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SOME DISTRIBUTIONAL PROPERTIES OF THE CONCOMITANTS OF RECORD STATISTICS FOR BIVARIATE PSEUDO–EXPONENTIAL DISTRIBUTION AND CHARACTERIZATION

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Abstract. A new class of distributions known as Bivariate Pseudo–Exponential distribution has been defined. The distribution of r–th concomitant and joint distribution of r–th and s–th concomitant of record statistics of the resulting distribution have been derived. Expression for single and product moments has also been obtained for the resulting distributions. A characterization of the k-th concomitant of record statistics for the Pseudo-exponential distribution by the conditional expectation is presented.

Key words: concomitants, record values, Pseudo–exponential distribution.
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1. Introduction

Record statistics has been a relatively new branch within the domain of ordered random variables. The distribution of record statistics is always needed when ever someone wants to model the records of some events like floods, track records and so on. The mathematical treatment of the subject was first developed by [4]. The moment properties of records have been provided by [1]. Sufficient material is now available in the field of records that can be found in [2]. Some characterizations by using the record statistics has been obtained

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by [3]. A comprehensive review of the mathematical foundation of the records has been given by [6].

We start by giving the distribution of the records in the case of a random sample from a continuous distribution. Specifically, the distribution of k–th upper record $X_{U(k)}$ is given by [1] as:

$$f_k\left(x_k\right) = \frac{1}{\Gamma\left(k\right)} f\left(x_k\right)\left[R\left(x_k\right)\right]^{k-1}, \tag{1}$$

where $R\left(x_k\right) = -\ln\left[1 - F\left(x_k\right)\right]$. Further, the joint distribution of k–th and m–th upper records; $X_{U(k)}$ and $X_{U(m)}$; is given by [1] as:$f\left(x_k, x_m\right) =$

$$\frac{1}{\Gamma\left(k\right)\Gamma\left(m-k-1\right)} r\left(x_k\right) f\left(x_m\right)\left[R\left(x_k\right)\right]^{k-1}\left[R\left(x_m\right) - R\left(x_k\right)\right]^{m-k-1}, \tag{2}$$

where $r\left(x_k\right) = R^{/}\left(x_k\right)$. Concomitants can be easily defined for records as they can be defined in case of order statistics. Not much work has been done in the concomitants of record values as compared with the concomitants of order statistics. The distribution of k–th concomitant of record values is given by [1] as:

$$f\left(y_k\right) = \int_{-\infty}^{\infty} f\left(y_k|x_k\right) f_k\left(x_k\right) dx_k, \tag{3}$$

where $f_k\left(x_k\right)$ is given in (1). The joint distribution of k–th and m–th concomitant of record statistics is given by [1] as:

$$f\left(y_k, y_m\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_m} f\left(y_k|x_k\right) f\left(y_m|x_m\right) f_{k,m}\left(x_k, x_m\right) dx_k dx_m, \tag{4}$$

where $f_{k,m}\left(x_k, x_m\right)$ is given in (2).

In this paper we have derived the distribution of the concomitants of record statistics for Bivariate Pseudo–Exponential distribution. Some distributional properties of the resulting distribution have also discussed along with a characterization.

The Bivariate Pseudo–Exponential distribution is given in the following section.

2. Bivariate Pseudo-exponential Distribution

A new class of multivariate distribution; the Pseudo–Weibull and Pseudo–Gamma distribution; as the linear combinations of random variables has been defined by [5]. In this section we have defined the Bivariate Pseudo-exponential Distribution as the compound distribution of two random variables. The distribution is given below:

Suppose a random variable X has an exponential distribution with parameter α. The probability density function of X is:

$$f(x;\alpha) = \alpha e^{-\alpha x}; \alpha, x > 0. \tag{5}$$

Further, suppose that the random variable Y also has the exponential distribution with parameter $\phi(x)$, where $\phi(x)$ is some function of random variable X. The density function of Y is therefore given as:

$$f(y;\phi(x)) = \phi(x) e^{-\phi(x)y}; \phi(x), y > 0. \tag{6}$$

The compound distribution of (5) and (6) will be referred to as the Bivariate Pseudo–Exponential distribution and has the density:

$$f(x,y) = \alpha\phi(x) \exp\left[-\left\{\alpha x + \phi(x) y\right\}\right]; x, y, \alpha, \phi(x) > 0. \tag{7}$$

Several distributions can be defined with the help of (7) depending upon the choice of $\phi(x)$. Using $\phi(x) = x$ in (7), the distribution becomes:

$$f(x,y) = \alpha x \exp\left[-x(\alpha + y)\right]; x, y, \alpha > 0. \tag{8}$$

In the following section the distribution of concomitant of record statistics has been derived for (8).

3. Distribution of k–th Concomitant and Moments

In this section distribution of k–th concomitants of record statistics for Bivariate Pseudo-exponential Distribution has been obtained.

The distribution of the k–th concomitants record statistics is given in (3). Now to obtain the distribution we first need the distribution of k–th record statistics for random variable X. The distribution is given below:

For the random variable X we have $R(x) = -\ln[1 - F(x)] = \alpha x$ and so the distribution of k–th record is:

$$f_k(x_k) = \frac{1}{\Gamma(k)} \alpha^k x^{k-1} e^{-\alpha x}, \alpha, x > 0. \tag{9}$$

Further, the conditional distribution $f(y|x)$ is:

$$f(y|x) = x e^{-xy}; x, y > 0. \tag{10}$$

Now using (9) and (10) in (3), the distribution of k–th concomitant of record statistic, $y_k = y$, is given as:

$$\begin{aligned} h(y) &= \frac{\alpha^k}{\Gamma(k)} \int_0^\infty x^k e^{-x(\alpha+y)} dx \\ &= \frac{k\alpha^k}{(\alpha + y)^{k+1}}; k, \alpha, y > 0. \end{aligned} \tag{11}$$

The distribution function for (11) is:

$$F(y) = 1 - \frac{\alpha^k}{(\alpha + y)^k}; k, \alpha, y > 0. \tag{12}$$

The r-th moment for (11) is readily obtained as:

$$\mu_r^{/} = E(Y^r) = \int_0^\infty y^r h(y)\, dy = \int_0^\infty y^r \frac{k\alpha^k}{(\alpha + y)^{k+1}} dy \tag{13}$$

$$= \frac{\alpha^k \Gamma(k - r)\Gamma(r + 1)}{\Gamma(k)}; k > r. \tag{14}$$

The mean and the variance of the concomitants of record statistics for Pseudo-exponential distribution is given as:

$$E(Y) = \frac{\alpha}{k-1} \text{ and } Var(Y) = \frac{k\alpha^2}{(k-1)^2(k-2)}. \tag{15}$$

The coefficient of Skewness and Kurtosis are:

$$\gamma_1 = \frac{1}{k}\sqrt{\frac{2(k+1)(k-1)^3(k-2)^3}{\alpha^3\{k(k-5)+6\}}},$$

$$\gamma_2 = \frac{6\left(k^3 + k^2 - 6k - 2\right)}{k(k-3)(k-4)}.$$

We can readily see that the Kurtosis does not depend upon α.We have also derived a recurrence relation for moments of concomitants of record statistics in terms of its lower moments. The recurrence relation has been derived by using the fact that:

$$\bar{F}(y) = h(y)\frac{(\alpha + y)}{k}; \tag{16}$$

where $\bar{F}(x) = 1 - F(x)$. Now consider r-th moment as given in (13). Using (16) in (13) and simplifying, we obtain following recurrence relation for moments of concomitants of record statistics:

$$\mu_r^{/} = \frac{r\alpha}{(k - r)}\mu_{r-1}^{/}. \tag{17}$$

The relation (17) can be used to obtain higher moments of concomitants of record statistics in terms of its lower moments.

4. Joint Distribution of the Concomitants and Moments

In this section we have obtained the joint distribution of the concomitants of the record statistics for Pseudo–Exponential Distribution. The Joint distribution of two concomitant is given in (4). To obtain the distribution we first obtain the joint distribution of k–th and m–th record statistics for random variable X by using (2). This distribution is:

$$\begin{aligned} f(x_k, x_m) &= \frac{\alpha^m}{\Gamma(k)\Gamma(k-m)} x_1^{k-1}(x_2 - x_1)^{m-k-1} e^{-\alpha x_2}; \\ \alpha &> 0, x_1 < x_2, x_2 > 0. \end{aligned} \tag{18}$$

where $X_1 = X_{U(k)}$ and $X_2 = X_{U(m)}$. Using (10) and (18) in (4) and simplifying, the joint distribution of two concomitants of records is:

$$g(y_1, y_2) = \frac{k\alpha^m}{\Gamma(m+1)} \int_0^\infty x_2^{m+1} e^{-x_2(\alpha+y_2)} \, {}_1F_1(k+1, m+1; -x_2 y_1)\, dx_2,$$

which, on further simplification, provide following distribution of two concomitants of records:

$$\begin{aligned} g(y_1, y_2) &= \frac{k(m+1)\alpha^m(\alpha+y_2)^{k-m-1}}{(y_1+y_2+\alpha)^{k+1}} - \frac{y_1(\alpha+y_2)^{k-m-1}}{(y_1+y_2+\alpha)^{k+2}}. \\ &= \frac{(\alpha+y_2)^{k-m-1}[k(m+1)\alpha^m - y_1(y_1+y_2+\alpha)]}{(y_1+y_2+\alpha)^{k+1}}, \end{aligned} \tag{19}$$

where $Y_1 = Y_k$ and $Y_2 = Y_m$.The product moments are readily obtained from (19) as:

$$\begin{aligned} \mu'_{p,q} &= E(Y_1^p, Y_2^q) = \int_0^\infty \int_0^\infty y_1^p y_2^q g(y_1, y_2)\, dy_1 dy_2 \\ &= k\alpha^{p+q}(m-p) \times \\ &\quad \frac{\Gamma(p+1)\Gamma(q+1)\Gamma(k-p)\Gamma(m-p-q)}{\Gamma(k+1)\Gamma(m-p+1)}. \end{aligned} \tag{20}$$

Using (20) and (14), the covariance between two concomitants is given as:

$$Cov(y_1, y_2) = \frac{\alpha^2(k-m)}{(k-1)^2(m-2)}. \tag{21}$$

Further, by using (21) and (14), the correlation coefficient between two concomitants is:

$$\rho_{y_1 y_2} = \frac{(k-m)(k-2)}{k(m-2)}. \tag{22}$$

Since $k < m$, so correlation coefficient between two concomitants will always be negative.

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