3 \end{cases}$$

Proof. We will consider the following cases:
Case 1: $m = 1$
Taking the direct product of any graph G with C_1 produces a disconnected graph $G \times C_1$, meaning that $V(G \times C_1)$ is an independent set. Because of this, by setting the color set such that $C = \{c_1\}$, it means that $x \notin DG(G \times C_1)$, $\forall x \in DG(G \times C_1)$. Thus Alice automatically wins with $|C| = 1$ colors, setting $X(G \times C_1) \leq 1$.
Meanwhile, Bob wins automatically with a color set of $C = \{\}$, meaning $X(G \times C_1) \geq 1$
So, $X(G \times C_1) = 1$, making $X(C_n \times C_1) = 1$

Case 2: $n = 2$ *and* $m = 2$
$K_2 \times K_2$ produces a disconnected graph consisting of two P_2 components. Here, $\Delta(K_2 \times K_2) = 1$, which means that by the inequality $\chi(G) \leq \chi_g(G) \leq \Delta(G)+1$, we have $\chi_g(K_2 \times K_2)$. To show that $X(K_2 \times K_2) \geq 2$, we show that Bob has a winning strategy with 1 color. By setting $C = \{c_1\}$, Alice automatically loses by coloring a vertex v_1 of $K_2 \times K_2$ with c_1, since $\exists v_2$ such that $v_2 \in Neighbourhood(v_1)$ and such that v_2 is not colored. Thus, Bob wins with 1 color, meaning $2 \leq \chi_g(K_2 \times K_2)$ Thus, $\chi_g(K_2 \times K_2) = 2$

Case 3: $n > 2$ *and* $m = 2$
Consider the inequality $\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$. Since $\Delta(C_n \times C_m) = 2$ in this case, we can then conclude that $\chi_g(C_n \times C_m) \leq 3$. To get the other inequality, we will consider a winning strategy for Bob for 2 or less colors. Set $C = \{c_1, c_2\}$. Let Alice color some vertex v_1 with c_1. Consider some vertex v_2 such that $v_2 \in Neighbourhood(v_1)$. Let Bob color some vertex v_3 such that $v_3 \in Neighbourhood(v_2)$ with c_2.

Because v_2 has two colors in its neighbourhood, it is no longer colorable, causing Bob to win. This gives us the inequality $\chi_g(C_m \times C_n) \geq 3$ in this case. Thus, $\chi_g(C_m \times C_n) = 3$ in this case.

Case 4: $n = 3$ and $m = 3$
Let the color set $C = \{c_1, c_2, c_3\}$, $V(C_n) = \{v_1, v_2, v_3\}$, $V(C_m) = \{w_1, w_2, w_3\}$ and $V(C_m \times C_n) = \{(v_i, w_j) \mid \forall i = 1, 2, 3 \text{ and } \forall j = 1, 2, 3\}$. Let us consider an Alice winning strategy: Let Alice color some vertex (v_a, w_{b1}) with c_1. Bob then colors some vertex (v_a, w_{b2}) such that $w_{b1} \neq w_{b2}$. Alice then colors the vertex (v_a, w_{b3}) with c_3 such that $w_{b3} \neq w_{b2} \neq w_{b1}$. One might notice that $\forall v \in V(C_n \times C_m)$ where v is uncolored, v is surrounded by two colors and that $\forall w \in Neighbourhood(v)$ is surrounded by the remaining color. That means that w cannot be colored by that remaining color, and thus Bob cannot surround v with all 3 colors. That means that all the vertices of the graph can be colored, giving Alice the win. So $\chi_g(C_n \times C_m) \leq 3$.

Let us now consider a winning strategy for Bob with 2 colors or less. Let $C = \{c_1, c_2\}$. Let Alice color a vertex v_1 with c_1. Consider vertices v_2 and v_3 such that $v_2 \in Neighbourhood(v_1)$, $v_3 \in Neighbourhood(v_2)$ and $v_3 \notin Neighbourhood(v_1)$. Let Bob color v_3 with c_2. This will surround v_2 with 2 colors, making v_2 uncolorable and thus giving Bob the win. So $\chi_g(C_n \times C_m) \geq 3$. Thus, $\chi_g(C_n \times C_m) = 3$ in this case.

Case 5: $n \geq 3$ and $m = 3$
Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$, $V(C_m) = \{w_1, w_2, w_3\}$ and $V(C_n \times C_m) = \{(v_i, w_j) \mid \forall i = 1, 2, \ldots, m$ and $\forall j = 1, 2, 3$. We will examine two subcases:

Case: 5.1: If n is even
Set $C = \{c_1, c_2, c_3\}$. Let Alice color some vertex (v_i, w_j) with some color, say c_1. We will then consider the following subsets of $V(C_n \times C_m)$
$L_1 = \{(v_i, w_j) \mid \forall j = 1, 2, 3\}$
$L_2 = \{(v_{i+1}, w_j), (v_{i-1}, w_j) \mid \forall j = 1, 2, 3\}$
$L_3 = \{(v_{i+2}, w_j), (v_{i-2}, w_j) \mid \forall j = 1, 2, 3\}$
$\vdots$
$L_{n-1} = \{(v_{i+n}, w_j), (v_{i-n}, w_j) \mid \forall j = 1, 2, 3\}$

Bob then colors some vertex $(v_a, w_b) \in L_2$ such that $b \neq j$. Alice then suitably colors a vertex of L_1 or L_3 with preferably the same color as (v_a, w_b). This repeats with Bob suitably coloring vertices of L_2 whilst Alice colors vertices of L_1 or L_3. If a vertex is under threat, then Alice colors it with a suitable color.

When L_2 is fully colored, Alice moves onto coloring L_5 or L_3 whilst Bob colors L_4. This pattern repeats, where Bob fully colors the even layers in order whilst Alice colors the odd layers around Bob's chosen layer at the current moment. After doing this, there will be only a few uncolored vertices left, in which Alice and Bob can suitably color. In the end, the graph will be fully colored, giving Alice the win. Thus, $\chi_g(C_n \times C_m) \leq 3$.

We will now consider a winning strategy for Bob. Let $C = \{c_1, c_2\}$. Let Alice color a vertex v_1 with c_1. Consider vertices v_2 and v_3 such that $v_2 \in Neighbourhood(v_1)$, $v_3 \in Neighbourhood(v_2)$ and $v_3 \notin Neighbourhood(v_1)$. Let Bob color v_3 with c_2. This will surround v_2 with 2 colors, making v_2 uncolorable and thus giving Bob the win. So $\chi_g(C_n \times C_m) \geq 3$.

Case: 5.2: If n is odd and $n \neq 3$
Let Alice color a vertex (v_i, w_j) with c_1. Let Bob suitably color a vertex (v_{i+1}, w_{b+1}). In response, Alice then colors some vertex (v_{i+1}, w_{b+1}) with the same color that Bob used. In general, this continues until the set of vertices $\{(v_a, w_{b+1}), (v_a, w_{b+2}) \mid \forall a = 1, 2, \ldots, n\}$ is fully colored. They then take turns coloring $\{(v_a, w_b) \mid \forall a = 1, 2, \ldots, n\}$ such that Alice and Bob color the vertices in a similar 'order' as was discussed before. In the end, $C_n \times Cm$ will be fully colored, giving Alice the win. Thus, $\chi_g(C_m \times C_n) \leq 3$

We will now consider a winning strategy for Bob. Let $C = \{c_1, c_2\}$. Let Alice color a vertex v_1 with c_1. Consider vertices v_2 and v_3 such that $v_2 \in Neighbourhood(v_1)$, $v_3 \in Neighbourhood(v_2)$ and

$v_3 \notin Neighbourhood(v_1)$. Let Bob color v_3 with c_2. This will surround v_2 with 2 colors, making v_2 uncolorable and thus giving Bob the win. So $\chi_g(C_n \times C_m) \geq 3$.

Case 6: $n \geq 3$ and $m \geq 3$
Let us first consider a winning strategy for Alice. Set $C = \{c_1, c_2, c_3, c_4\}$. Let Alice color some vertex v_1 with c_1. Bob then suitably colors vertex v_2 with color c_2 such that $v_2 \in Neighbourhood(v_1)$. Alice then suitably colors a vertex v_3 such that $v_3 \notin Neighbourhood(v_2)$ with preferably c_1, and Bob suitably colors v_4 such that $v_4 \in Neighbourhood(v_3)$ afterwards. This repeats until all vertices are colored.

If at any point Bob colors a vertex v_j independent from Alice's previously colored vertex, Alice can proceed to suitably color a vertex v_i disconnected from v_j with preferably c_1. The graph will be fully colored in the end, giving Alice the win. This means that $\chi_g(C_n \times C_m) \leq 4$.

We can then consider a winning strategy for Bob. Let $C = \{c_1, c_2, c_3\}$. Let Alice color some vertex v_1 with c_1. Bob then colors v_2 with c_2 such that $v_2 \in Neighbourhood(v_1)$, causing some vertex v_3 to become under threat. After Alice colors v_3 with the appropriate color, Bob then colors a neighbour of v_3 with c_3, causing two vertices to become under threat such that these two vertices don't share a vertex in common. As Alice can not block both threats at once, Bob wins the game, meaning $\chi_g(C_n \times C_m) \geq 4$. Thus, $\chi_g(C_n \times C_m) = 4$ □

Theorem 2.2. *The game chromatic number of product of a cycle and complete graph is given by :*

$$\chi_g(C_n \times K_m) = \begin{cases} 1 & \text{if } m = 1 \\ 2 & \text{if } n \geq 2 \text{ and } m = 2 \\ 3 & \text{if } n \geq 2 \text{ and } m \geq 3 \end{cases}$$

Proof. We need to consider the following cases:
Case 1: $m = 1$
Please see the proof of Theorem 1, Case 1 for the graph K_1.
Case 2: $n \geq 2$ and $m = 2$
Since $K_2 = P_2$, the required result follows from the corresponding arguments in Theorem 1.
Case 3: If $n \geq 2$ and $m \geq 3$
Let $C = \{c_1, c_2, c_3\}$. Let $V(K_m) = \{v_1, v_2, \ldots, v_m\}$, $V(C_n) = \{w_1, w_2, \ldots, w_n\}$ and $V(K_m \times C_n) = \{(v_j, w_i) \mid \forall j = 1, 2, \ldots, m\ \forall i = 1, 2, \ldots, n\}$. We will consider two cases for Alice's strategy:
Case: 3.1: If n and m are even
Let Alice color a vertex (v_j, w_i) with the color c_1. Let us then consider the following subsets of $V(K_m \times C_n)$:

$$\begin{aligned} V_1 &= \{(v_j, w_i) \mid \forall i = 1, 2, \ldots, n\} \\ V_2 &= \{(v_{j+1}, w_i), (v_{j-1}, w_i) \mid \forall i = 1, 2, \ldots, n\} \\ V_3 &= \{(v_{j+2}, w_i), (v_{j-2}, w_i) \mid \forall i = 1, 2, \ldots, n\} \\ &\vdots \\ V_m &= \{(v_{j+m}, w_i) \mid \forall i = 1, 2, \ldots, n\}, \end{aligned} \tag{2.1}$$

After Alice colors $(v_j, w_i) \in V_1$ with c_1, Bob then colors $(v_{j+m}, w_{i+1}) \in V_m$ or $(v_{j+m}, w_{i-1}) \in V_m$ with c_2. Alice then suitably colors $(v_{j+m}, w_{j+2}) \in V_m$ or $(v_{j+m}, w_{j-2}) \in V_m$ with c_1, in which Bob introduces c_3 afterwards by suitably coloring $(v_{j+m}, w_{i+3}) \in V_m$ or $(v_{j+m}, w_{i-3}) \in V_m$. Alice and Bob keep on suitably coloring elements of V_m until V_m is fully colored.

Alice and Bob then suitably color the other levels in the order of $V_{m-1}, V_{m-2}, \ldots, V_2, V_1$. Note that when dealing with levels with two 'categories' of vertices, Alice and Bob must first color $\{(v_j, w_i) \mid \forall i = 1, 2, \ldots, n\}$ for a specific i, then they can color the other category of vertices. In the end, the graph becomes fully colored with 3 colors, giving Alice the win. Thus, $\chi_g(K_m \times C_n) \leq 3$.

Case: 3.2: Otherwise
Let Alice color a vertex (v_j, w_i) with c_1. Let us then consider the following subsets of $V(K_m \times C_n)$:

$$\begin{aligned} V_1 &= \{(v_j, w_i) \mid \forall i = 1, 2, \dots, n\} \\ V_2 &= \{(v_{j+1}, w_i) \mid \forall i = 1, 2, \dots, n\} \\ V_3 &= \{(v_{j+2}, w_i) \mid \forall i = 1, 2, \dots, n\} \\ &\vdots \\ V_m &= \{(v_{j+m}, w_i) \mid \forall i = 1, 2, \dots, n\}, \end{aligned} \tag{2.2}$$

Let Bob suitably color some vertex $v_1 \in V_l$ such that $l \neq 1$. Alice then suitably colors another vertex $v_2 \in V_l$ with preferably c_1. This continues until V_l is fully colored.

Similarly, Alice and Bob color each subset until they are fully colored in the order of $V_{l-1}, V_{l-2}, \dots, V_2, V_1$. In the end, the graph $K_m \times C_n$ will be fully colored with 3 colors, giving Alice the win. Thus $\chi_g(K_m \times C_n) \leq 3$

Let us now consider a winning strategy for Bob with 2 colors or less. Let $C = \{c_1, c_2\}$. Let Alice color a vertex v_1 with c_1. Consider vertices v_2 and v_3 such that $v_2 \in Neighbourhood(v_1)$, $v_3 \in Neighbourhood(v_2)$ and $v_3 \notin Neighbourhood(v_1)$. Let Bob color v_3 with c_2. This will surround v_2 with 2 colors, making v_2 uncolorable and thus giving Bob the win. So $\chi_g(C_n \times C_m) \geq 3$. □

Theorem 2.3. *The game chromatic number of a product of a star and path is given by:*

$$\chi_g(S_n \times P_m) = \begin{cases} 1 & \text{if } m = 1 \text{ or } n = 1 \text{ and } m \geq 1 \\ 2 & \text{if } n \geq 1 \text{ and } m = 2 \\ 3 & \text{Otherwise} \end{cases}$$

Proof. Let $S_n = \{s, l_1, l_2, \dots, l_n\}$, $Pm = \{p_1, p_2, \dots, p_m\}$ and $S_n \times P_m = \{(s, p_j), (l_i, p_j) \mid \forall i = 1, 2, \dots, n \; \forall j = 1, 2, \dots, m\}$. We will consider 4 cases:

Case 1: $m = 1$ and $n \geq 1$ or $n = 1$ and $m \geq 1$
The graph $S_n \times P_1$ is a completely disconnected graph, meaning that V$(S_n \times P_1)$ is an independent set. This is also the case when with $S_1 \times P_n$, as well as $S_1 \times P_1$. Please see Theorem 1, Case 1 for a proof regarding such cases.

Case 2: $n \geq 1$ and $m = 2$
In this case, $S_n \times P_m$ creates two disconnected S_n components.

Let us consider a winning strategy for Alice. Set $C = \{c_1, c_2\}$. On her turns, let Alice color (s, p_1) and (s, p_2) with c_1. This will force Bob to color the other vertices with c_2, and since they are leaf nodes, they can't threaten any other vertex with c_2. If Bob on his turn colors (s, p_2) with c_2, Alice still wins since the rest of the vertices are leaf nodes that can just be colored with c_1 or c_2. In the end, the graph will be fully colored with 2 colors, meaning $\chi_g(S_n \times P_m) \leq 2$

Now we will consider a winning strategy for Bob. Set $C = \{c_1\}$. Since $deg(s, p_j) \geq 1$ and $\deg(l_i, p_j) = 1$, $\forall i = 1, 2, \dots, n$ and $\forall j = 1, 2$, if Alice colors a vertex, she automatically loses since there exists a neighbour that is uncolored and can now no longer be colored with c_1. So, $\chi_g(S_n \times P_m) \geq 2$ and hence $\chi_g(S_n \times P_m) = 2$.

Case 3: Otherwise
We will first discuss a winning strategy for Alice. Let $C = \{c_1, c_2, c_3\}$, such that the danger zone of $S_n \times P_m$ will accordingly be
$DG(S_n \times P_m) = \{(s, p_2), (s, p_3), \dots, (s, p_{m-1})\}$. Bob's only chance at winning is to surround some vertex $v_a \in DG(S_n \times P_m)$ with all 3 colors by coloring three vertices $w_1, w_2, w_3 \notin DG(S_n \times P_m)$ such that $w_1, w_2, w_3 \in Neighbourhood(V_a)$.

Let Alice color some vertex $v_1 \in DG(S_n \times P_m)$ at the start of the game. The default strategy for Alice is to color $v \in DG(S_n \times P_m)$ whilst Bob colors $w \notin DG(S_n \times P_m)$. Assume at some point in the game, the vertex $v_a \in DG(S_n \times P_m)$ has two neighbours $w_1, w_2 \in DG(S_n \times P_m)$ colored with c_1 and c_2 respectively, with no loss of generality. Alice can then color v_a in her turn with the appropriate color, blocking the threat. This sequence of events of repeats with Alice prioritizing fully coloring $DG(C_n \times P_m)$ first, and then moving onto coloring vertices outside of the danger zone alongside Bob. In the end, the graph $S_n \times P_m$ will be fully colored, giving Alice the win. Thus, $\chi_g(S_n \times P_m) \leq 3$

Let us now consider a winning strategy for Bob with 2 colors or less. Let $C = \{c_1, c_2\}$. Let Alice color a vertex v_1 with c_1. Consider vertices v_2 and v_3 such that $v_2 \in Neighbourhood(v_1)$, $v_3 \in Neighbourhood(v_2)$ and $v_3 \notin Neighbourhood(v_1)$. Let Bob color v_3 with c_2. This will surround v_2 with 2 colors, making v_2 uncolorable and thus giving Bob the win. So $\chi_g(S_n \times P_m) \geq 3$. Thus, $\chi_g(S_n \times P_m) = 3$

□

Theorem 2.4. *The game chromatic number of a product of a start and cycle is given by:*

$$\chi_g(S_n \times C_m) = \begin{cases} 1 & \text{if } m = 1 \\ 2 & \text{if } m = 2 \\ 3 & \text{if } m \geq 3 \end{cases}$$

Proof. We will consider 3 cases:
Case 1: $m = 1$
$V(S_n \times C_1)$ is an independent set. Please see Theorem 1, Case 1 for the proof. Thus, $\chi_g(S_n \times C_m) = 1$ in this case.
Case 2: $m = 2$
$S_n \times C_2$ is equivalent to $S_n \times P_2$. Please see Theorem 3, Case 2 for the proof.
Case 3: $m \geq 3$
Let $V(S_n) = \{s, p_1, p_2, \ldots, p_n\}$, $V(C_m) = \{c_1, c_2, \ldots, c_m\}$ and $V(S_n \times C_m) = \{(s, c_j), (p_i, c_j) \mid \forall i = 1, 2, \ldots, n\ \forall j = 1, 2, \ldots, m\}$

We will first discuss a winning strategy for Alice. Let $C = \{c_1, c_2, c_3\}$ where $DG(S_n \times C_m) = \{(s, p_j) \mid \forall j = 1, 2, \ldots, m\}$. Let Alice color a vertex $v_1 \in DG(S_n \times C_m)$ with c_1. Bob's only way of winning is to by coloring the neighbours of a vertex $v_a \in DG(S_n \times C_m)$ with 3 colors. However, in the case that the vertex v_a falls under threat due to 2 colors existing in its neighbourhood, Alice can simply color v_a with an appropriate color to block the threat.

In general, Alice colors the vertices of $DG(S_n \times C_m)$ with preferably c_1, in which Bob can either color vertices inside $DG(S_n \times C_m)$ or vertices outside of the set. In the end, the graph $S_n \times C_m$ becomes fully colored with 3 colors, giving Alice the win. Thus, $\chi_g(S_n \times C_m) \leq 3$

Let us now consider a winning strategy for Bob with 2 colors or less. Let $C = \{c_1, c_2\}$. Let Alice color a vertex v_1 with c_1. Consider vertices v_2 and v_3 such that $v_2 \in Neighbourhood(v_1)$, $v_3 \in Neighbourhood(v_2)$ and $v_3 \notin Neighbourhood(v_1)$. Let Bob color v_3 with c_2. This will surround v_2 with 2 colors, making v_2 uncolorable and thus giving Bob the win. So $\chi_g(S_n \times C_m) \geq 3$. Thus, $\chi_g(S_n \times C_m) = 3$ in this case. □

Theorem 2.5. *The game chromatic number of products of two paths*

$$\chi_g(P_n \times P_m) = \begin{cases} 1 & \text{if } n = 1 \text{ or } m = 1 \\ 2 & \text{if } n = m = 2 \text{ or } n = 2 \text{ and } m = 3 \\ 3 & \text{Otherwise} \end{cases}$$

Proof. Let $V(P_n) = \{a_1, a_2, \ldots, a_n\}$, $V(P_m) = \{b_1, b_2, \ldots, b_m\}$ and $V(P_n \times P_m) = \{(a_i, b_j) \mid \forall i = 1, 2, \ldots, n$ and $\forall j = 1, 2, \ldots, m\}$. We will consider 3 cases:

Case 1: $n = 1$ *or* $m = 1$

$P_1 \times P_m$, $P_n \times P_1$ and $P_1 \times P_1$ all have fully independent vertex sets. please see Theorem 1, Case 1 for the proof, since $P_1 = K_1$

Case 2: $n = m = 2$
$P_2 \times P_2$ is equivalent to $K_2 \times K_2$. Please see Theorem 2, Case 2 for the proof

Case 3: $n = 2$ *and* $m = 3$
We can immediately see that $P_2 \times P_3$ consists of two disconnected P_3 graphs. Let us first discuss a winning strategy for Alice:

Let $C = \{c_1, c_2\}$, in which $DG(P_n \times P_m) = \{(a_n, b_2) \mid \forall i = 1, 2\}$. Let Alice color a vertex $v_1 \in DG(P_n \times P_m)$ with a color c_1. There is only one uncolored vertex left in $DG(P_n \times P_m)$ which we will denote by v_c. If Bob colors v_a, then Alice and Bob can simply color the rest of the vertices with suitable colors. If Bob colors a vertex outside of $DG(P_n \times P_m)$ and threatens v_c, Alice can simply color v_c with a suitable color in her turn. In the end, the graph $P_n \times P_m$ will be fully colored with 2 colors, which means Alice wins and that $\chi_g(P_n \times P_m) \leq 2$

We will now discuss a winning strategy for Bob. By setting $C = \{c_1\}$, Alice automatically loses by coloring a vertex since all vertices have a degree of 1 or more. That means that all vertices have at least one neighbour such that if Alice colors a vertex with c_1, it's neighbour(s) can no longer be colored with c_1 and the graph is made uncolorable. This gives Bob the win, thus $\chi_g(P_n \times P_m) \geq 2$

Thus, $\chi_g(P_n \times P_m) = 2$ in this case.

Case 4: Otherwise
We will first discuss a winning strategy for Alice. Let $C = \{c_1, c_2, c_3\}$ which results to $DG(P_n \times P_m) = \{(v_a, w_b) \mid \forall a = 2, 3, \ldots, n-1 \text{ and } \forall b = 2, 3, \ldots, m-1\}$. Alice's general strategy is to color elements of $DG(P_n \times P_m)$ with the same color. The only way Bob can cause a threat is to surround a vertex v with 2 colors such that $v \in DG(P_n \times P_m)$. If that happens, then Alice can simply color that vertex with a suitable color. In the end, the graph $P_n \times P_m$ becomes fully colored, giving Alice the win. Thus, $\chi_g(P_n \times P_m) \leq 3$

Let us now consider a winning strategy for Bob with 2 colors or less. Let $C = \{c_1, c_2\}$. Let Alice color a vertex v_1 with c_1. Consider vertices v_2 and v_3 such that $v_2 \in Neighbourhood(v_1)$, $v_3 \in Neighbourhood(v_2)$ and $v_3 \notin Neighbourhood(v_1)$. Let Bob color v_3 with c_2. This will surround v_2 with 2 colors, making v_2 uncolorable and thus giving Bob the win. So $\chi_g(P_n \times P_m) \geq 3$. Thus, $\chi_g(P_n \times P_m) = 3$ in this case. □

Theorem 2.6. *The game chromatic number of products of path and a cycle is:*

$$\chi_g(P_n \times C_m) = \begin{cases} 1 & \text{if } n = 1 \text{ or } m = 1 \\ 2 & \text{if } n = 2 \text{ and } m \geq 2 \text{ or } m = 2 \text{ and } n = 3 \\ 3 & \text{Otherwise} \end{cases}$$

Proof.
Case 1: $m = 1$ *or* $n = 1$
If $m = 1$, then P_1 is equivalent to K_1, and we know from Theorem 1 that $\chi_g(K_1 \times C_m) = 1$. Meanwhile, if $n = 1$, then we discussed in the proof of Theorem 1, Case 1 that $\chi_g(G \times C_1) = 1$. Thus, $\chi_g(P_n \times C_m) = 1$ in this case.

Case 2: $n = 2$
Since $P_2 = K_2$, we have

$$P_2 \times C_n \cong K_2 \times C_n.$$

We know that $\chi_g(C_2 \times C_m)$=2 from Theorem 1, thus $\chi_g(P_n \times C_m) = 2$ in this case.

Case 3: $m = 2$

Since $K_2 = P_2$, we obtain
$$\chi_g(P_n \times K_2) = \chi_g(P_n \times P_2).$$

Case 4: Otherwise

Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$, $V(C_m) = \{w_1, w_2, \dots, w_m\}$ and $V(C_n \times C_m) = \{(v_i, w_j) | \forall i = 1, 2, \dots, n$ and $\forall j = 1, 2, \dots, m\}$. We will first consider a winning strategy for Alice. Consider these subsets of $V(P_n \times C_m)$:
$$O = \{(v_a, w_j) \ \ \forall\ a \text{ is odd and } \ \forall j = 1, 2, \dots, m\}$$
$$E = \{(v_b, w_j) \ \ \forall\ b \text{ is even and } \ \forall j = 1, 2, \dots, m\}$$
The general strategy is that Alice needs to color one subset with preferably one color whilst Bob needs to color the other subset. If a vertex becomes under threat at any point during the game, Alice can simply color that vertex with the appropriate color. In the end, $P_n \times C_m$ will be fully colored with 3 colors, giving Alice the win. Thus, $\chi_g(P_n \times C_m) \leq 3$.

Let us now consider a winning strategy for Bob with 2 colors or less. Let $C = \{c_1, c_2\}$. Let Alice color a vertex v_1 with c_1. Consider vertices v_2 and v_3 such that $v_2 \in Neighbourhood(v_1)$, $v_3 \in Neighbourhood(v_2)$ and $v_3 \notin Neighbourhood(v_1)$. Let Bob color v_3 with c_2. This will surround v_2 with 2 colors, making v_2 uncolorable and thus giving Bob the win. So $\chi_g(P_n \times C_m) \geq 3$. Thus, $\chi_g(P_n \times C_m) = 3$ in this case. □

Theorem 2.7. $\chi_g(K_n \times K_m) \leq \lceil \frac{nm}{2} \rceil$

Proof. Let $C = \{c_1, c_2, \dots, c_{\frac{nm}{2}}\}$, $V(K_n) = \{v_1, v_2, \dots, v_n\}$,
$V(K_m) = \{w_1, w_2, \dots, w_m\}$ and $V(K_n \times K_m) = \{(v_i, w_j) \mid \forall i = 1, 2, \dots, n$ and $\forall j = 1, 2, \dots, m\}$. Consider the following subsets of $V(K_n \times K_m)$:
$$\begin{aligned} L_1 &= \{(v_1, w_j) \mid \forall j = 1, 2, \dots, m\} \\ L_2 &= \{(v_2, w_j) \mid \forall j = 1, 2, \dots, m\} \\ &\ \ \vdots \\ L_n &= \{(v_n, w_j) \mid \forall j = 1, 2, \dots, m\}, \end{aligned} \tag{2.3}$$
Let Alice color some vertex $(v_i, w_j) \in L_i$ with c_1. Bob then colors $(v_{i+1}, w_{j+1}) \in L_{i+1}$. Alice then colors $(v_{i+1}, w_{j+2}) \in L_{i+1}$ with the same color that Bob previously used (c_2 in this case). Bob then colors $(v_{i+1}, w_{j+3}) \in L_{i+1}$ with c_3, in which Alice then colors $(v_{i+1}, w_{j+4}) \in L_{i+1}$ with the same color c_3. This continues until L_{i+1} is fully colored. The process is then continued such that Alice and Bob color the subsets $L_{i+2}, L_{i+3}, \dots, L_{i+n}, L_1$ in that order until all but one vertex is colored. The final vertex is then colored with c_1 □

Theorem 2.8. *Let G be either a star graph S_n, wheel graph W_n or a friendship graph F_n. Then*
$$\chi_g(G \times C_m) = \begin{cases} 1 & \text{if } m = 1 \\ 2 & \text{if } m = 2 \\ 3 & \text{If } m \geq 3 \end{cases}$$

Proof. Let $V(G) = \{a, v_1, v_2, \dots, v_n\}$, $V(C_m) = \{w_1, w_2, \dots, w_m\}$ and $V(G \times C_m) = \{(a, w_j), (v_i, c_j) \ \ \forall i = 1, 2, \dots, n$ and $\forall j = 1, 2, \dots, m\}$. Let us consider the following subsets of $V(G)$: $A = \{a\}$ and $B = \{v_1, v_2, \dots, v_n\}$, where $\deg(a) = n$ and $\deg(a) > \deg(v_i)$ $\forall i = 1, 2, \dots, n$. Similarly, we can consider the following subsets of $V(G \times C_m)$:
$$\begin{aligned} V(A \times C_m) &= \{(a, w_j) \ \ \forall j = 1, 2, \dots, m\} \\ V(B \times C_m) &= \{(v_i, w_j) \ \ \forall i = 1, 2, \dots, n and \forall j = 1, 2, \dots, m\} \end{aligned} \tag{2.4}$$

Let Alice color some vertex $(a, w_j) \in V(A \times C_m)$ with c_1. Bob then colors a vertex $(a, w_{j+1}) \in V(A \times C_m)$ with c_2, in which Alice then colors $(a, c_{j+2}) \in V(A \times C_m)$ with c_1. Bob then colors (a, c_{j+3}) with c_3, in which Alice then colors (a, c_{j+4}) in the order of $c_1, c_2, c_3, c_2, c_1, \ldots, c_2, c_1$ until all of $V(A \times C_m)$ is colored.

We can then consider these following subsets:

$$\begin{aligned} L_1 &= \{(v_1, w_j) \ \ \forall j = 1, 2, \ldots, m\} \\ L_2 &= \{(v_2, w_j) \ \ \forall j = 1, 2, \ldots, m\} \\ &\vdots \\ L_n &= \{(v_n, w_j) \ \ \forall j = 1, 2, \ldots, m\}, \end{aligned} \tag{2.5}$$

Alice and Bob color each level, in which the order of colors used should be the same as the one used in the level $V(A \times C_m)$. The order of levels that need to be colored is $L_1, L_2, \ldots, L_{m-1}, L_m$. In the end, the graph will be fully colored by 3 colors, which gives Alice the win. Thus, $\chi_g(G \times C_m) \leq 3$

Let us now consider a winning strategy for Bob with 2 colors or less. Let $C = \{c_1, c_2\}$. Let Alice color a vertex v_1 with c_1. Consider vertices v_2 and v_3 such that $v_2 \in Neighbourhood(v_1)$, $v_3 \in Neighbourhood(v_2)$ and $v_3 \notin Neighbourhood(v_1)$. Let Bob color v_3 with c_2. This will surround v_2 with 2 colors, making v_2 uncolorable and thus giving Bob the win. So $\chi_g(P_n \times C_m) \geq 3$. Thus, $\chi_g(G \times C_m) = 3$

We can use these exact strategies for $m = 2$. When it comes to $m = 1$, then $V(G \times C_1)$ is a completely independent set. Please look at Theorem 1, Case 1 for such a case. □

Theorem 2.9. *Let G be either a complete bipartite graph $K_{n,m}$ or a path of order n P_n. Then*

$$\chi_g(G \times C_m) = \begin{cases} 1 & \text{if } m = 1 \\ 2 & \text{if } m = 2 \\ 3 & \text{If } m \geq 3 \end{cases}$$

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$, $V(C_m) = \{w_1, w_2, \ldots, w_m\}$ and $V(G \times C_m) = \{(v_i, w_j) \mid \forall i = 1, 2, \ldots, n$ and $\forall j = 1, 2, \ldots, m\}$. Consider the following subsets of $V(G \times C_m)$

$$\begin{aligned} E &= \{(v_i, w_j) \mid \forall i = 2, 4, \ldots, 2m \text{ and } \forall j = 1, 2, \ldots, m\} \\ O &= \{(v_i, w_j) \mid \forall i = 1, 3, 5, \ldots, 2n+1 \text{ and } \forall j = 1, 2, \ldots, m\}, \end{aligned} \tag{2.6}$$

Alice chooses a layer by coloring a vertex in either E or O with c_1. For the rest of the game, Alice colors only her chosen layer whilst Bob colors only the other layer. In the end, the graph will be fully colored with 3 colors, giving Alice the win. Thus, $\chi_g(G \times C_m) \leq 3$.

Let us now consider a winning strategy for Bob with 2 colors or less. Let $C = \{c_1, c_2\}$. Let Alice color a vertex v_1 with c_1. Consider vertices v_2 and v_3 such that $v_2 \in Neighbourhood(v_1)$, $v_3 \in Neighbourhood(v_2)$ and $v_3 \notin Neighbourhood(v_1)$. Let Bob color v_3 with c_2. This will surround v_2 with 2 colors, making v_2 uncolorable and thus giving Bob the win. So $\chi_g(G \times C_m) \geq 3$.

Thus, $\chi_g(G \times C_m) = 3$

Note that can use these exact strategies for when $m = 2$. When it comes to $m = 1$, then $V(G \times C_1)$ is a completely independent set. Please look at Theorem 1, Case 1 for such a case. □

Theorem 2.10. *Let $SL_{n,m}$ be a bipartite graph such that $V(SL_{n,m}) = \{u\} \cup B_{n,m}$. The vertex u is a universal vertex, whilst B_n contains n copies of K_m. The graph SL_n is called a star-like graph. Let C_a be a cycle graph of order m. Then $SL_{n,m} \times C_a = 3 + (m-2) = m+1$ for $n \geq 3, m \geq 3$*

Proof. Let $V(SL_{n,m}) = \{u, k_1, k_2, \ldots, k_m, k_{m+1}, k_{m+2}, \ldots, k_{2m}, \ldots, k_{nm}\}$ and $V(C_a) = \{c_1, c_2, \ldots, c_a\}$. We will create the following subsets of $V(SL_{n,m} \times C_a)$:

$$\begin{aligned} U &= \{(u, c_j) \mid j = 1, 2, \ldots, a\} \\ L_1 &= \{(k_i, c_j) \mid i = 1, m+1, 2m+1, \ldots, (n-1)m+1,\ j = 1, , 2, \ldots, a\} \\ L_2 &= \{(k_i, c_j) \mid i = 2, m+2, 2m+2, \ldots, (n-1)m+2,\ j = 1, , 2, \ldots, a\} \\ &\vdots \\ L_n &= \{(k_i, c_j) \mid i = m, 2m, \ldots, nm,\ j = 1, , 2, \ldots, a\}, \end{aligned} \tag{2.7}$$

Let us first consider a winning strategy for Alice. The strategy will be similar to that of $\chi_g(S_n \times C_m)$ and $\chi_g(F_n \times C_m)$, where Alice and Bob will color U first with h_1, h_2 and h_3 first, then the other vertices with the rest of the vertices. Let the color set $C(G) = \{h_1, h_2, \ldots, h_{m+2}\}$

Let Alice start by coloring some vertex (u, c_j) for some $j \in Z_a$ with some color h_1. Bob then colors (u, c_{j+1}) with a new color h_2, in which Alice then responds by coloring (u, c_{j+2}) with h_1. Bob then colors (u, c_{j+3}) with a new color h_3, in which Alice then colors (u, c_{j+4}) with h_1. This continues, with U being colored in the following color order: $h_1, h_2, h_3, h_2, h_1, h_s, \ldots, h_1, h_2, h_1$ for any s.

Alice and Bob then start coloring the other layers in the order of $L_1, L_2, \ldots, L_n$. Note that if we kept the game chromatic number as 3 and color the layers with the same order as above, the graph will become uncolorable during the coloring of L_2 (this idea will be expanded upon when we discuss Bob's winning strategy below). Thus, we add $(m-2)$ colors to be able to fully color the graph. The order in which the other layers are colored is not important. In the end, the graph $SL_{n,m} \times C_a$ will be fully colored with $m+1$ colors. Thus, $\chi_g(SL_{n,m} \times C_a) \leq m+1$

We will now discuss a winning strategy for Bob. For this, let us consider a scenario where the same strategy as above is followed with n colors, and let us consider the moment when Alice first colors a vertex in L_{m-1}, say v. Then there exists a vertex $w \in L_n$ such that $w \in Neighbourhood(v)$ and w is uncolored. The vertex v is surrounded by $m-1$ colors at this point, so Bob can color w at his next turn to surround v with n colors, rendering v uncolorable and giving Bob the win. Thus, $\chi_g(SL_{n,m} \times C_a) \geq m+1$.

Therefore, $\chi_g(SL_{n,m} \times C_a) = m+1$.

□

References

[1] R. Alagammai and V. Vijayalakshmi. Game chromatic number of lexicographic product graphs. *AKCE International Journal of Graphs and Combinatorics*, 12(2-3):216–220, 2015. 1

[2] R. Alagammai and V. Vijayalakshmi. Game chromatic number of lexicographic product graphs. *AKCE International Journal of Graphs and Combinatorics*, 12(2-3):216–220, 2015. 1

[3] R. Alagammai and V. Vijayalakshmi. The game chromatic number of corona of two graphs. *AKCE International Journal of Graphs and Combinatorics*, 17(3):899–904, 2020. 1

[4] T. Bartnicki, J. Grytczuk, H. A. Kierstead, and X. Zhu. The map-coloring game. *The American Mathematical Monthly*, 114(9):793–803, 2007. 1

[5] T. Bohman, A. Frieze, and B. Sudakov. The game chromatic number of random graphs. *Random Structures & Algorithms*, 32(2):223–235, 2008. 1

[6] S. A. U. H. Bokhary, T. Iqbal, and U. Ali. Game chromatic number of cartesian and corona product graphs. *Journal of Algebra Combinatorics Discrete Structures and Applications*, 5(3):129–136, 2018. 1

[7] S. A. U. H. Bokhary, T. Iqbal, and U. Ali. Game chromatic number of cartesian and corona product graphs. *Journal of Algebra Combinatorics Discrete Structures and Applications*, 5(3):129–136, 2018. 1

[8] G. Chen, R. H. Schelp, and W. E. Shreve. A new game chromatic number. *European Journal of Combinatorics*, 18(1):1–9, 1997. 1

[9] H. Enomoto, J. Fujisawa, and N. Matsumoto. Game chromatic number of strong product graphs. *Discrete Mathematics*, 346(1):113162, 2023. 1

[10] U. Faigle, W. Kern, H. Kierstead, and W. T. Trotter. On the game chromatic number of some classes of graphs. *Ars combinatoria*, 35:143–150, 1993. 1

[11] L. Hollom. On graphs with maximum difference between game chromatic number and chromatic number. *Discrete Mathematics*, 348(2):114271, 2025. 1