Journal of Prime Research in Mathematics Vol. **6***(2010), 38-55*

SOME EXACT SOLUTIONS FOR THE FLOW OF A NEWTONIAN FLUID WITH HEAT TRANSFER VIA PRESCRIBED VORTICITY

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Abstract. Two-dimensional , steady, laminar equations of motion of an incompressible fluid with variable viscosity and heat transfer equations are considered. The problem investigated is the flow for which the vorticity distribution is proportional to the stream function perturbed by a sinusoidal stream. Employing transformation variable, the governing Navier-Stokes Equations are transformed into the ordinary differential equations and exact solutions are obtained. Finally, the influence of different parameters of interest on the velocity, temperature and pressure profiles are plotted and discussed.

Key words: exact solutions, Navier-Stokes equations, steady plane flows, incompressible fluid, variable viscosity, heat transfer, prescribed vorticity.
AMS SUBJECT: Primary 35Q30, 76D05.

1. Introduction

Due to the non-linearity of the Navier-Stokes equations and the inapplicability of the superposition principle for non-linear partial differential equations, exact solutions are difficult to obtain. For this reason, there exist only a limited number of exact solutions under certain conditions such that a number of terms in the equations of motion either disappear automatically or may be neglected, and the resulting equations reduce to a form that can be readily solved. Exact solutions are very important not only because they are solutions of some fundamental flows but also because they serve as accuracy

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checks for experimental, numerical or empirical and asymptotic methods. Although computer techniques make the complete integration of the equation of motion feasible, the accuracy of the results can be established by comparison with an exact solution. Wang [1] has given an excellent review of these solutions of the Navier-Stokes Equation. These known solutions of viscous incompressible Newtonian fluid may be classified into three types from Chandna and Oku-Ukpong [2]:

(i)- Flows for which the non-linear inertial terms in the linear momentum equations vanish identically. Parallel flows and flow with uniform suction are examples of these flows.

(ii)- Flow with similarity properties such that the flow equations reduce to a set of ordinary differential equations. Stagnation point flow is an example of such flow.

(iii)- Flow for which the vorticity function is chosen so that the governing equations in terms of the stream function reduce to a linear equation.

By considering the vorticity distribution directly proportional to the stream function $\nabla^2\psi = K\psi$, Taylor [3] showed that the non-linearities are self-canceling and obtained an exact solution which represent the decay of the double array of vortices. Kampe-De-Feriet [4] generalized the Taylor's idea by taking the vorticity of the form $\nabla^2\psi = f(\psi)$. Kovasznay [5] extended Taylor's idea by taking the vorticity to be proportional to the stream function perturbed by a uniform stream of the form $\nabla^2\psi = y + (K^2 - 4\pi^2)\psi$. Kovaszany was able to linearize the Navier-Stokes equation and determine an exact solution for steady flow, which resembles that the downstream of a two-dimensional gird. Wang [6] was able to linearize the Navier-Stokes equations and showed that the result established by Taylor and Kovasznay could be obtained from his finding as special case by taking the vorticity $\nabla^2\psi = Cy + A\psi$. Lin and Tobak [7], Hui [8] and Naeem and Jamil [9] obtained more results by studing similar flows, taking $\nabla^2\psi = K(\psi - Rz)$, $\nabla^2\psi = K(\psi - Ry)$ and $\nabla^2\psi = K(\psi - Uy)$. Recently Islam and Zhou [10] obtained some exact solution for couple stress fluids by taking $\nabla^2\psi = K(\psi - Uy)$.

By assuming certain form of vorticity distribution or stream function, solutions for Newtonian and non-Newtonian fluid are obtained by researchers such as Jeffrey [11], Riabouchinsky [12], Nemenyi [13], Ting [14], Rajagopal [15], Rajagopal and Gupta [16], Siddiqui and Kaloni [17], Wang [18], Benharbit and Siddiqui [19], Chandna and Oku-Ukpong [20], Oku-Ukpong and Chandna [21], Scconmandi [22], Labropulu [23], Labropulu [24], Mohyuddin et al. [25] and more recently Islam et al. [26] and Hayat et al. [27].

The exact solutions of the Navier-Stokes equations when the viscosity is variable are rare, however the literature in which the viscosity is variable is dependent upon the space, time, temperature, and pressure etc. Martin [28] first time used an elegant method in the study of the Navier-Stokes equations for an incompressible fluid of variable viscosity. Martin reduced the order of the governing equations from second order to first order by introducing the vorticity function and the generalized energy function. Martin there introduced the curvilinear coordinates ϕ, ψ in the plane flow, in which the coordinate lines ψ equal to constant are the streamlines of the flow and the coordinate lines ϕ constant are left arbitrary. Martin discussed some flows and suggested fifteen different types of flows, which could be studied. Naeem et al. [29] generalized Martin approach to study the steady-state, plane, variable viscosity, incompressible Navier-Stokes equations. Naeem et al. transformed the equations to a new system with, viscosity, vorticity, speed, energy function and the transformation matrices as the unknown functions, and determined some exact solutions for vortex, radial and parallel flows. Naeem [30], utilizing one parameter group of transformations, transformed the equations describing steady plane flow of an incompressible fluid of variable viscosity into a system of ordinary differential equations of second order. Naeem, utilizing particular method for finding the solutions of second order differential equations, determined new exact solutions and indicated that utilizing other one parameter groups one can determine some other solutions to the flow equations not determinable through other known methods. Later on the same fluid applied one-parameter group of transformations, von-Mises variables, Hodograph transformations etc; and obtained exact solution for example see Naeem et al. [31] and references there in. More recently work appeared on Newtonian fluid with variable viscosity is given by Hayat et al. [32] and Nadeem et al. [33].

In this paper, we present some new exact solutions to the equation governing the steady plane flows of an incompressible fluid with variable viscosity and heat transfer for which the vorticity distribution is proportional to the stream function perturbed by a sinusoidal stream of the form $\nabla^2\psi = K\big(\psi - U\sin(ax+by)\big)$. We point out that the exact solutions obtained by taking this form of vorticity to the best of our knowledge is yet not consider either in Newtonian or non-Newtonian flows.

The plan of this paper is as follows: In section 2 basic flow equations are considered and are transformed into a new system of equations. In section 3, some exact solutions of the new system of equations are determined. The method used in determining the exact solutions to these equations is straightforward.

2. Basic Governing Equations

The non-dimensional equations governing the steady plane flow of an incompressible fluid of variable viscosity, in the absence of external force and with no heat addition from Naeem and Jamil [9] are:

$$u_x + v_y = 0, \tag{1}$$

$$uu_x + vu_y = -p_x + \frac{1}{Re}\Big[(2\mu u_x)_x + \big(\mu(u_y + v_x)\big)_y\Big], \tag{2}$$

$$uv_x + vv_y = -p_y + \frac{1}{Re}\Big[(2\mu v_y)_y + \big(\mu(u_y + v_x)\big)_x\Big], \tag{3}$$

$$uT_x + vT_y = \frac{1}{RePr}(T_{xx} + T_{yy}) + \frac{Ec}{Re}\mu\Big[2(u_x^2 + v_y^2) + (u_y + v_x)^2\Big], \tag{4}$$

where u, v are the velocity components, x, y are cartesian coordinates, t is the time, p the pressure, T the temperature, μ the coefficient of viscosity, Re, Pr and Ec are Reynolds, Prandtl and Eckert numbers respectively. Equation (1) implies the existence of the stream function ψ such that

$$u = \psi_y,\ v = -\psi_x. \tag{5}$$

The system of Eqs.(1-4) on utilizing Eq.(5), transform into the following system of equations:

$$J_x = -\psi_x\omega + \frac{1}{Re}\Big[\mu(\psi_{yy} - \psi_{xx})\Big]_y, \tag{6}$$

$$J_y = -\psi_y\omega - \frac{4}{Re}(\mu\psi_{xy})_y + \frac{1}{Re}\Big[\mu(\psi_{yy} - \psi_{xx})\Big]_x, \tag{7}$$

$$\psi_y T_x - \psi_x T_y = \frac{1}{RePr}(T_{xx} + T_{yy}) + \frac{Ec}{Re}\mu\Big[4(\psi_{xy})^2 + (\psi_{yy} - \psi_{xx})^2\Big], \tag{8}$$

where the vorticity function ω and the generalized energy function J are defined by

$$\omega = -(\psi_{xx} + \psi_{yy}), \tag{9}$$

$$J = p + \frac{1}{2}(\psi_x^2 + \psi_y^2) - \frac{2\mu\psi_{xy}}{Re}. \tag{10}$$

Once a solution of system of Eqs.(6-8) is determined, the pressure p is obtained form Eq.(10). We shall investigate fluid motion for which the vorticity distribution is proportional to the stream function perturbed by a sinusoidal stream. This is given by

$$\psi_{xx} + \psi_{yy} = K\big(\psi - U\sin(ax + by)\big), \tag{11}$$

where $K, a, b \neq 0$,$a \neq b$ and U are real constants. On substituting

$$\Psi = \psi - U\sin(ax + by), \tag{12}$$

and employing Eq.(11), the Eq.(9) becomes

$$\omega = -K\Psi. \tag{13}$$

Equation(6) and (7), utilizing Eqs.(12) and (13), become

$$J_x = \Big(\frac{K\Psi^2}{2}\Big)_x - aUK\Psi\cos(ax+by) + \frac{1}{Re}\Big[\mu\big(\Psi_{yy} - \Psi_{xx} - (b^2 - a^2)\times \times U\sin(ax+by)\big)\Big]_y, \tag{14}$$

$$J_y = \Big(\frac{K\Psi^2}{2}\Big)_y - bUK\Psi\cos(ax+by) - \frac{4}{Re}\Big[\mu\big(\Psi_{xy} - abU\sin(ax+by)\big)\Big]_y + + \frac{1}{Re}\Big[\mu\big(\Psi_{yy} - \Psi_{xx} - (b^2 - a^2)U\sin(ax+by)\big)\Big]_x. \tag{15}$$

Equation(14) and (15), on using the compatibility condition $J_{xy} = J_{yx}$, provide

$$H_{xx} - H_{yy} + UK(a\Psi_y - b\Psi_x)\cos(ax+by) - \frac{4}{Re}\Big[\mu\Big\{\Psi_{xy} - -abU\sin(ax+by)\Big\}\Big]_{xy} = 0, \tag{16}$$

where

$$H = \frac{\mu\big(\Psi_{yy} - \Psi_{xx} - (b^2 - a^2)U\sin(ax+by)\big)}{Re}.$$

Equation(16) is the equation that must be satisfied by the function Ψ and the viscosity μ for the motion of an steady incompressible fluid of variable viscosity in which the vorticity distribution is proportional to the stream function perturbed by a sinusoidal stream. Equation(8), employing Eq.(12), becomes

$$\big(\Psi_y + bU\cos(ax+by)\big)T_x - \big(\Psi_x + aU\cos(ax+by)\big)T_y = = \frac{1}{RePr}(T_{xx} + T_{yy}) + \frac{Ec}{Re}\mu\Big[4\Big\{\Psi_{xy} - abU\sin(ax+by)\Big\}^2 + + \Big\{\Psi_{yy} - \Psi_{xx} - (b^2 - a^2)U\sin(ax+by)\Big\}^2\Big]. \tag{17}$$

Equation(11), employing equation(12), becomes

$$\Psi_{xx} + \Psi_{yy} - K\Psi = (a^2 + b^2)U\sin(ax+by). \tag{18}$$

Introducing new variable

$$\xi = ax + by.$$

Transforming the Equations(16), (17) and (18), into new independent variable ξ, we get

$$\Psi_{\xi\xi} - \Lambda\Psi = U \sin\xi, \tag{19}$$

where

$$\Lambda = \frac{K}{a^2 + b^2},$$

and

$$\left(\mu\Psi\right)_{\xi\xi} = 0, \tag{20}$$

$$T_{\xi\xi} + EcPr\Lambda^2(a^2 + b^2)\mu\Psi^2 = 0. \tag{21}$$

3. Exact Solutions

In this section we present some exact solution of the system of equations(19-21) as follows: We consider the following three cases:
Case-I: $\Lambda = -n^2, n > 0$
Case-II: $\Lambda = m^2, m > 0$
Case-III: $\Lambda = 0$
We now consider these cases separately and determine the solution of the equations(19-21). Our strategy is that first we find Ψ from equation(19) and use this Ψ to determine μ, T, ψ, u, v and p from system of Eqs.(20), (21), (12), (5) and (10).

Case-I

For this case the solution of Eq.(19) in the physical plane is given by

$$\Psi = A_1 \cos(n(ax + by) + A_2) - \frac{U \sin(ax + by)}{1 - n^2}, \tag{22}$$

where A_1 and A_2 are real constants. Equation(20), utilizing Eq.(22), gives

$$\mu = \frac{A_3(ax + by) + A_4}{A_1 \cos(n(ax + by) + A_2) - \frac{U \sin(ax+by)}{1-n^2}}, \tag{23}$$

where A_3 and A_4 are real constants. Equation(21), using Eq.(23), becomes

$$T_{\xi\xi} + EcPrn^4(a^2 + b^2)\Psi(A_3\xi + A_4) = 0. \tag{24}$$

The solution of Eq.(24) is

$$T = EcPrn^4(a^2+b^2)\Bigg[\frac{A_1A_4}{n^2}\cos(n(ax+by)+A_2)+\frac{A_4U}{n^2-1}\sin(ax+by)+$$

$$+\frac{A_1A_3}{n^3}\Big\{n(ax+by)\cos(n(ax+by)+A_2)-2\sin(n(ax+by)+A_2)\Big\}+$$

$$+\frac{A_3U}{n^2-1}\Big\{2\cos(ax+by)+(ax+by)\sin(ax+by)\Big\}\Bigg]+A_5(ax+by)+A_6, \quad (25)$$

where A_5 and A_6 are real constants. The stream function ψ for this case is given by

$$\psi = \frac{Un^2}{n^2-1}\sin(ax+by)+A_1\cos(n(ax+by)+A_2). \quad (26)$$

It represent a sinusoidal stream $\frac{Un^2}{n^2-1}\sin(ax+by)$ in the positive x-direction plus a perturbation that is periodic in x and y. The component of velocity distribution from Eqs.(5) and (26), and pressure from Eq.(10), are given by

$$u = \frac{Ubn^2}{n^2-1}\cos(ax+by)-A_1nb\sin(n(ax+by)+A_2), \quad (27)$$

$$v = -\frac{Uan^2}{n^2-1}\cos(ax+by)+A_1na\sin(n(ax+by)+A_2), \quad (28)$$

$$p = \frac{U^2n^4}{4(n^2-1)}\cos(2(ax+by)+\frac{A_1Un^2}{2(n-1)}\sin\big((n-1)(ax+by)+A_2\big)+$$

$$+\frac{A_1Un^2}{4(n+1)}\sin\big((n+1)(ax+by)+A_2\big)+\frac{A_3n^2(a^2-b^2)}{Re}(ay+bx)-$$

$$-\frac{(a^2+b^2)n^2}{2(n^2-1)^2}\Big\{Un\cos(ax+by)-A_1(n^2-1)\sin\big(n(ax+by)+A_2\big)\Big\}^2+$$

$$+\frac{2abn^2}{Re}\big(A_3(ax+by)+A_4\big)+A_7, \quad (29)$$

where A_7 is real constant. The compatibility condition $J_{xy}=J_{yx}$, implies $p_{xy}=p_{yx}$, which is obviously satisfied for this case.

Case-II

For this case

$$\Psi = B_1 e^{m(ax+by)} + B_2 e^{-m(ax+by)} - \frac{U\sin(ax+by)}{1+m^2}, \tag{30}$$

where B_1 and B_2 are real constants. Equation(20), utilizing Eq.(30), gives

$$\mu = \frac{B_3(ax+by)+B_4}{B_1 e^{m(ax+by)} + B_2 e^{-m(ax+by)} - \frac{U\sin(ax+by)}{1+m^2}}, \tag{31}$$

where B_3 and B_4 are real constants. Equation(21), using Eq.(31), becomes

$$T_{\xi\xi} + EcPrm^4(a^2+b^2)\Psi(B_3\xi + B_4) = 0. \tag{32}$$

The solution of Eq.(32) is

$$T = EcPrm^4(a^2+b^2)\Bigg[\frac{B_1B_4}{m^2}e^{m(ax+by)} + \frac{B_2B_4}{m^2}e^{-m(ax+by)} +$$

$$+\frac{B_1B_3}{m^3}\Big(m(ax+by)-2\Big)e^{m(ax+by)} + \frac{B_2B_3}{m^3}\Big(m(ax+by)+2\Big)e^{-m(ax+by)} -$$

$$-\frac{B_3U}{m^2+1}\Big\{2\cos(ax+by) + (ax+by)\sin(ax+by)\Big\} - \frac{B_3U}{m^2+1}\sin(ax+by)\Bigg] +$$

$$+B_5(ax+by) + B_6, \tag{33}$$

where B_5 and B_6 are real constants. For this case stream function

$$\psi = \frac{Um^2}{m^2+1}\sin(ax+by) + B_1 e^{m(ax+by)} + B_2 e^{-m(ax+by)}, \tag{34}$$

represent a sinusoidal stream $\frac{Um^2}{m^2+1}\sin(ax+by)$ in the positive x-direction plus a perturbation that is not periodic in x and y. The components of velocity distribution and pressure, are given by

$$u = \frac{Ubm^2}{m^2+1}\cos(ax+by) + B_1 mb e^{m(ax+by)} - B_2 mb e^{-m(ax+by)}, \tag{35}$$

$$v = -\frac{Uam^2}{m^2+1}\cos(ax+by) - B_1 ma e^{m(ax+by)} + B_2 ma e^{-m(ax+by)}, \tag{36}$$

$$p = \frac{B_3 m^2(b^2-a^2)}{Re}(ay+bx) - \frac{2abm^2}{Re}\Big(B_3(ax+by)+B_4\Big) -$$

$$-\frac{U^2m^2}{4(m^2+1)}\cos(2(ax+by)) - \frac{(a^2+b^2)m^2e^{-2m(ax+by)}}{2(m^2+1)^2}\times$$

$$\times\Big\{Ume^{m(ax+by)}\cos(ax+by) + B_1(m^2+1)e^{2m(ax+by)} - B_2(m^2+1)\Big\}^2 -$$

$$-\frac{B_2Um^2}{m^2+1}e^{-m(ax+by)}\Big\{\sin(ax+by) - m\cos(ax+by)\Big\} -$$

$$-\frac{B_1Um^2}{m^2+1}e^{m(ax+by)}\Big\{m\cos(ax+by) + \sin(ax+by)\Big\} + B_7, \tag{37}$$

where B_7 is real constant. The compatibility condition is also satisfied for this case.

Case-III

For this case, we have

$$\Psi = C_1(ax+by) + C_2 - U\sin(ax+by), \tag{38}$$

$$\mu = \frac{C_3(ax+by) + C_4}{C_1(ax+by) + C_2 - U\sin(ax+by)}, \tag{39}$$

$$T = C_5(ax+by) + C_6, \tag{40}$$

where C_1, C_2,..., C_6 are real constants. For this case, we have $\Lambda = 0$, which corresponds to an irrotational flow and it is the following uniform flow

$$\psi = C_1(ax+by) + C_2. \tag{41}$$

The components of velocity distribution and pressure, in this case are given by

$$u = C_1b, \tag{42}$$

$$v = -C_1a, \tag{43}$$

$$p = -\frac{(a^2+b^2)C_1^2}{2} + C_7, \tag{44}$$

where C_7 is real constant.

4. Results and Discussion

This section deals with the influence of the parameters n, m, a and b on the velocity, temperature and pressure distribution profiles. Figures(1-8) are for case-I and figures(9-16) are for case-II. The effect of Prandtl number Pr and Eckert number Ec on the temperature profile is also included in this section. Figs.1 and 2 show the velocity profile u in the direction of y for different values of parameter n and b. It is clear form these figures that velocity has oscillating behavior and magnitude of velocity or amplitude of the oscillation increase with increase of parameter n and b. It is noted that increase in these parameter increase the velocity in the most narrow position. Similar effects are observed for the velocity component v in the direction of x as shown in Figs. 3 and 4, and Figs. 5 and 6. shows temperature is increase with increase of parameter n and b. Figs. 7 and 8 shows the pressure profile p in the direction of y. In Fig. 7 pressure increase with an increase of parameter n, however pressure has mixed behavior for parameter b in Fig. 8.
Figs. 9 and 10 shows the velocity profile u increase with increase of the parameter m and b, however velocity component v have quite opposite behavior as shown in Figs. 11 and 12. It can be further seen form Figs. 13, 14, 15 and 16 the effect of the parameter m and b on the temperature and pressure profile are again quite opposite.

The effect of Prandtl and Eckert numbers on the temperature form Eqs. (25) and (33) are clear that temperature is directly proportional to the Prandtl and Eckert numbers and temperature increase or decrease with increase or decrease of Prandtl and Eckert numbers.

5. Concluding Remarks

Some exact solutions of the equations governing the steady laminar plane motion of an incompressible fluid with variable viscosity and heat transfer are determined. These solutions consist of flows for which the vorticity distribution is proportional to the streamfunction perturbed by a sinusoidal stream.

In order to determine the exact solutions, the flow equations are first written in terms of the streamfunction ψ, the vorticity function ω and the generalized energy function J. Employing the compatibility condition on the generalized energy function J, an equation is determined that must be satisfied by the function Ψ and the viscosity μ for the flow under consideration.

The solutions are obtained through the procedure described in section 3. All the solutions satisfy the compatibility condition $J_{xy} = J_{yx} (\Rightarrow p_{xy} = p_{yx})$. The solutions in case-I represents a sinusoidal stream plus a perturbation that is periodic in x and y. The solution in case-II, in general, represents a sinusoidal stream plus a perturbation that is not periodic in x and y. When $B_1 = 0$, in the solution of case-II, the solution represents a sinusoidal flow in the region

$x > 0$, $y > 0$ perturbed by a part which decays and grows exponentially as x , y increase for $m > 0$ and $m < 0$, respectively. Similarly, we can give description for flow in other regions. Finally the effect of various parameters of interest on the velocity components, temperature and pressure are plotted and discussed.

Acknowledgement

The authors would like to take great pleasure in thanking the reviewers for their valuable and helpful comments and suggestions for improvement of the earlier draft of this work.

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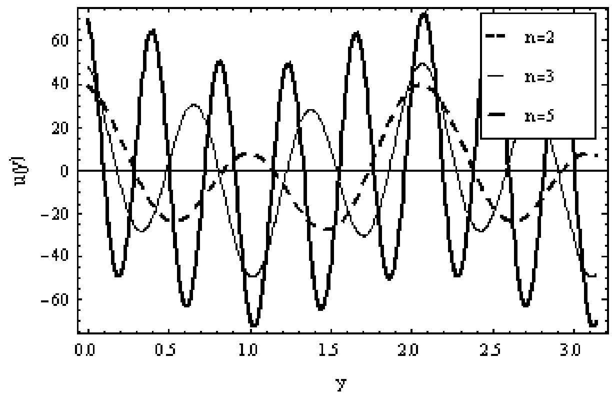

Fig.1. Velocity profiles of u in the direction of y with different values of the parameter "n" for $A_1 = 4$, $A_2 = 5$, $U = 4$, $b = 3$.

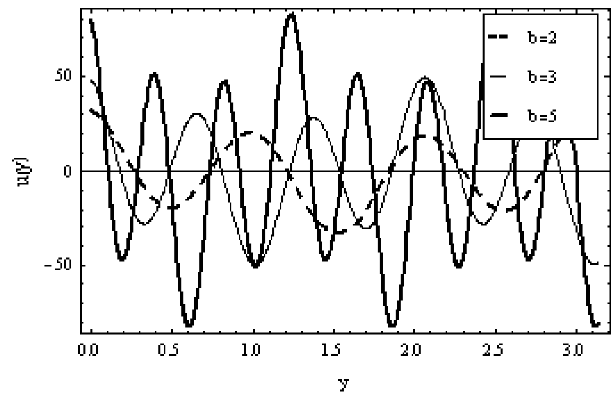

Fig.2. Velocity profiles of u in the direction of y with different values of the parameter "b" for $A_1 = 4$, $A_2 = 5$, $U = 4$, $n = 3$.

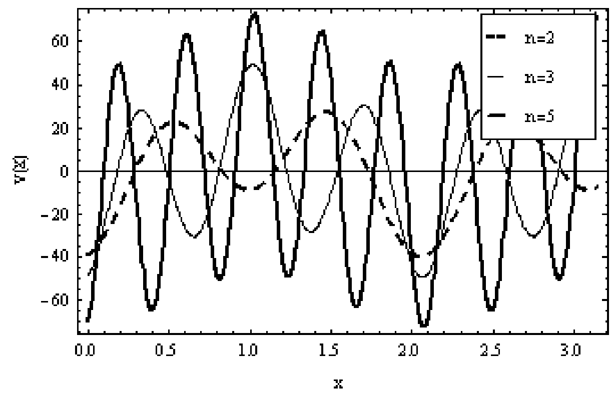

Fig.3. Velocity profiles of v in the direction of x with different values of the parameter "n" for $A_1 = 4$, $A_2 = 5$, $U = 4$, $a = 3$.

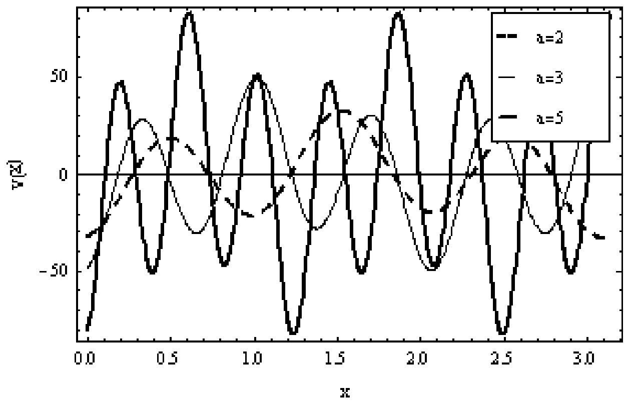

Fig.4. Velocity profiles of v in the direction of x with different values of the parameter "a" for $A_1 = 4$, $A_2 = 5$, $U = 4$, $n = 3$.

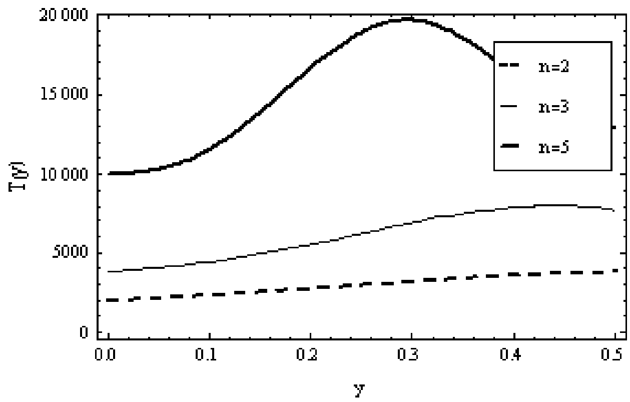

Fig.5. Temperature profiles T in the direction of y with different values of the parameter "n" for $A_1 = 1$, $A_2 = 2$, $A_3 = 3$, $A_4 = 4$, $A_5 = 5$, $A_6 = 6$, $U = 4$, $b = 3$, $Ec = 1$, $Pr = 2$.

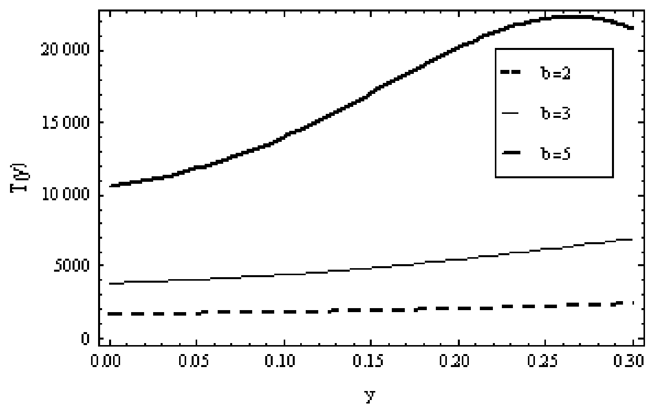

Fig.6. Temperature profiles T in the direction of y with different values of the parameter "b" for $A_1 = 1$, $A_2 = 2$, $A_3 = 3$, $A_4 = 4$, $A_5 = 5$, $A_6 = 6$, $U = 4$, $n = 3$, $Ec = 1$, $Pr = 2$.

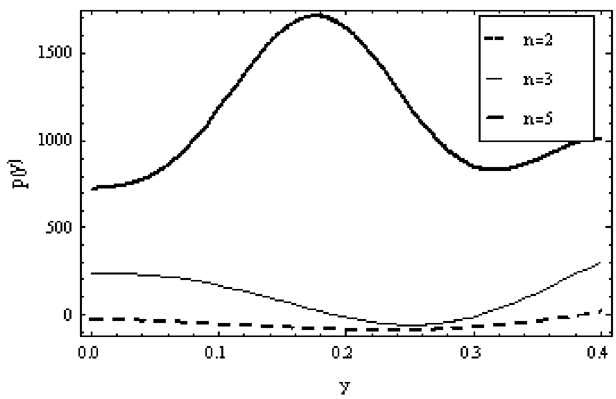

Fig.7. Pressure profiles p in the direction of y with different values of the parameter “n” for $A_1 = 1$, $A_2 = 2$, $A_3 = 3$, $A_4 = 4$, $A_5 = 5$, $A_6 = 6$, $A_7 = 7$, $U = 4$, $a = 4$, $b = 3$, $Re = 4$.

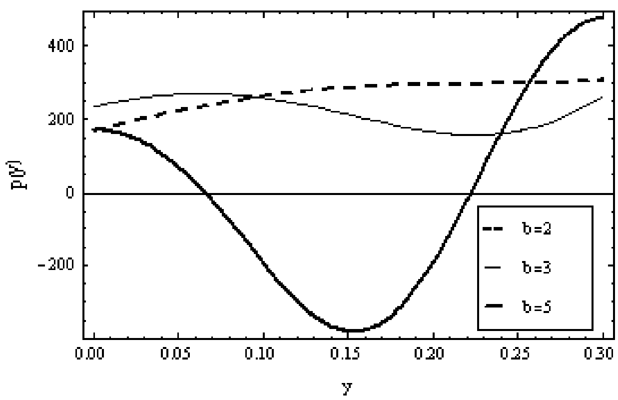

Fig.8. Pressure profiles p in the direction of y with different values of the parameter “b” for $A_1 = 1$, $A_2 = 2$, $A_3 = 3$, $A_4 = 4$, $A_5 = 5$, $A_6 = 6$, $A_7 = 7$, $U = 4$, $a = 4$, $n = 3$, $Re = 4$.

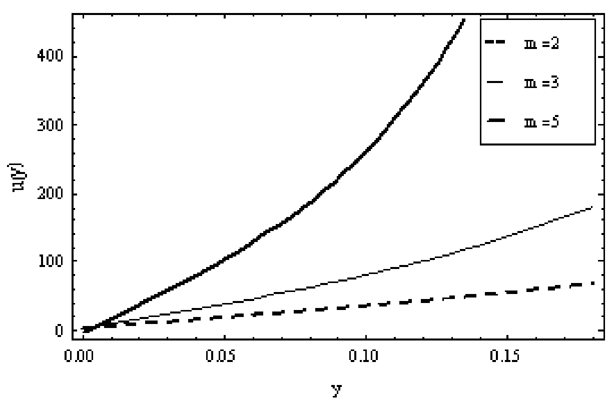

Fig.9. Velocity profiles of u in the direction of y with different values of the parameter “m” for $B_1 = 4$, $B_2 = 5$, $U = 4$, $b = 3$.

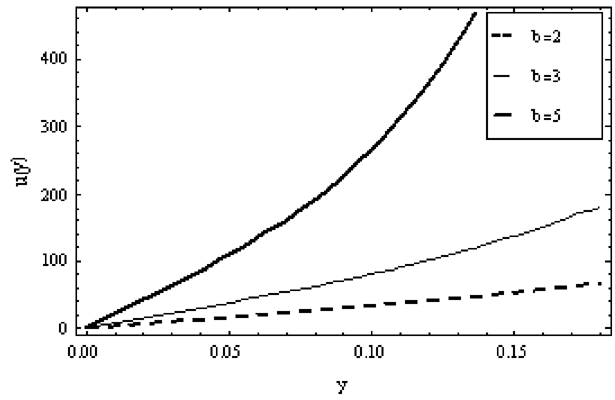

Fig.10. Velocity profiles of u in the direction of y with different values of the parameter "b" for $B_1 = 4$, $B_2 = 5$, $U = 4$, $m = 3$.

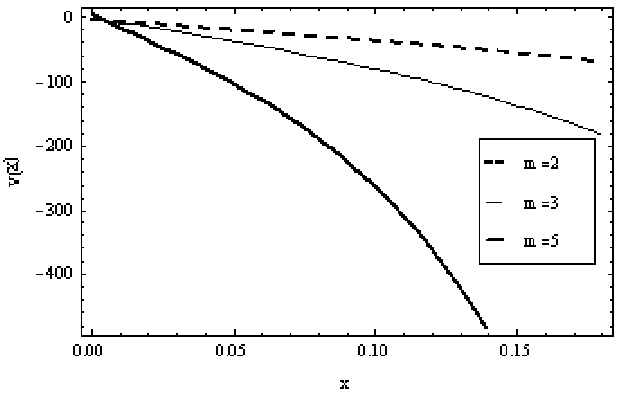

Fig.11. Velocity profiles of v in the direction of x with different values of the parameter "m" for $B_1 = 4$, $B_2 = 5$, $U = 4$, $b = 3$.

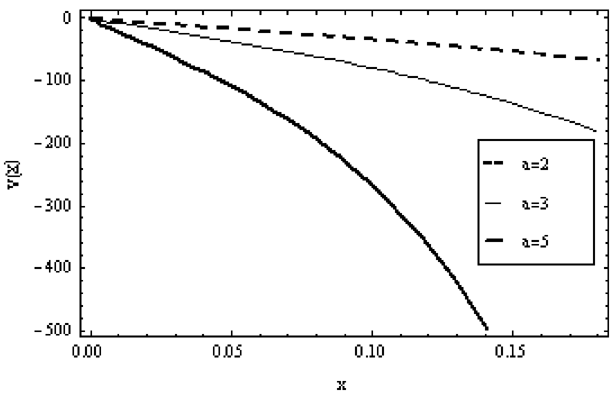

Fig.12. Velocity profiles of v in the direction of x with different values of the parameter "a" for $B_1 = 4$, $B_2 = 5$, $U = 4$, $m = 3$.

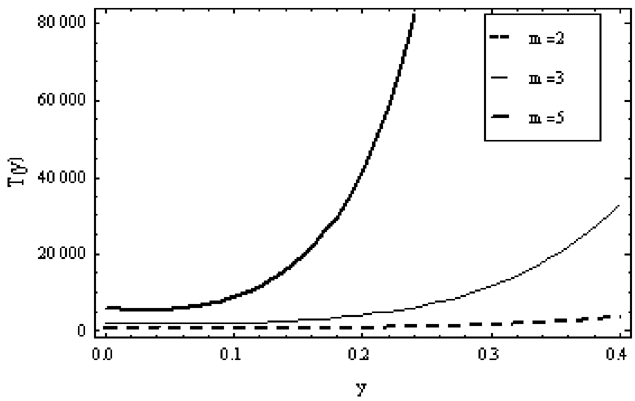

Fig.13. Temperature profiles T in the direction of y with different values of the parameter "m" for $B_1 = 1$, $B_2 = 2$, $B_3 = 3$, $B_4 = 4$, $B_5 = 5$, $B_6 = 6$, U = 4, b = 3, Ec = 1, Pr = 2.

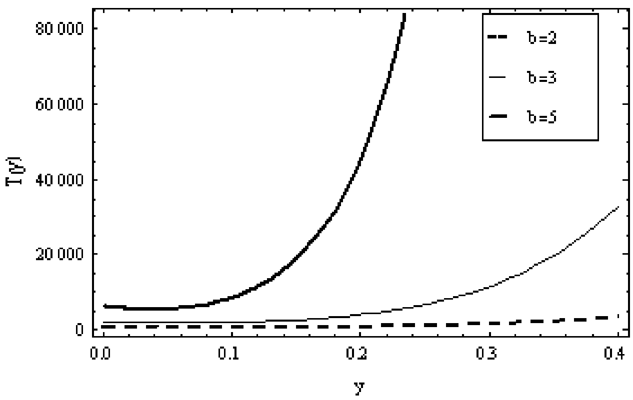

Fig.14. Temperature profiles T in the direction of y with different values of the parameter "b" for $B_1 = 1$, $B_2 = 2$, $B_3 = 3$, $B_4 = 4$, $B_5 = 5$, $B_6 = 6$, U = 4, m = 3, Ec = 1, Pr = 2.

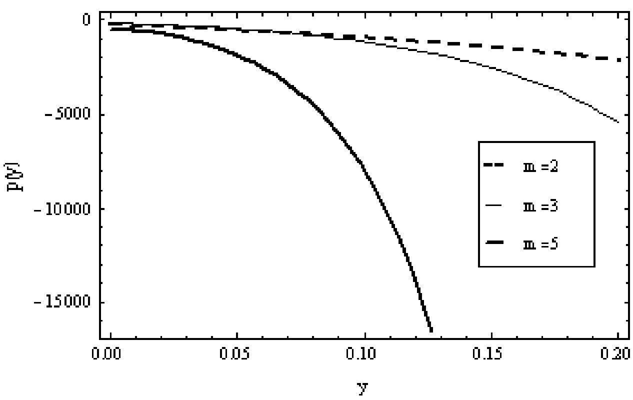

Fig.15. Pressure profiles p in the direction of y with different values of the parameter "m" for $B_1 = 1$, $B_2 = 2$, $B_3 = 3$, $B_4 = 4$, $B_5 = 5$, $B_6 = 6$, $B_7 = 7$, U = 4, a = 4, b = 3, Re = 4.

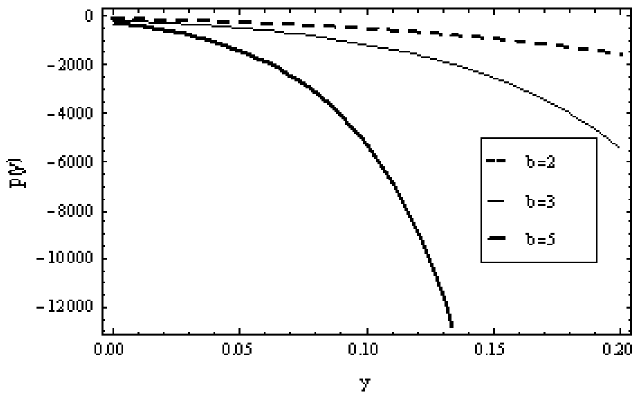

Fig.16. Pressure profiles p in the direction of y with different values of the parameter "b" for $B_1 = 1$, $B_2 = 2$, $B_3 = 3$, $B_4 = 4$, $B_5 = 5$, $B_6 = 6$, $B_7 = 7$, $U = 4$, $a = 4$, $m = 3$, $Re = 4$.