

# COMMON FIXED POINTS BY ONE STEP ITERATIVE SCHEME WITH ERROR FOR ASYMPTOTICALLY QUASI-NONEXPANSIVE TYPE NONSELF MAPPINGS

PANKAJ KUMAR JHADE<sup>1</sup>, A. S. SALUJA<sup>2</sup>

**ABSTRACT.** The aim of this paper is to define a new one step iterative scheme with error for approximating common fixed points of asymptotically quasi-nonexpansive type non-self mappings in Banach space.

*Key words:* asymptotically quasi-nonexpansive type mappings, One step iterative scheme, common fixed point.

*AMS SUBJECT CLASSIFICATION 2010:* 47H10, 47H09, 54H25.

## 1. INTRODUCTION & PRELIMINARIES

Let  $X$  be a real Banach space and let  $C$  be a nonempty subset of  $X$ . Further, for a mapping  $T : C \rightarrow C$ , let  $\varphi \neq F(T)$  be the set of all fixed points of  $T$ . A mapping  $T : C \rightarrow C$  is said to be asymptotically quasi-nonexpansive if there exists a sequence  $\{k_n\}$  of real numbers with  $k_n \geq 1$  and  $\lim_n k_n = 1$  such that for all  $x \in C, q \in F(T)$ ,

$$\|T^n x - q\| \leq k_n \|x - q\| \quad (1)$$

for all  $n \geq 1$

$T$  is called asymptotically quasi-nonexpansive type [5] provided  $T$  is uniformly continuous and

$$\limsup_n \{ \sup_{x \in C} (\|T^n x - q\| - \|x - q\|) \} \leq 0 \quad (2)$$

for all  $q \in F(T)$ .

The mapping  $T$  is called uniformly L-Lipschitzian if there exists a positive constant  $L$  such that

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<sup>1</sup>NRI Institute of Information Science & Technology, Bhopal, India. Email: pmath-sjhade@gmail.com

<sup>2</sup>Government J. H. (PG) College, Betul, India. Email: dssaluja@rediffmail.com.

$$\|T^n x - T^n y\| \leq L\|x - y\| \quad (3)$$

for all  $x, y \in C$  and all  $n \geq 1$

The mapping  $T : C \rightarrow C$  is completely continuous [6] if for any bounded sequence  $\{x_n\} \subset C$  there exists a convergent subsequence of  $\{Tx_n\}$ .

Again a Banach space  $X$  is called uniformly convex [4] if for every  $0 < \varepsilon \leq 2$ , there exists  $\delta = \delta(\varepsilon) > 0$  such that  $\|\frac{x+y}{2}\| \leq 1 - \delta$  for every  $x, y \in S_X$  and  $\|x - y\| \geq \varepsilon$ , where  $S_X = \{x \in X : \|x\| = 1\}$

A Banach space  $X$  is called smooth [8], if  $\lim_{t \rightarrow 0} \frac{\|x+ty\| - \|x\|}{t}$  exists for all  $x, y \in S_X$ ,  $S_X = \{x \in X : \|x\| = 1\}$ .

Recently, H. Y. Zhou, Y. J. Cho and S. M. Kang [8] gave a new iterative scheme for approximating common fixed points of two asymptotically nonexpansive nonself mappings with respect to  $P$  as follows: For a given  $x_1 \in C$ , compute the sequence  $\{x_n\}$  by the iterative scheme

$$x_{n+1} = \alpha_n x_n + \beta_n (PT_1)^n x_n + \gamma_n (PT_2)^n x_n \quad (4)$$

for all  $n \geq 1$

where  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  are real sequences in  $[0, 1]$ . They satisfying  $\alpha_n + \beta_n + \gamma_n = 1$  and prove some weak and strong convergence theorems for such mappings in a uniformly convex Banach space.

In this paper, we define a new iterative scheme with error and study the approximating common fixed point of two asymptotically quasi- nonexpansive type nonself mappings with respect to  $P$ .

Now, we recall the well known concepts to prove our main result.

**Definition 1.** A subset  $C$  of  $X$  is called retract of  $X$  if there exists a continuous mapping  $P : X \rightarrow C$  such that  $Px = x$  for all  $x \in C$ . Every closed convex subset of a uniformly convex Banach space is a retract. A mapping  $P : X \rightarrow C$  is called retraction if  $P^2 = P$ . Note that, if a mapping  $P$  is a retraction, then  $Pz = z$  for all  $z$  in the range of  $P$ .

**Definition 2.** Let  $D, E$  be subsets of a Banach space  $X$ . Then, a mapping  $P : D \rightarrow E$  is said to be sunny if  $P(Px + t(x - Px)) = Px$  whenever  $Px + t(x - Px) \in D$  for all  $x \in D$  and  $t \geq 0$ .

**Definition 3.** Let  $C$  be a subset of a Banach space  $X$ . For all  $x \in C$  is defined a set  $I_C(x)$  by  $I_C(x) = \{x + \lambda(y - x) : \lambda > 0, y \in C\}$ . A nonself mapping  $T : C \rightarrow X$  is said to be inward if  $Tx \in I_C(x)$  for all  $x \in C$ , and  $T$  is said to be weakly inward if  $Tx \in \overline{I_C(x)}$  for all  $x \in C$ .

Let  $X$  be a real normed linear space and let  $C$  be a nonempty closed convex subset of  $X$ . Let  $P : X \rightarrow C$  be the nonexpansive retraction of  $X$  onto  $C$  and

let  $T_1 : C \rightarrow X$  and  $T_2 : C \rightarrow X$  be two asymptotically quasi- nonexpansive type nonself mappings.

**Algorithm 1:** For a given  $x_1 \in C$ , compute the sequence  $\{x_n\}$  by the iterative scheme

$$x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_n(PT_1)^n x_n + \gamma_n(PT_2)^n x_n + \delta_n u_n \quad (5)$$

for all  $n \geq 1$

where  $\{\alpha_n\}, \{\gamma_n\}, \{\delta_n\}$  are real sequences in  $[0, 1]$  and satisfying  $\alpha_n + \gamma_n + \delta_n = 1$  and  $\{u_n\}$  is bounded sequence in  $C$ . The iterative scheme (5) is called the one-step iterative scheme with error.

**Definition 4.** [8] Let  $C$  be a nonempty subset of real normed linear space  $X$ . Let  $P : X \rightarrow C$  be a nonexpansive retraction of  $X$  onto  $C$ . Then  $T : C \rightarrow X$  is said to be  $L$ -Lipschitzian nonself mapping with respect to  $P$  if there exists a constant  $L > 0$  such that

$$\|(PT)^n x - (PT)^n y\| \leq \|x - y\| \quad (6)$$

for all  $x, y \in C$ , and  $n \geq 1$

**Lemma 1.** [7] Let  $\{a_n\}$  and  $\{b_n\}$  be sequences of nonnegative real numbers such that  $a_{n+1} \leq a_n + b_n$  for all  $n \geq 1$ . If  $\sum_{n=1}^{\infty} b_n < \infty$ , then  $\lim_n a_n$  exists.

**Lemma 2.** [2] Let  $X$  be a uniformly convex Banach space and  $B_r = \{x \in X : \|x\| < r, r > 0\}$ . Then there exists a continuous strictly increasing convex function  $g : [0, \infty) \rightarrow [0, \infty)$  with  $g(0) = 0$  such that

$$\|\lambda x + \mu y + \xi z + \nu w\|^2 \leq \lambda \|x\|^2 + \mu \|y\|^2 + \xi \|z\|^2 + \nu \|w\|^2 - \lambda \mu g(\|x - y\|)$$

for all  $x, y, z, w \in B_r$  and  $\lambda, \mu, \xi, \nu \in [0, 1]$  with  $\lambda + \mu + \xi + \nu = 1$ .

**Lemma 3.** [8] Let  $X$  be a real smooth Banach space and let  $C$  be a nonempty closed subset of  $X$  with  $P$  as a sunny nonexpansive retraction and let  $T : C \rightarrow X$  be a mapping satisfying weakly inward condition. Then  $F(PT) = F(T)$ .

**Definition 5.** Let  $C$  be a nonempty subset of a real normed linear space  $X$ . Let  $P : X \rightarrow C$  be a nonexpansive retraction of  $X$  onto  $C$ , and  $F(T) \neq \varphi$  be the set of all fixed points of  $T$ . Then  $T : X \rightarrow C$  is called asymptotically quasi- nonexpansive type nonself mapping with respect to  $P$  if  $T$  is uniformly continuous and

$$\limsup_n \left\{ \sup_{x \in C} (\|(PT)^n x - q\| - \|x - q\|) \right\} \leq 0 \quad (7)$$

for all  $q \in F(T)$

**Remark 1.** If  $T$  is self mapping, then  $P$  become the identity mapping, so that (6) and (7) are reduced to (3) and (4).

## 2. MAIN RESULTS

**Lemma 4.** *Let  $X$  be a uniformly convex Banach space,  $C$  a nonempty closed convex subset of  $X$  and  $T_1, T_2 : C \rightarrow X$  be two asymptotically quasi-nonexpansive type nonself mappings with respect to  $P$ , a nonexpansive retraction of  $X$  onto  $C$ . Put*

$$G_n = \max\{0, \sup_{x \in C} (\|(PT_1)^n x - q\| - \|x - q\|)\}$$

and

$$K_n = \max\{0, \sup_{x \in C} (\|(PT_2)^n x - q\| - \|x - q\|)\}$$

such that  $\sum_{n=1}^{\infty} G_n < \infty$  and  $\sum_{n=1}^{\infty} K_n < \infty$  respectively. Suppose that  $\{x_n\}$  is a sequence defined by (5) with  $\sum_{n=1}^{\infty} \delta_n < \infty$ . If  $F(T_1) \cap F(T_2) \neq \varphi$ , then  $\lim_{n \rightarrow \infty} \|x_n - q\|$  exists for any  $q \in F(T_1) \cap F(T_2)$ .

*Proof.* For any  $q \in F(T_1) \cap F(T_2)$ , by (5) we have ,

$$\begin{aligned} \|x_{n+1} - q\| &= \|(1 - \alpha_n - \gamma_n)x_n + \alpha_n(PT_1)^n x_n + \gamma_n(PT_2)^n x_n + \delta_n u_n - q\| \\ &\leq (1 - \alpha_n - \gamma_n)\|x_n - q\| + \alpha_n\|(PT_1)^n x_n - q\| \\ &\quad + \gamma_n\|(PT_2)^n x_n - q\| + \delta_n\|u_n - q\| \\ &\leq (1 - \alpha_n - \gamma_n)\|x_n - q\| + \alpha_n(\|(PT_1)^n x_n - q\| - \|x_n - q\|) \\ &\quad + \alpha_n\|x_n - q\| + \gamma_n(\|(PT_2)^n x_n - q\| - \|x_n - q\|) + \gamma_n\|x_n - q\| \\ &\quad + \delta_n\|u_n - q\| \\ &\leq (1 - \alpha_n - \gamma_n + \alpha_n + \gamma_n)\|x_n - q\| \\ &\quad + \alpha_n \sup_{x \in C} (\|(PT_1)^n x - q\| - \|x - q\|) \\ &\quad + \gamma_n \sup_{x \in C} (\|(PT_2)^n x - q\| - \|x - q\|) + \delta_n\|u_n - q\| \\ &\leq \|x_n - q\| + \alpha_n G_n + \gamma_n K_n + \delta_n\|u_n - q\| \\ &\leq \|x_n - q\| + d_n \end{aligned}$$

where  $d_n = \alpha_n G_n + \gamma_n K_n + \delta_n\|u_n - q\|$ . Since  $\sum_{n=1}^{\infty} G_n < \infty, \sum_{n=1}^{\infty} K_n < \infty$  &  $\sum_{n=1}^{\infty} \delta_n < \infty$  this implies that  $\sum_{n=1}^{\infty} d_n < \infty$ . Hence it follows from Lemma 1 that  $\lim_{n \rightarrow \infty} \|x_n - q\|$  exists. This completes the proof of the Lemma.  $\square$

**Lemma 5.** *Let  $X$  be a uniformly convex Banach space,  $C$  a nonempty closed convex subset of  $X$  and  $T_1, T_2 : C \rightarrow X$  be two asymptotically quasi-nonexpansive type nonself mappings with respect to  $P$ , a nonexpansive retraction of  $X$  onto  $C$ . Put*

$$G_n = \max\{0, \sup_{x \in C} (\|(PT_1)^n x - q\| - \|x - q\|)\}$$

and

$$K_n = \max\{0, \sup_{x \in C} (\|(PT_2)^n x - q\| - \|x - q\|)\}$$

such that  $\sum_{n=1}^{\infty} G_n < \infty$  and  $\sum_{n=1}^{\infty} K_n < \infty$  respectively. Suppose that  $\{x_n\}$  is a sequence defined by (5) with  $\sum_{n=1}^{\infty} \delta_n < \infty$  and the additional assumption that  $0 < \lim_n \inf \alpha_n, 0 < \lim_n \inf \gamma_n$  and  $0 < \lim_n \inf \delta_n$ . If  $F(T_1) \cap F(T_2) \neq \varphi$ , then  $\lim_n \|x_n - (PT_1)x_n\| = \lim_n \|x_n - (PT_2)x_n\| = 0$ .

*Proof.* By Lemma 4,  $\lim_{n \rightarrow \infty} \|x_n - q\|$  exists for any  $q \in F(T_1) \cap F(T_2)$ . It follows that  $\{x_n - q\}$ ,  $\{(PT_1)^n x_n - q\}$  and  $\{(PT_2)^n x_n - q\}$  are bounded. Also,  $\{u_n - q\}$  is bounded by assumption. We may assume that such sequences belong to  $B_r$ , where  $B_r = \{x \in X : \|x\| < r, r > 0\}$ .

From (5) and Lemma 2, we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|(1 - \alpha_n - \gamma_n)x_n + \alpha_n(PT_1)^n x_n + \gamma_n(PT_2)^n x_n + \delta_n u_n - q\|^2 \\ &\leq \|(1 - \alpha_n - \gamma_n)(x_n - q) + \alpha_n((PT_1)^n x_n - q) \\ &\quad + \gamma_n((PT_2)^n x_n - q) + \delta_n(u_n - q)\|^2 \\ &\leq (1 - \alpha_n - \gamma_n)\|x_n - q\|^2 + \alpha_n\|(PT_1)^n x_n - q\|^2 \\ &\quad + \gamma_n\|(PT_2)^n x_n - q\|^2 + \delta_n\|u_n - q\|^2 \\ &\quad - (1 - \alpha_n - \gamma_n)\alpha_n g(\|x_n - (PT_1)^n x_n\|) \\ &\leq (1 - \alpha_n - \gamma_n)\|x_n - q\|^2 + \alpha_n(\|(PT_1)^n x_n - q\|^2 - \|x_n - q\|^2) \\ &\quad + \alpha_n\|x_n - q\|^2 + \gamma_n(\|(PT_2)^n x_n - q\|^2 - \|x_n - q\|^2) + \gamma_n\|x_n - q\|^2 \\ &\quad + \delta_n\|u_n - q\|^2 - (1 - \alpha_n - \gamma_n)\alpha_n g(\|x_n - (PT_1)^n x_n\|) \end{aligned} \quad (8)$$

and

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|(1 - \alpha_n - \gamma_n)x_n + \alpha_n(PT_1)^n x_n + \gamma_n(PT_2)^n x_n + \delta_n u_n - q\|^2 \\ &\leq \|(1 - \alpha_n - \gamma_n)(x_n - q) + \alpha_n((PT_1)^n x_n - q) \\ &\quad + \gamma_n((PT_2)^n x_n - q) + \delta_n(u_n - q)\|^2 \\ &\leq (1 - \alpha_n - \gamma_n)\|x_n - q\|^2 + \alpha_n\|(PT_1)^n x_n - q\|^2 \\ &\quad + \gamma_n\|(PT_2)^n x_n - q\|^2 + \delta_n\|u_n - q\|^2 \\ &\quad - (1 - \alpha_n - \gamma_n)\gamma_n g(\|x_n - (PT_2)^n x_n\|) \\ &\leq (1 - \alpha_n - \gamma_n)\|x_n - q\|^2 + \alpha_n(\|(PT_1)^n x_n - q\|^2 - \|x_n - q\|^2) \\ &\quad + \alpha_n\|x_n - q\|^2 + \gamma_n(\|(PT_2)^n x_n - q\|^2 - \|x_n - q\|^2) + \gamma_n\|x_n - q\|^2 \\ &\quad + \delta_n\|u_n - q\|^2 - (1 - \alpha_n - \gamma_n)\gamma_n g(\|x_n - (PT_2)^n x_n\|) \end{aligned} \quad (9)$$

From 8,

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq (1 - \alpha_n - \gamma_n)\|x_n - q\|^2 \\ &\quad + \alpha_n(\|(PT_1)^n x_n - q\| - \|x_n - q\|)(\|(PT_1)^n x_n - q\| + \|x_n - q\|) \\ &\quad + \gamma_n(\|(PT_2)^n x_n - q\| - \|x_n - q\|)(\|(PT_2)^n x_n - q\| + \|x_n - q\|) \\ &\quad + \alpha_n\|x_n - q\|^2 + \gamma_n\|x_n - q\|^2 + \delta_n\|u_n - q\|^2 \\ &\quad - (1 - \alpha_n - \gamma_n)\alpha_n g(\|x_n - (PT_1)^n x_n\|) \\ &\leq \|x_n - q\|^2 \end{aligned}$$

$$\begin{aligned}
& +\alpha_n(\|(PT_1)^n x_n - q\| + \|x_n - q\|) \sup_{x \in C} (\|(PT_1)^n x_n - q\| - \|x_n - q\|) \\
& +\gamma_n(\|(PT_2)^n x_n - q\| + \|x_n - q\|) \sup_{x \in C} (\|(PT_2)^n x_n - q\| - \|x_n - q\|) \\
& +\delta_n \|u_n - q\|^2 - (1 - \alpha_n - \gamma_n) \alpha_n g(\|x_n - (PT_1)^n x_n\|) \\
& \leq \|x_n - q\|^2 + \alpha_n(\|(PT_1)^n x_n - q\| + \|x_n - q\|) G_n \\
& +\gamma_n(\|(PT_2)^n x_n - q\| + \|x_n - q\|) K_n + \delta_n \|u_n - q\|^2 \\
& +\delta_n \|u_n - q\|^2 - (1 - \alpha_n - \gamma_n) \alpha_n g(\|x_n - (PT_1)^n x_n\|)
\end{aligned}$$

Hence,

$$\begin{aligned}
\|x_{n+1} - q\|^2 & \leq \|x_n - q\|^2 + \omega_n G_n + \nu_n K_n + \delta_n \|u_n - q\|^2 \\
& - (1 - \alpha_n - \gamma_n) \alpha_n g(\|x_n - (PT_1)^n x_n\|)
\end{aligned}$$

where  $\omega_n = \alpha_n(\|(PT_1)^n x_n - q\| + \|x_n - q\|)$  and  $\nu_n = \gamma_n(\|(PT_2)^n x_n - q\| + \|x_n - q\|)$ .

Then,

$$(1 - \alpha_n - \gamma_n) \alpha_n g(\|x_n - (PT_1)^n x_n\|) \leq$$

$$\|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \omega_n G_n + \nu_n K_n + \delta_n \|u_n - q\|^2$$

Since  $\sum_{n=1}^{\infty} G_n < \infty$ ,  $\sum_{n=1}^{\infty} K_n < \infty$  &  $\sum_{n=1}^{\infty} \delta_n < \infty$  and  $\lim_{n \rightarrow \infty} \|x_n - q\|$  exists.

It follows that

$$\limsup_n (1 - \alpha_n - \gamma_n) \alpha_n g(\|x_n - (PT_1)^n x_n\|) = 0$$

Since  $g$  is continuous strictly increasing with  $g(0) = 0$  and  $0 < \lim_n \inf \alpha_n, 0 < \lim_n \inf \gamma_n$ , we have  $\lim_n \|x_n - (PT_1)^n x_n\| = 0$ .

By using a similar method together with (9), it can be show that  $\lim_n \|x_n - (PT_2)^n x_n\| = 0$ .

Next, we show that  $\lim_n \|x_n - (PT_1)x_n\| = 0$  and  $\lim_n \|x_n - (PT_2)x_n\| = 0$ .

Using (5), we have

$$\|x_{n+1} - x_n\| \leq$$

$$(1 - \alpha_n - \gamma_n) \|x_n - x_n\| + \alpha_n \|(PT_1)^n x_n - x_n\| + \gamma_n \|(PT_2)^n x_n - x_n\| + \delta_n \|u_n - x_n\|$$

Since  $\sum_{n=1}^{\infty} \delta_n < \infty$ ,  $\|(PT_1)^n x_n - x_n\| \rightarrow 0$  and  $\|(PT_2)^n x_n - x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . Then  $\|x_{n+1} - x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ .

Now we consider,

$$\begin{aligned}
\|x_n - (PT_1)x_n\| & \leq \|x_n - x_{n+1}\| + \|x_{n+1} - (PT_1)^{n+1} x_{n+1}\| \\
& + \|(PT_1)^{n+1} x_{n+1} - (PT_1)^{n+1} x_n\| + \|(PT_1)^{n+1} x_n - (PT_1)x_n\| \\
& \leq \|x_n - x_{n+1}\| + \|x_{n+1} - (PT_1)^{n+1} x_{n+1}\| \\
& + L \|x_{n+1} - x_n\| + L \|(PT_1)^n x_n - x_n\|
\end{aligned} \tag{10}$$

for some  $L \in R^+$

$$\begin{aligned} \|x_n - (PT_2)x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - (PT_2)^{n+1}x_{n+1}\| \\ &\quad + \|(PT_2)^{n+1}x_{n+1} - (PT_2)^{n+1}x_n\| + \|(PT_2)^{n+1}x_n - (PT_2)x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - (PT_2)^{n+1}x_{n+1}\| \\ &\quad + L\|x_{n+1} - x_n\| + L\|(PT_2)^n x_n - x_n\| \end{aligned} \quad (11)$$

for some  $L \in R^+$

Since  $\|(PT_1)^n x_n - x_n\| \rightarrow 0$ ,  $\|(PT_2)^n x_n - x_n\| \rightarrow 0$ , and  $\|x_{n+1} - x_n\| \rightarrow 0$  as  $n \rightarrow \infty$  and uniform continuity of  $T_1, T_2$  together with (10) and (11) it can be shown that  $\|(PT_1)x_n - x_n\| \rightarrow 0$  and  $\|(PT_2)x_n - x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ , respectively.  $\square$

**Remark 2.** Expressions (10) and (11) mean that the assumption  $T_1$  and  $T_2$  are uniformly  $L$ -Lipschitzian nonself mappings with respect to  $P$  is the same as to assume that the mappings  $PT_1$  and  $PT_2$  are uniformly  $L$ -Lipschitzian.

**Theorem 6.** Let  $C$  be a nonempty closed convex subset of a real smooth and uniformly convex Banach space  $X$ . Let  $T_1, T_2 : C \rightarrow X$  be two weakly inward and asymptotically quasi-nonexpansive type nonself mappings with respect to  $P$ , as a sunny nonexpansive retraction of  $X$  onto  $C$  and  $T_1, T_2$  be uniformly  $L$ -Lipschitzian nonself mappings with respect to  $P$ . Put

$$G_n = \max\{0, \sup_{x \in C} (\|(PT_1)^n x - q\| - \|x - q\|)\}$$

and

$$K_n = \max\{0, \sup_{x \in C} (\|(PT_2)^n x - q\| - \|x - q\|)\}$$

such that  $\sum_{n=1}^{\infty} G_n < \infty$  and  $\sum_{n=1}^{\infty} K_n < \infty$  respectively. Suppose that  $\{x_n\}$  is the sequence defined by (5) with  $\sum_{n=1}^{\infty} \delta_n < \infty$  and the additional assumption that  $0 < \liminf \alpha_n, 0 < \liminf \gamma_n$  and  $0 < \liminf \delta_n$ .

If one of  $T_1$  or  $T_2$  is completely continuous and  $F(T_1) \cap F(T_2) \neq \varphi$ , then  $\{x_n\}$  converges strongly to a common fixed point of  $T_1$  and  $T_2$ .

*Proof.* From Lemma 4, for any  $q \in F(T_1) \cap F(T_2)$ ,  $\lim_n \|x_n - q\|$  exists, therefore  $\{x_n\}$  is bounded. Again by Lemma 5, we have

$$\lim_{n \rightarrow \infty} \|(PT_1)x_n - x_n\| = 0, \text{ and } \lim_{n \rightarrow \infty} \|(PT_2)x_n - x_n\| = 0 \quad (12)$$

Now if  $T_1$  is completely continuous and  $\{x_n\}$  is bounded, we conclude that there exists sub sequences  $\{PT_1 x_{n_j}\}$  of  $\{PT_1 x_n\}$  such that  $\{PT_1 x_{n_j}\}$  converges.

Therefore, from (12),  $\{x_{n_j}\}$  is convergent.

Let  $x_{n_j} \rightarrow r$ , as  $j \rightarrow \infty$ . By the continuity of  $P, T_1, T_2$  and (12) we have  $r = PT_1 r = PT_2 r$ . Since  $F(PT_1) = F(T_1)$  and  $F(PT_2) = F(T_2)$  and by Lemma 3, we have  $r = T_1 r = T_2 r$ . Thus  $\{x_n\}$  converges strongly to a common fixed point  $r$  of  $T_1$  and  $T_2$ . This completes the proof of the theorem.  $\square$

**Acknowledgments:** The authors are very grateful to the referees for their valuable comments and good suggestions.

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