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SHAPE PRESERVING CONSTRAINED DATA VISUALIZATION USING RATIONAL FUNCTIONS

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Abstract. This work has been contributed on the visualization of curves and surfaces for constrained data. A rational cubic function, with free shape parameters in its description, has been introduced and used. This function has been constrained to visualize the preservation of shape of the data by imposing constraints on free parameters. The rational cubic curve case has also been extended to a rational bi-cubic partially blended surface to visualize the shape preserving surface to constrained data.

Key words: rational function, constrained curve data, constrained surface data, free parameters.

1. Introduction

Shape preserving data visualization is a key issue in Computer Graphics, Computer Aided Geometric Design, Computer Aided Design and Computer Aided Manufacturing, etc. Quite a few methods have been developed in the last two decades [1-11]. Spline interpolation methods have been given much attention by the researchers due to their effectiveness and hence have been applied widely. Parametric rational spline interpolation has got good attention due to the presence of parameters in the description of interpolation functions. These parameters can guide and help to control and constrain the shape of the resultant curves and surfaces.

In the last few years, a good amount of work has been done in the field of shape preserving data [1-11]. Asim et al [1] presented an algorithm for the visualization of positive data. They used modified quadratic Shepard method to

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obtain non-negative curve. Brodlie et al [2] discussed the problem of visualizing scattered data having underlying constraints. They proved that modified quadratic Shepard method, interpolates the given scattered data of any dimensionality. They applied positive constraints on quadratic basis function to attain the desired positive interpolanting function. Furthermore, they restricted the interpolant within the range from 0 to 1 to interpolate the data where 0 is lower bound and 1 is upper bound of the interpolanting function.

Brodlie et al [3] addressed with two problems; one is the interpolation subject to simple linear constraints and other for constructing a positive piecewise bi-cubic function over a rectangular mesh. They derived constraints in the form of the first partial derivatives and mixed partial derivatives at knots. They generalized the problem of linearly constrained interpolation, where it is expected that bi-cubic function lies between the bounds of linear functions. Chan and Ong [4] used cubic Bzier triangular interpolant to interpolate the range restricted irregular surface data. The interpolating surface was a convex combination of cubic Bzier triangular patches. They derived sufficient conditions on Bzier ordinates to obtain non-negative cubic Bzier triangular patch. If the Bzier ordinates fail to satisfy the sufficient conditions then it was modified by scaling of first order partial derivatives at the vertices. Constantini and Manni [5] proposed a local method for the construction of monotonic functions defined over the regular data. Their suggested scheme is based upon the Boolean sum of cubic interpolating operators. These operators blend together $1D$ shape-preserving interpolating polynomials of adaptive degree.

Hussain and Maria [7] presented a scheme to preserve the shape of positive surface data by using rational bi-cubic spline (with eight parameters in its description); they also presented a method to preserve the shape of surface data lying above the plane. In [7] authors derived constraints on eight parameters to attain the desired shape. Hussain et al [8] developed a scheme for the visualization of constrained curve data by using rational cubic function with four parameters in its description. They derived constraints on two parameters and kept the remaining two as free parameters. The authors extended this work for the visualization of constrained surface data.

Hussain et al [9] introduced a rational cubic function (cubic/linear) with one free parameter and developed three schemes; first for the visualization of positive data, second for the visualization of constrained data and third for the visualization of monotone data by making constraints on one parameter. Piah et al [11] worked on the problem of range restricted interpolation using quartic Bzier triangular interpolanting function; they achieved positivity of surfaces by applying the constraints on Bzier ordinates.

In this paper, the authors have developed a smooth rational cubic function (cubic over quadratic) with two free parameters in its description to preserve the shape of constrained data of positive nature. The advantages of the developed schemes in the paper are as follows:

- The shape of the data is preserved by imposing conditions on free parameters in the description of rational cubic function without inserting any extra knot.
- The schemes developed in this paper work for both equally and unequally spaced data.
- The desired shape of the data is attained by imposing the data dependent constraints on the free parameters in the description of rational cubic function. So, certain function values and derivative values are not assumed for shape control.
- Finding the values of the derivatives parameters is computationally economical.

This paper is designed as follows. Section 2 is the review of the rational cubic function developed in [10] together with an extension to rational bicubic partially blended function. In Section 3, two schemes are proposed: one for the visualization of shape preserving constrained curve data and other for the visualization of shape preserving constrained surface data. Numerical examples are also illustrated in this section. Section 4 concludes the work.

2. Rational Interpolation

This section is devoted to the interpolation methods used for curve and surface descriptions for the corresponding constrained data sets. The rational cubic function, for curve manipulation, is defined in Section 2.1 and its extended bicubic version is defined in Section 2.2.

2.1. Rational Cubic Function.

Let $(x_i, f_i), i = 1, 2, 3, ..., n$ be given set of data points where $x_1 < x_2 < ... < x_n$. The rational cubic function is defined as:

$$S(x) = S(x_i) = \frac{\sum_{i=1}^{4} \omega_i \theta^{i-1}(1-\theta)^{4-i}}{q_i(\theta)}, \tag{1}$$

where
$\omega_1 = \mu_i f_i$
$\omega_2 = (2\mu_i + \nu_i) f_i + \mu_i h_i d_i$
$\omega_3 = (\mu_i + 2\nu_i) f_{i+1} - \nu_i h_i d_{i+1}$
$\omega_4 = \nu_i f_{i+1}$
$q_i(\theta) = \mu_i(1-\theta)^2 + (\mu_i + \nu_i)\theta(1-\theta) + \nu_i \theta^2$
and
$\theta = \frac{x - x_i}{h_i}$, $h_i = x_{i+1} - x_i$, $\forall i = 1, 2, 3, ..., n-1$.

The rational cubic function (1)has the following interpolatory properties:

$$S(x_i) = f_i, \quad S(x_{i+1}) = f_{i+1}$$
$$S^{(1)}(x_i) = d_i, \quad S^{(1)}(x_{i+1}) = d_{i+1}$$

where $S^{(1)}(x)$ denotes the first order derivative with respect to x. The d_i denotes derivative value at the knot x_i. The rational cubic function (1) reduces to standard cubic Hermite function when we take the values of free parameters $\mu_i = \nu_i = 1$.

2.2. **Rational Bi-Cubic Function.** Let $\pi : a = x_0 < x_1 < x_2 < \cdots < x_n$ be partition of $[a, b]$ and $\hat{\pi} : c = y_0 < y_1 < y_2 < \cdots < y_m$ be partition of $[c, d]$. The rational bi-cubic partially blended function is defined over each rectangular patch $[x_i, x_{i+1}] \times [y_j, y_{j+i}]$, $i = 0, 1, 2, \cdots, n-1, j = 0, 1, 2, \cdots, m-1$ as:

$$S(x, y) = -PFQ^t \tag{2}$$

where

$$F = \begin{pmatrix} 0 & S(x, y_j) & S(x, y_{j+1}) \\ S(x_i, y) & S(x_i, y_j) & S(x_i, y_{j+1}) \\ S(x_{i+1}, y) & S(x_{i+1}, y_j) & S(x_{i+1}, y_{j+1}) \end{pmatrix}$$

$$P = [-1 \quad p_0(\theta) \quad p_1(\theta)],$$
$$Q = [-1 \quad q_0(\phi) \quad q_1(\phi)]$$

with

$$p_0 = (1-\theta)^2(1+2\theta), p_1 = \theta^3(3-2\theta), q_0 = (1-\phi)^2(1+2\phi), q_1 = \phi^2(3-2\phi).$$

$$\theta = \frac{x - x_i}{h_i}, h_i = x_{i+1} - x_i, \phi = \frac{y - y_j}{\hat{h}_j}, \hat{h}_j = y_{j+1} - y_j, 0 \le \theta, \phi \le 1.$$

$S(x, y_j)$, $S(x, y_{j+1})$, $S(x_i, y)$ and $S(x_{i+1}, y)$ are the ratioanl cubic function (1) defined over the boundry of rectangular patch $[x_i, x_{i+1}] \times [y_j, y_{j+i}]$ as:

$$S(x, y_j) = \frac{\sum_{i=1}^{4} A_{i,j}\theta^{i-1}(1-\theta)^{4-i}}{q_{i,j}(\theta)}, \tag{3}$$

with

$A_{1,j} = \mu_{i,j}^2 F_{i,j}$,
$A_{2,j} = (2\mu_{i,j} + \nu_{i,j})F_{i,j} + \mu_{i,j}h_i F_{i,j}^x$,
$A_{3,j} = (\mu_{i,j} + 2\nu_{i,j})F_{i+1,j} - \nu_{i,j}h_i F_{i+1,j}^x$,
$A_{4,j} = \nu_{i,j}^2 F_{i+1,j}$,
$q_{i,j} = \mu_{i,j}(1-\theta)^2 + (\mu_{i,j} + \nu_{i,j})\theta(1-\theta) + \nu_{i,j}\theta^2$,

$$S(x, y_{j+1}) = \frac{\sum_{i=1}^{4} B_{i,j}\theta^{i-1}(1-\theta)^{4-i}}{q_{i,j+1}(\theta)}, \tag{4}$$

with
$B_{1,j} = \mu_{i,j+1}^2 F_{i,j+1}$,
$B_{2,j} = (2\mu_{i,j+1} + \nu_{i,j+1})F_{i,j+1} + \mu_{i,j+1}h_i F_{i,j+1}^x$,
$B_{3,j} = (\mu_{i,j+1} + 2\nu_{i,j+1})F_{i+1,j+1} - \nu_{i,j+1}h_i F_{i+1,j+1}^x$,
$B_{4,j} = \nu_{i,j+1}^2 F_{i+1,j+1}$,
$q_{i,j+1} = \mu_{i,j+1}(1-\theta)^2 + (\mu_{i,j+1} + \nu_{i,j+1})\theta(1-\theta) + \nu_{i,j+1}\theta^2$,

$$S(x_i, y) = \frac{\sum_{i=1}^{4} C_{i,j}\phi^{i-1}(1-\phi)^{4-i}}{\hat{q}_{i,j}(\phi)}, \tag{5}$$

with
$C_{1,j} = \hat{\mu}_{i,j}^2 F_{i,j}$,
$C_{2,j} = (2\hat{\mu}_{i,j} + \hat{\nu}_{i,j})F_{i,j} + \hat{\mu}_{i,j}\hat{h}_j F_{i,j}^y$,
$C_{3,j} = (\hat{\mu}_{i,j} + 2\hat{\nu}_{i,j})F_{i,j+1} - \hat{\nu}_{i,j}\hat{h}_j F_{i,j+1}^y$,
$C_{4,j} = \hat{\nu}_{i,j}^2 F_{i,j+1}$,
$\hat{q}_{i,j} = \hat{\mu}_{i,j}(1-\phi)^2 + (\hat{\mu}_{i,j} + \hat{\nu}_{i,j})\phi(1-\phi) + \hat{\nu}_{i,j}\phi^2$,

$$S(x_i + 1, y) = \frac{\sum_{i=1}^{4} D_{i,j}\phi^{i-1}(1-\phi)^{4-i}}{\hat{q}_{i+1,j}(\phi)}, \tag{6}$$

with
$D_{1,j} = \hat{\mu}_{i+1,j}^2 F_{i+1,j}$,
$D_{2,j} = (2\hat{\mu}_{i+1,j} + \hat{\nu}_{i+1,j})F_{i+1,j} + \hat{\mu}_{i+1,j}\hat{h}_j F_{i+1,j}^y$,
$D_{3,j} = (\hat{\mu}_{i+1,j} + 2\hat{\nu}_{i+1,j})F_{i+1,j+1} - \hat{\nu}_{i+1,j}\hat{h}_j F_{i+1,j+1}^y$,
$D_{4,j} = \hat{\nu}_{i,j}^2 F_{i,j+1}$,
$\hat{q}_{i+1,j} = \hat{\mu}_{i+1,j}(1-\phi)^2 + (\hat{\mu}_{i+1,j} + \hat{\nu}_{i+1,j})\phi(1-\phi) + \hat{\nu}_{i+1,j}\phi^2$,

3. Visualization by Curves and Surfaces

This section is devoted to the proposed constrained interpolation methods used for curve and surface descriptions for the corresponding constrained data sets. The rational cubic function, for curve manipulation, is devised in Section 3.1 and its extended bicubic version is devised in Section 3.2.

3.1. **Constrained Curve Data Visualization.**

Let $(x_i, f_i), i = 1, 2, 3, ..., n$ be the given data points which lie above the straight line $f(x) = mx + c$ i.e. $f_i > mx_i + c, \forall i = 1, 2, ...n$ we required $S(x) \equiv S_i(x) > mx_i + c$, let us assume that $m > 0$ (The case when $m < 0$ can be dealt in the same fashion). In each interval $x \in [x_i, x_i+1], mx+c$ can be written as:

$$a_i(1-\theta)+b_i\theta \tag{7}$$

where

$$a_i = mx_i + c, b_i = mx_{i+1} + c.$$

We required that

$$S_i(x) > a_i(1-\theta)+b_i\theta, \quad \forall i = 1,2,...,n. \tag{8}$$

Equation (8) can be written as:

$M_i = f_i(1-\theta)^3 + W_i\theta(1-\theta)^2 + T_i\theta^2(1-\theta) + f_{i+1}\theta^3 - \{a_i(1-\theta)+b_i\theta\} > 0.$

This implis:

$$\mu_i(f_i - a_i)(1-\theta)^3 + (W_i - a_i - \mu_i b_i)\theta(1-\theta)^2 + (T_i - \nu_i a_i - b_i)\theta^2(1-\theta) + \nu_i(f_{i+1} - b_i)\theta^3 > 0$$

as $(f_i - a_i) > 0$, and $(f_{i+1} - b_i) > 0$,

Now $M_i > 0$ iff $(W_i - a_i - \mu_i b_i) > 0$, and $(T_i - \nu_i a_i - b_i) > 0$

$$(W_i - a_i - \mu_i b_i) > 0 \Longrightarrow \mu_i > \frac{a_i - \nu_i f_i}{2f_i + h_i d_i - b_i} \tag{9}$$

$$(T_i - b_i - \nu_i a_i) > 0 \Longrightarrow \mu_i > \frac{a_i - 2f_{i+1} + h_i d_{i+1}\nu_i b_i}{f_{i+1}} \tag{10}$$

The above discussion can be written in the form of the following theorem:

Theorem 1. The rational cubic function defined in (1) lies above the given straight line if and only if

$$\nu_i > 0,$$

$$\mu_i = g_i + max\{\frac{a_i - \nu_i f_i}{2f_i + h_i d_i - b_i}, \frac{a_i - 2f_{i+1} + h_i d_{i+1}\nu_i b_i}{f_{i+1}}\}$$

The above discussion can be summarized in the following algorithm for computation purposes:

Algorithm 1

Step 1. Enter n data points $(x_i, f_i), i = 1,2,3,...,n,$that lie above the line.

Step 2. Estimate the derivative parameters $d_i^{,}s$ at knots $x_i^{,}s$ by Arithmetic Mean Method [9].

Step 3. Calculate the values of shape parameters $\mu_i^{,}s$ and $\nu_i^{,}s$ using Theorem 1.

Step 4. Substitute the values of $f_i^{,}s, d_i^{,}s \quad \forall i = 1,2,...,n$; and $\mu_i^{,}s,$, $\nu_i^{,}s \forall i = 1,2,...,n$ in rational cubic function (1) to obtain a curve lying above the line.

3.2. **Constrained Surface Data Visualization.** In this section we construct a rational bi-cubic function that preserves the shape of data that lies above a plane $Z(x, y)$. Let $(x_i, y_i, F_{i,j}) : i = 0, 1, 2, ..., n-1; \quad j = 0, 1, 2, ..., m-1$ be the data defined over rectangular grid $I_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+i}]$, $i = 0, 1, 2, \cdots, n-1, \quad j = 0, 1, 2, \cdots, m-1$ that is lying above the following plane

$$Z = C[1 - \frac{x}{A} - \frac{y}{B}],$$

i.e.

$$F_{i,j} > Z_{i,j}, \forall i, j. \tag{11}$$

The bi-cubic partially blended rational function (2) will lie above the plane if each of the boundary curve $S(x, y_j)$, $\quad S(x, y_{j+1})$, $\quad S(x_i, y)$, and $S(x_{i+1}, y)$, lie above the plane. The boundary curve $S(x, y_j)$ will lie above the plane if

$$S(x, y_j) > (1 - \theta)Z_{i,j} + \theta Z_{i+1,j}, \tag{12}$$

Substituting the value of $S(x, y_j)$ in the above

$$\frac{\sum_{i=1}^{4} A_{i,j}\theta^{i-1}(1-\theta)^{4-i}}{q_{i,j}(\theta)} > (1 - \theta)Z_{i,j} + \theta Z_{i+i,j} \tag{13}$$

Now $q_{i,j}(\theta) > 0$ if $\mu_{i,j} > 0$ and $\nu_{i,j} > 0$.
After some manipulation (13) can be written as

$$U_i(\theta) = \sum_{i=1}^{4} (1 - \theta)^{4-i}\theta^i \omega_{i,j} \tag{14}$$

with
$\omega_{1,j} = \mu_{i,j}(\mu_{i,j}F_{i,j} - Z_{i,j})$
$\omega_{2,j} = (2\mu_{i,j} + \nu_{i,j})F_{i,j} + \mu_{i,j}h_iF^x_{i,j} - (\mu_{i,j} + \nu_{i,j})Z_{i,j} - \mu_{i,j}Z_{i+1,j}$
$\omega_{3,j} = (2\mu_{i,j} + \nu_{i,j})F_{i+1,j} + \nu_{i,j}h_iF^x_{i+1,j} - (\mu_{i,j} + \nu_{i,j})Z_{i+1,j} - \mu_{i,j}Z_{i,j}$
$\omega_{4,j} = \nu_{i,j}(\nu_{i,j}F_{i+1,j} - Z_{i+1,j})$
$U_i(\theta) > 0$ if $\omega_{i,j} > 0, \forall i = 1, 2, 3, 4.$
$\omega_{i,j} > 0, i = 1, 2, 3, 4$ if $\nu_{i,j} > max\left\{0, \frac{Z_{i+1,j}}{F_{i+1,j}}\right\}$ and
$\mu_{i,j} > max\left\{0, \frac{Z_{i,j}}{F_{i,j}}, \frac{\nu_{i,j}(Z_{i,j}-F_{i,j})}{2F_{i,j}+h_iF^x_{i,j}-Z_{i,j}-Z_{i+1,j}}, \frac{\nu_{i,j}(Z_{i,j}+Z_{i+1,j}-2F_{i+1,j}+h_iF^x_{i+1,j})}{F_{i+1,j}+Z_{i+1,j}}\right\}$
Similarly the boundary curves $S(x, y_{j+1})$, $\quad S(x_i, y)$, and $S(x_{i+1}, y)$, will lie above the plane if free parameters $\mu_{i,j+1}, \nu_{i,j+1}, \hat{\mu}_{i,j}, \hat{\nu}_{i,j}, \hat{\mu}_{i+1,j}, \hat{\nu}_{i+1,j}$ satisfy

the following constraints:

$$
\begin{aligned}
\nu_{i,j+1} &> \{0, \frac{Z_{i+1,j+1}}{F_{i+1,j+1}}\}, \\
\hat{\nu}_{i,j} &> \{0, \frac{Z_{i,j+1}}{F_{i,j+1}}\}, \\
\hat{\nu}_{i+1,j} &> \{0, \frac{Z_{i+1,j+1}}{F_{i+1,j+1}}\}, \\
\mu_{i,j+1} &> \max\{0, \frac{Z_{i,j+1}}{F_{i,j+1}}, \frac{\nu_{i,j+1}(Z_{i,j+1} - F_{i,j+1})}{2F_{i,j+1} + h_i F^x_{i,j+1} - Z_{i,j+1} - Z_{i+1,j+1}}, \\
&\quad \frac{\nu_{i,j+1}(Z_{i,j+1} + Z_{i+1,j+1} - 2F_{i+1,j+1} + h_i F^x_{i+1,j+1})}{F_{i+1,j+1} + Z_{i+1,j+1}}\}, \\
\hat{\mu}_{i,j} &> \max\{0, \frac{Z_{i,j}}{F_{i,j}}, \frac{\hat{\nu}_{i,j}(Z_{i,j} - F_{i,j})}{2F_{i,j} + h_j F^y_{i,j} - Z_{i,j} - Z_{i,j+1}}, \\
&\quad \frac{\hat{\nu}_{i,j}(Z_{i,j} + Z_{i,j+1} - 2F_{i,j+1} + \hat{h}_j F^y_{i,j+1})}{F_{i,j+1} + Z_{i,j+1}}\} \\
\hat{\mu}_{i+1,j} &> \{0, \frac{Z_{i+1,j}}{F_{i+1,j}}, \frac{\hat{\nu}_{i+1,j}(Z_{i+1,j} - F_{i+1,j})}{2F_{i+1,j} + \hat{h}_j F^y_{i+1,j} - Z_{i+1,j} - Z_{i+1,j+1}}, \\
&\quad \frac{\hat{\nu}_{i+1,j}(Z_{i+1,j} + Z_{i+1,j+1} - 2F_{i+1,j+1} + \hat{h}_j F^y_{i+1,j+1})}{F_{i+1,j+1} + Z_{i+1,j+1}}\}
\end{aligned}
$$

All the above discussion can be summarized in the form of following theorem:

Theorem 2. The rational bi-cubic partially blended function defined in (2) visualizes the shape of the surface that lies above the plane, if in each rectangular patch $I_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$,the shape parameters $\mu_{i,j}, \nu_{i,j}, \mu_{i,j+1}, \nu_{i,j+1}$, $\hat{\mu}_{i,j}, \hat{\nu}_{i,j}, \hat{\mu}_{i+1,j}$, and $\hat{\nu}_{i+1,j}$ satisfy the following conditions:

$$
\begin{aligned}
\nu_{i,j} &= n_{i,j} + \{0, \frac{Z_{i+1,j}}{F_{i+1,j}}\} \\
\nu_{i,j+1} &= n_{i,j+1} + \{0, \frac{Z_{i+1,j+1}}{F_{i+1,j+1}}\}, \\
\hat{\nu}_{i,j} &= \hat{n}_{i,j} + \{0, \frac{Z_{i,j+1}}{F_{i,j+1}}\}, \\
\hat{\nu}_{i+1,j} &= \hat{n}_{i+1,j} + \{0, \frac{Z_{i+1,j+1}}{F_{i+1,j+1}}\},
\end{aligned}
$$

$$\mu_{i,j} = k_{i,j} + \max\{0, \frac{Z_{i,j}}{F_{i,j}}, \frac{\nu_{i,j}(Z_{i,j} + Z_{i+1,j} - 2F_{i+1,j} + h_i F^x_{i+1,j})}{F_{i+1,j} + Z_{i+1,j}}\},$$

$$\frac{\nu_{i,j}(Z_{i,j} - F_{i,j})}{2F_{i,j} + h_i F^x_{i,j} - Zi,j - Z_{i+1,j}}\},$$

$$\mu_{i,j+1} = k_{i,j+1} + \max\{0, \frac{Z_{i,j+1}}{F_{i,j+1}}, \frac{\nu_{i,j+1}(Z_{i,j+1} - F_{i,j+1})}{2F_{i,j+1} + h_i F^x_{i,j+1} - Z_{i,j+1} - Z_{i+1,j+1}},$$

$$\frac{\nu_{i,j+1}(Z_{i,j+1} + Z_{i+1,j+1} - 2F_{i+1,j+1} + h_i F^x_{i+1,j+1})}{F_{i+1,j+1} + Z_{i+1,j+1}}\},$$

$$\hat{\mu}_{i,j} = \hat{k}_{i,j} + \max\{0, \frac{Z_{i,j}}{F_{i,j}}, \frac{\hat{\nu}_{i,j}(Z_{i,j} - F_{i,j})}{2F_{i,j} + h_j F^y_{i,j} - Z_{i,j} - Z_{i,j+1}},$$

$$\frac{\hat{\nu}_{i,j}(Z_{i,j} + Z_{i,j+1} - 2F_{i,j+1} + \hat{h}_j F^y_{i,j+1})}{F_{i,j+1} + Z_{i,j+1}}\}$$

$$\hat{\mu}_{i+1,j} = \hat{k}_{i+1,j} + \{0, \frac{Z_{i+1,j}}{F_{i+1,j}}, \frac{\hat{\nu}_{i+1,j}(Z_{i+1,j} - F_{i+1,j})}{2F_{i+1,j} + \hat{h}_j F^y_{i+1,j} - Z_{i+1,j} - Z_{i+1,j+1}},$$

$$\frac{\hat{\nu}_{i+1,j}(Z_{i+1,j} + Z_{i+1,j+1} - 2F_{i+1,j+1} + \hat{h}_j F^y_{i+1,j+1})}{F_{i+1,j+1} + Z_{i+1,j+1}}\}$$

where $n_{i,j}, n_{i,j+1}, \hat{n}_{i,j}, \hat{n}_{i+1,j}, k_{i,j}, k_{i,j+1}, \hat{k}_{i,j}, \hat{k}_{i+1,j} > 0$. The above discussion can be summarized in the following algorithm for computation purposes:

Algorithm 2

Step 1. Enter the $(n+1) \times (m+1)$ positive data points $(x_i, y_j, F_{i,j}), i = 0,1,2,...,n, j = 0,1,2,..,m$,that lie above the plan.

Step 2. Estimate the derivatives $F^x_{i,j}, F^y_{i,j}, F^{xy}_{i,j}$ at knots by Arithmetic Mean Method [9].

Step 3. Calculate the values of shape parameters $\mu_{i,j}, \nu_{i,j}, \mu_{i,j+1}, \nu_{i,j+1}, \hat{\mu}_{i,j}, \hat{\nu}_{i,j}$, $\hat{\mu}_{i+1,j}$, and $\hat{\nu}_{i+1,j}$ using Theorem 2.

Step 4. Substitute the values of $F_{i,j}, F^x_{i,j}, F^y_{i,j}, F^{xy}_{i,j}, \forall i = 0,1,2,...,n, j = 0,1, 2,...m$ and $\mu_{i,j}, \nu_{i,j}, \mu_{i,j+1}, \nu_{i,j+1}, \hat{\mu}_{i,j}, \hat{\nu}_{i,j}, \hat{\mu}_{i+1,j}$, and $\hat{\nu}_{i+1,j} \forall i = 0,1,2,...,n-1, j = 0,1,2,...,m-1$ in rational bi-cubic function (3) to obtain a surface lying above the plane.

3.3. **Demonstration.** A graphical comparison between Cubic Hermite Function and Constrained Rational Cubic Function is demonstrated in Figures 1-4. Consider the data in the Table 1 which lies above the straight line $f_1 = \frac{x}{4} - 1$.

Table 1. A data above the straight line $f_1 = \frac{x}{4} - 1$.

x	2	3	7	8	9	13	14
f_i	12	4.5	6.5	12	7.5	9.5	18

Table 2. A data above the straight line $f_2 = \frac{x}{4} - \frac{13}{5}$.

x	2	2.3	2.7	4	7.3	8.5	10
f_i	8	7	6	4	4.8	5.5	8

Table 3. (A) Numerical results for Figure 2, (B) Numerical results for Figure 4.

(A)

i	d_i	ν_i	μ_i
1	-9.10	0.21	0.50
2	-5.8462	0.21	0.51
3	5.5538	0.21	0.48
4	0.5000	0.21	0.52
5	-4.5538	0.21	0.55
6	6.8462	-	-

(B)

i	d_i	ν_i	μ_i
1	-3.6905	3.2	3.50
2	-3.0004	3.2	3.44
3	-1.9257	3.2	3.20
4	-0.1098	3.2	3.56
5	0.2888	3.2	3.48
6	0.8228	-	-

Figure 1 is obtained by using cubic Hermite function which does not preserve the shape of the data given in Table 1 that lies above the line $f_1 = \frac{x}{4} - 1$. Figure 2 is obtained using Theorem 1, it is observed that Figure 2 preserve the shape of the data taken in Table 1.
In Table 2, we take a data which lie above the line $f_2 = \frac{x}{4} - \frac{13}{5}$. Figure 3 is obtained by using cubic Hermite function which does not preserve the shape of the data taken in Table 2. Figure 4 preserves the shape of the data taken in Table 2, which is constructed using Theorem 1 of Section 3. The numerical results corresponding to the curves in Figure 2 and Figure 4, for different parameters, are shown in Table 3(A) and Table 3(B) respectively.

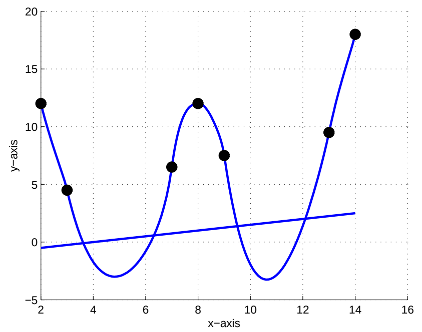

Figure 1: Cubic Hermite Function

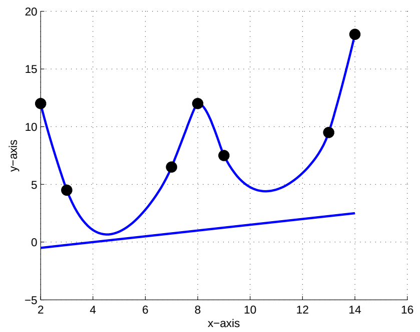

Figure 2: Constrained rational Cubic Function

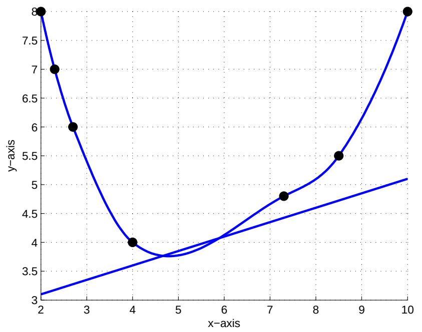

Figure 3: Cubic Hermite Function

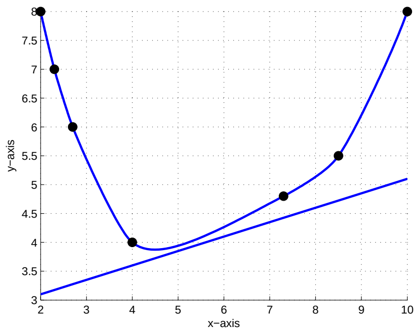

Figure 4: Constrained rational Cubic Function

A graphical comparison between Bi-cubic Hermite Function and Constrained Rational Bi-cubic Function is demonstrated in Figures 5-10. A data set is considered in Table 4, this is generated by the following plane:

$$Z(x, y) = (1 - \frac{x}{8} - \frac{y}{8}),$$

All the values of the plane $Z(x, y)$ in Table 4 are truncated up to three decimal places. Another data shown in in Table 5 is computed by the following function:

$$F_1(x, y) = 2sinxcosy + 2.1755$$

This is defined over the rectangular grid $[1, 8] \times [1, 8]$. All values of the function $F_1(x, y)$ in Table 5 are truncated up to four decimal places. One can observe that all the surface data in Table 5 lies above the plane data in Table 4. But when we interpolate the data in Table 4 and Table 5 using bi-cubic Hermite function, some part of the surface lies below the plane as in Figure 5, which didn't preserve the shape of the data generated by $F_1(x, y)$. To overcome this problem we used Theorem 2 developed in Section 3.1 to generate the Figure 8,

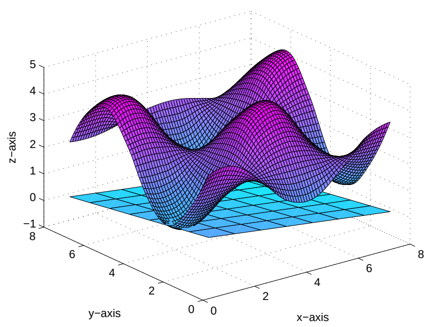

Figure 5: Bi-cubic Hermite Function

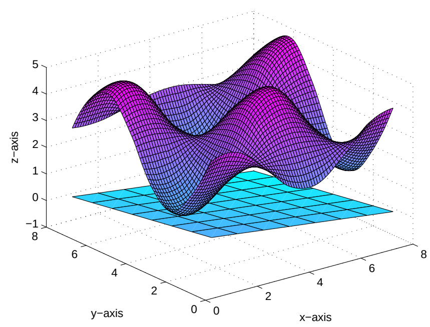

Figure 8: Constrained Rational Bi-cubic Function

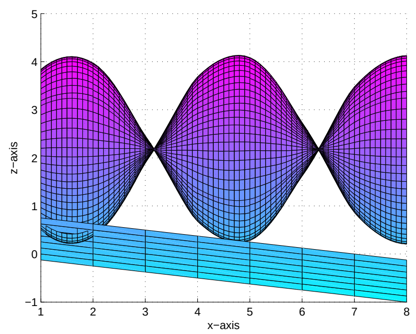

Figure 6: xz-view of Figure 5.

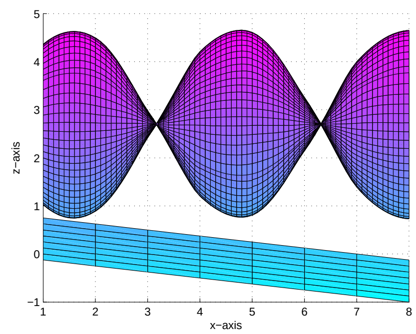

Figure 9: xz-view of Figure 8.

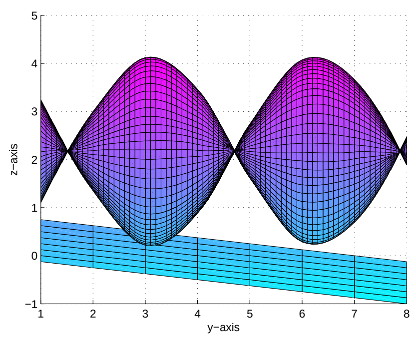

Figure 7: yz-view of Figure 5.

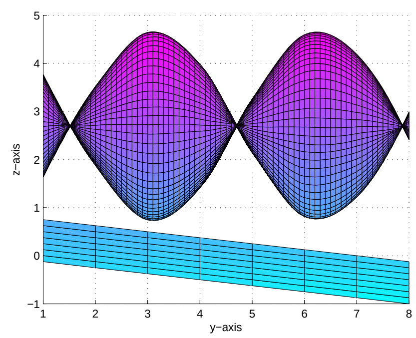

Figure 10: yz-view of Figure 8.

which clearly visualizes the shape of the data generated by $F_1(x, y)$. Figures 6-7 and Figures 9-10 present different views of their counter parts Figure 5 and Figure 8 respectively.

TABLE 4. A data set generated by the plane $Z(x,y) = (1 - \frac{x}{8} - \frac{y}{8}), 1 \leq x, y \leq 8$.

y/x	1	2	3	4	5	6	7	8
1	0.750	0.625	0.500	0.375	0.250	0.125	0	-0.125
2	0.625	0.500	0.375	0.250	0.125	0	-0.125	-0.250
3	0.500	0.375	0.250	0.125	0	-0.125	-0.250	-0.375
4	0.375	0.250	0.125	0	-0.125	-0.250	-0.375	-0.500
5	0.250	0.125	0	-0.125	-0.250	-0.375	-0.500	-0.625
6	0.125	0	-0.125	-0.250	-0.375	-0.500	-0.625	-0.750
7	0	-0.125	-0.250	-0.375	-0.500	-0.625	-0.750	-0.875
8	-0.125	-0.250	-0.375	-0.500	-0.625	-0.750	-0.875	-1.000

TABLE 5. A data set generated by the function $F_1(x,y) = 2xcosy + 2.1755$.

y/x	1	2	3	4	5	6	7	8
1	3.0848	3.1581	2.3280	1.3577	1.1393	1.8736	2.8854	3.2446
2	1.4751	1.4187	2.0580	2.8054	2.9736	2.4081	1.6287	1.3521
3	0.5094	0.3751	1.8961	3.6740	4.0742	2.7287	0.8747	0.2166
4	1.0755	0.9868	1.9910	3.1649	3.4291	2.5408	1.3166	0.8821
5	2.6529	2.6914	2.2556	1.7461	1.6315	2.0170	2.5482	2.7368
6	3.7914	3.9217	2.4465	0.7222	0.3340	1.6389	3.4371	4.0754
7	3.4443	3.5465	2.3883	1.0344	0.7296	1.7542	3.1661	3.6673
8	1.9306	1.9109	2.1344	2.3957	2.4545	2.2568	1.9843	1.8876

The numerical results corresponding to the surface in Figure 8, for different parameters, are shown in Table 6. All the values in Table 6 are truncated up to four decimal places.

TABLE 6. Numerical results of Figure 8.

$F^x_{i,j}$	1	2	3	4	5	6	7	8
1	0.5250	-0.3784	-0.9002	-0.5944	0.2579	0.8731	0.6855	0.0328
2	-0.4044	0.2914	0.6933	0.4578	-0.1987	-0.6725	-0.5280	-0.0253
3	-0.9619	0.6933	1.6494	1.0890	-0.4726	-1.5997	-1.2561	-0.0601
4	-0.6351	0.4578	1.0890	0.7190	-0.3120	-1.0562	-0.8293	-0.0397
5	0.2756	-0.1987	-0.4726	-0.3120	0.1354	0.4584	0.3599	0.0172
6	0.9330	-0.6725	-1.5997	-1.0562	0.4584	1.5515	1.2182	0.0583
7	0.7325	-0.5280	-1.2561	-0.8293	0.3599	1.2182	0.9565	0.0458
8	-0.1414	1.9109	0.2424	0.1601	-0.0695	-0.2351	-0.1846	-0.0088
$F^y_{i,j}$	1	2	3	4	5	6	7	8
1	-1.9316	-2.0873	-0.3239	1.7372	2.2012	0.6414	-1.5081	-2.2711
2	-1.2877	-1.3915	-0.2160	1.1581	1.4674	0.4276	-1.00547	-1.5140
3	-0.1998	-0.2160	-0.0335	0.1797	0.2277	0.0664	-0.1560	-0.2350
4	1.0717	1.1581	0.1797	-0.9639	-1.2213	-0.3559	0.8368	1.2601
5	1.3580	1.4674	0.2277	-1.2213	-1.5475	-0.4509	1.0603	1.5966
6	0.3957	0.4276	0.0664	-0.3559	-0.4509	-0.1314	0.3089	0.4652
7	-0.9304	-1.0054	-0.1560	0.8368	1.0603	0.3089	-0.7264	-1.0939
8	-2.0969	-2.2659	-0.3517	1.8859	2.3896	0.6963	-1.6372	-2.4654
$\nu_{i,j}$	1	2	3	4	5	6	7	8
1	0.2697	0.2753	0.2992	0.2503	0.1521	0.1000	0.1000	-
2	0.3573	0.2452	0.1751	0.1357	0.1000	0.1000	0.1000	-
3	0.5168	0.2033	0.1298	0.1000	0.1000	0.1000	0.1000	-
4	0.2654	0.1497	0.1000	0.1000	0.1000	0.1000	0.1000	-
5	0.1389	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	-
6	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	-
7	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	-
8	-	-	-	-	-	-	-	-
$\mu_{i,j}$	1	2	3	4	5	6	7	8
1	0.4578	0.4197	0.4253	0.4492	0.4003	0.3021	0.2500	-
2	0.5626	0.5073	0.3952	0.3251	0.2857	0.2500	0.2500	-
3	0.7336	0.6668	0.3533	0.2798	0.2500	0.2500	0.2500	-
4	0.4844	0.4154	0.2997	0.2500	0.2500	0.2500	0.2500	-
5	0.3287	0.2889	0.2500	0.2500	0.2500	0.2500	0.2500	-
6	0.2790	0.2500	0.2500	0.2500	0.2500	0.2500	0.2500	-
7	0.2500	0.2500	0.2500	0.2500	0.2500	0.2500	0.2500	-
8	-	-	-	-	-	-	-	-

$\nu_{i,j+1}$	1	2	3	4	5	6	7	8
1	0.4573	0.3452	0.2751	0.2357	0.2000	0.2000	0.2000	-
2	0.6168	0.3033	0.2298	0.2000	0.2000	0.2000	0.2000	-
3	0.3654	0.2497	0.2000	0.2000	0.2000	0.2000	0.2000	-
4	0.2389	0.2000	0.2000	0.2000	0.2000	0.2000	0.2000	-
5	0.2000	0.2000	0.2000	0.2000	0.2000	0.2000	0.2000	-
6	0.2000	0.2000	0.2000	0.2000	0.2000	0.2000	0.2000	-
7	0.2000	0.2000	0.2000	0.2000	0.2000	0.2000	0.2000	-
8	-	-	-	-	-	-	-	-
$\mu_{i,j+1}$	1	2	3	4	5	6	7	8
1	0.8776	0.8223	0.7102	0.6401	0.6007	0.5650	0.5650	-
2	1.0486	0.9818	0.6683	0.5948	0.5650	0.5650	0.5650	-
3	0.7994	0.7304	0.6147	0.5650	0.5650	0.5650	0.5650	-
4	0.6437	0.6039	0.5650	0.5650	0.5650	0.5650	0.5650	-
5	0.5940	0.5650	0.5650	0.5650	0.5650	0.5650	0.5650	-
6	0.5650	0.5650	0.5650	0.5650	0.5650	0.5650	0.5650	-
7	0.5650	0.5650	0.5650	0.5650	0.5650	0.5650	0.5650	-
8	-	-	-	-	-	-	-	-
$\hat{\nu}_{i,j+1}$	1	2	3	4	5	6	7	8
1	0.6126	0.5573	0.4452	0.3751	0.3357	0.3000	0.3000	-
2	0.7836	0.7168	0.4033	0.3298	0.3000	0.3000	0.3000	-
3	0.5344	0.4654	0.3497	0.3000	0.3000	0.3000	0.3000	-
4	0.3787	0.3389	0.3000	0.3000	0.3000	0.3000	0.3000	-
5	0.3290	0.3000	0.3000	0.3000	0.3000	0.3000	0.3000	-
6	0.3000	0.3000	0.3000	0.3000	0.3000	0.3000	0.3000	-
7	0.3000	0.3000	0.3000	0.3000	0.3000	0.3000	0.3000	-
8	-	-	-	-	-	-	-	-
$\hat{\mu}_{i,j+1}$	1	2	3	4	5	6	7	8
1	0.8578	0.8197	0.8253	0.8492	0.8003	0.7021	0.6500	-
2	0.9626	0.9073	0.7952	0.7251	0.6857	0.6500	0.6500	-
3	1.1336	1.0668	0.7533	0.6798	0.6500	0.6500	0.6500	-
4	0.8844	0.8154	0.6997	0.6500	0.6500	0.6500	0.6500	-
5	0.7287	0.6889	0.6500	0.6500	0.6500	0.6500	0.6500	-
6	0.6790	0.6500	0.6500	0.6500	0.6500	0.6500	0.6500	-
7	0.6500	0.6500	0.6500	0.6500	0.6500	0.6500	0.6500	-
8	-	-	-	-	-	-	-	-
$\hat{\nu}_{i+1,j}$	1	2	3	4	5	6	7	8
1	0.6573	0.5452	0.4751	0.4357	0.4000	0.4000	0.4000	-
2	0.8168	0.5033	0.4298	0.4000	0.4000	0.4000	0.4000	-
3	0.5654	0.4497	0.4000	0.4000	0.4000	0.4000	0.4000	-
4	0.4389	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000	-
5	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000	-
6	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000	-
7	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000	-
8	-	-	-	-	-	-	-	-

$\hat{\mu}_{i+1,j}$	1	2	3	4	5	6	7	8
1	0.7197	0.7253	0.7492	0.7003	0.6021	0.5500	0.5500	-
2	0.8073	0.6952	0.6251	0.5857	0.5500	0.5500	0.5500	-
3	0.9668	0.6533	0.5798	0.5500	0.5500	0.5500	0.5500	-
4	0.7154	0.5997	0.5500	0.5500	0.5500	0.5500	0.5500	-
5	0.5889	0.5500	0.5500	0.5500	0.5500	0.5500	0.5500	-
6	0.5500	0.5500	0.5500	0.5500	0.5500	0.5500	0.5500	-
7	0.5500	0.5500	0.5500	0.5500	0.5500	0.5500	0.5500	-
8	-	-	-	-	-	-	-	-

4. Conclusion

In this paper two schemes are proposed; first for the visualization of shape preserving curve data and second for the visualization of shape preserving surface data. A piecewise rational cubic function with two shape parameters is used for the visualization of constrained curve data and a piecewise rational bi-cubic partially blended function is used for the visualization of constrained surface data. Data dependent constraints are derived on free parameters to restrict the visualized curve and surface lie above the line and above the plane respectively. These schemes are local, computationally economical, and visualize the shape of constrained data. In [4, 10] constraints are derived on derivatives, when derivatives are given with data then these schemes are not helpful. But in this paper, the developed schemes are acceptable to both the data with derivatives and data without derivatives. The proposed schemes are also simpler than those in [8] where a piecewise rational cubic function was used to visualize curve data and surface data with four and eight free parameters respectively. Numerical examples are illustrated to show the worth of proposed schemes. Degree of smoothness attained is C^1 .

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