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MULTIVARIABLE AND SCATTERED DATA INTERPOLATION FOR SOLVING MULTIVARIABLE INTEGRAL EQUATIONS

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Abstract. In this paper we use radial basis functions in one of the projection methods to solve integral equations of the second kind with two or more variables. This method implemented without needing any introductory algorithms. Relatively good error bound and the numerical experiments show the accuracy of the method.

Key words: collocation method, multivariable integral equation, radial basis functions, scattered data interpolation.
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1. Introduction

In many literatures univariable integral equations have been solved with projection methods as collocation and Galerkin methods and with different types of basis functions such as wavelets or other (orthogonal) basis functions [1-7,9,11,13,14,16]. In some projection methods such as collocation method one can use interpolation scheme; but as we know in the case of two or more variable cases there is not any natural generalization of interpolation [8,10,12]. Other methods such as finite element which uses (orthogonal) basis functions actually needs mesh generation for domain of integration. Mesh dependent methods need some triangulation (or rectangulation) and coding for nodes of each triangle (or rectangle) therefore some introductory algorithms should be executed before implementation of the underlying method [1]. Here we solve multi variable I.E. with some mesh less methods.

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To simplify the notation we consider only functions of two variables. Generalizations of functions in more than two variables should be fairly straightforward.

2. Multivariable I.E. and Some Kinds of Interpolations

To simplify the notation and to make more intuitive development we consider only the following I.E. of the second kind with two variables

$$\lambda\rho(x,y) - \int_R k(x,y,\xi,\eta)\rho(\xi,\eta)d\xi d\eta = \psi(x,y), \quad (x,y) \in R \tag{1}$$

where R, is a bounded region in the plane $\mathbb{R}^2$. In this section we apply (for comparison) two methods, firstly multivariable interpolation and secondly scattered data interpolation method to the above equation. The second one need not any triangulation of R.

2.1. **Multivariable interpolation and collocation method.** Interpolation for functions of more than one variables is large topic with applications to many areas [18]. One of them is obtaining numerical solution of integral equations. Applications of multi variable interpolation are generally based on first breaking up a large planar region R into smaller ones of an especially simple form and then polynomial interpolation is carried out over these smaller regions. The two most common shapes for these smaller regions are the rectangle and triangle.

For polynomial interpolation one can use two dimensional Lagrange's form

$$P_{m,n} = \sum_{i=0}^{m}\sum_{j=0}^{n} g(x_i,y_j)l_{i,m}(x)l_{j,n}(y) \tag{2}$$

with

$$l_{i,m} = \prod_{\substack{j=0\\ j\neq i}}^{m} \frac{x-x_j}{x_i-x_j}, \qquad l_{j,n} = \prod_{\substack{i=0\\ i\neq j}}^{n} \frac{y-y_i}{y_j-y_i} \tag{3}$$

and $R = [a,b]\times[c,d]$, $a \le x_0 < \ldots x_m \le b$, $c \le y_0 < \ldots y_n \le d$. This polynomial $P_{m,n}$ interpolates $g(x,y)$ at $(x,y) = (x_i,y_j)$, $i = 0,\ldots,m$, $j = 0,\ldots,n$; it is of degree $r = m+n$. A popular form of (2) appears by taking $m = n = 1$, yielding the *bilinear interpolation* polynomial.

For interpolation over triangles, firstly denote $\triangle$ a planar triangle in the xy-plane and let $g(x,y)$ be a continuous function on $\triangle$. In this case a polynomial interpolant $p(x,y)$ of degree r, for some $r \ge 0$ can be as

$$p(x,y) = \sum_{\substack{i,j\ge 0\\ i+j\le r}} c_{i,j}x^i y^j. \tag{4}$$

To determine coefficients $c_{i,j}$ we require

$$p(x_k, y_k) = g(x_k, y_k), \qquad k = 1, \ldots, f_r$$

for some choice of $f_r = \frac{(r+1)(r+2)}{2}$ interpolation nodes $\{(x_k, y_k) \mid 1 \leq k \leq f_r\} \subset \triangle$.
In some literatures to simplify the notation and to lead to a form more readily adaptable to implementation in computer languages, use a sequential ordering of the nodes $q_1, \ldots, q_{f_r}$ for a unit simplex

$$\{(s,t) \mid s,t \geq 0,\ s+t \leq 1\},$$

and define an 1-1 and onto affine mapping $T : \sigma \longrightarrow \triangle$ by

$$(x,y) = T(s,t) \equiv uv_1 + tv_2 + sv_3, \quad u = 1 - s - t, \tag{5}$$

of a point $(x,y) \in \triangle$ with v_1, v_2, v_3 the vertices of $\triangle$. Now given a function $g \in C(\triangle)$, the unique polynomial of degree $\leq r$ that interpolate the nodes $v_1, \ldots, v_{f_r}$ of $v_i = T(q_i), \quad i = 1, \ldots, f_r$ is given by

$$p_r(x,y) = \sum_{i=1}^{f_r} g(T(q_i)) l_i(s,t) = \sum_{i=1}^{f_r} g(v_i) l_{\triangle,i}(x,y), \tag{6}$$

with $l_{\triangle,i}(x,y) = l_i(s,t), \quad (x,y) = T(s,t)$. Now the interpolatory operator $\mathcal{P}_n$ on $C(R)$ is introducing as

$$\mathcal{P}_n \rho(x,y) = \rho_n(x,y) = \sum_{i=1}^{f_r} \rho_n(v_{k,i}) l_i(s,t), \ \ (x,y) = T_k(s,t) \in \triangle_k, \ k = 1, \ldots, n. \tag{7}$$

with

$$v_{k,i} = T_k(q_i), \quad i = 1, \ldots, f_r, \quad k = 1, \ldots, n. \tag{8}$$

Substituting (8) into (1) and then collocating at the node points $v_n = \{v_1, \ldots, v_{n_v}\}$ of (8) we obtain the linear system

$$\lambda \rho_n(v_i) - 2 \sum_{k=1}^{n} Area(\triangle_k) \sum_{j=1}^{f_r} \rho_n(v_{k,j}) \int_\sigma k(v_i, T_k(s,t)) l_j(s,t) d\sigma = \psi(v_i), \ \ i = 1, \ldots, n_v. \tag{9}$$

Now the collocation method for (1) can be written symbolically as

$$\mathcal{P}_n(\lambda - \mathcal{K})\rho_n = \mathcal{P}_n \psi, \quad \rho_n \in \chi_n, \tag{10}$$

where

$$\mathcal{K}\rho = \int_R k(x,y,\xi,\eta)\rho(\xi,\eta) d\xi d\eta.$$

However if we write the equation (1) symbolically as

$$(\lambda - \mathcal{K})\rho = \psi, \tag{11}$$

we will have the error bound as

Theorem 1. *Let R be a polygonal region in R^2, and let $\mathcal{T}_n$ be a sequence of triangulations of R. Assume $\delta_n = \max_{k=1,\ldots,n} diameter\{\triangle_k\} \to 0$ as $n \to \infty$ and the integral equation (11) is uniquely solvable with $\mathcal{K}$ a compact operator on $C(R)$. Then for all sufficiently large n, $n \geq N$, the approximating equation (10) is uniquely solvable and we have*

$$\|\rho - \rho_n\|_\infty \leq \mid \lambda \mid \|(\lambda - \mathcal{P}_n\mathcal{K})^{-1}\|\|\rho - \mathcal{P}_n\rho\|_\infty, \ n \geq N. \tag{12}$$

Proof: See[1].

Also by polynomial interpolation (6) of degree r, we can see ([1]) that,

$$\|\rho - \rho_n\| \leq c\delta_n^{r+1}, \ n \geq N, \ \rho \in C^{r+1}(R). \tag{13}$$

2.2. **Scattered data interpolation and collocation method.** Given a region R in plane $\mathbb{R}^2$, and a set of data (measurements and locations at which these measurements were obtained) we want to find a rule which exactly match the given measurements at the corresponding locations. If the locations at which the measurements are taken, are not on a uniform or regular grid then the process is called *scattered data interpolation.*

A common approach to solving the scattered data problem is to make the assumption that the function $\mathcal{P}f$ is a linear combination of certain basis functions B_k, i.e.

$$\mathcal{P}f(\mathbf{x}) = \sum_{k=1}^{N} c_k B_k(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^s. \tag{14}$$

Here we use scattered data interpolation to find the solution of multivariable integral equation. In the univariate setting it is well known that one can interpolate to arbitrary data at N distinct data sites using a polynomial of degree $N-1$. For multivariate setting however, there is the following negative result due to Mairhuber and Curtis in 1956, [15].

Theorem 2. *If $\Omega \subset \mathbb{R}^s$, $s \geq 2$ contains an interior point then there exist no Haar spaces of continuous functions except for one dimensional ones.*

Proof: See [15].

Remarks:

1. Note that existence of a Haar space guarantees invertibility of the interpolation matrix.

2. The Mairhuber-Curtis theorem implies that in the multivariable setting we can no longer expect this to be the case, e.g., it is not possible to construct unique interpolation with multivariate polynomials of degree N to data given at arbitrary locations in $\mathbb{R}^2$.

Definition 1. *A complex valued continuous function* Φ *is called positive definite on* $\mathbb{R}^s$ *if*

$$\sum_{j=1}^{N}\sum_{k=1}^{N} c_j\bar{c}_k\Phi(\mathbf{x}_j-\mathbf{x}_k)\geq 0, \tag{15}$$

for any N *pairwise different* $\mathbf{x}_1,\ldots,\mathbf{x}_N\in\mathbb{R}^s$ *and* $\mathbf{c}=[c_1,\ldots,c_N]^t\in\mathbb{C}^N$. *The function* Φ *is called strictly positive definite on* $\mathbb{R}^s$, *if the only vector that turns eq. (15) into an equality is the zero vector.*

Definition 1 and the discussion preceding it, suggest that we should use strictly positive definite functions as basis functions in eq. (14), i.e. $B_k(\mathbf{x})=\Phi(\mathbf{x}-\mathbf{x}_k)$ or

$$\mathcal{P}_N f(\mathbf{x})=\sum_{k=1}^{N} c_k\Phi(\mathbf{x}-\mathbf{x}_k),\quad \mathbf{x}\in\mathbb{R}^s. \tag{16}$$

Definition 2. *A function* $\Phi:\mathbb{R}^s\to\mathbb{R}$ *is called Radial provided that there exist a univariate function* $\varphi:[0,\infty)\to\mathbb{R}$, *such that* $\Phi(\mathbf{x})=\varphi(r)$, *where* $r=\|\mathbf{x}\|$, *and* $\|.\|$ *is some norm on* $\mathbb{R}^s$ *usually the Euclidean norm.*

Some radial functions that are useful for interpolation are as bellow

1. $\phi(r)=\exp(-\alpha r^2),\quad \alpha>0,\quad Gaussian(GA)$

2. $\phi(r)=(c^2+r^2)^{\beta},\quad \beta>0,\quad \beta\notin\mathbf{N},\ Multiquadric\ (MQ)$

3. $\phi(r)=(c^2+r^2)^{\beta},\quad \beta<0,\quad Inverse\ Multiquadric\ (IM)$

4. $\phi(r)=r^2\ln(r),\quad Thin\ plate\ spline\ (TPS)$

5. $\phi(r)_{s,k}=(1-r)_+^m p(r),\quad Wendland\ functions,\ \ where\ (.)_+\ is\ defined\ by$

$$(x)_+^m=\begin{cases} x & x\geq 0\\ 0 & x<0,\end{cases}$$

and p(r) is a suitable polynomial of degree at most k [18].

To define the collocation method for solving eq. (1), proceed as follows. Use a Radial interpolation method over R by introducing the interpolatory operator $\mathcal{P}_n$ on $C(R)$ as

$$\mathcal{P}_n\rho(x,y)=\rho_n(x,y)=\sum_{j=1}^{n} c_j\Phi(\mathbf{x}-\mathbf{x}_j),\qquad \mathbf{x}=(x,y), \tag{17}$$

where $\Phi(\mathbf{x}) = \varphi(\|\mathbf{x})\|)$, and ϕ is a radial function as in preceding section and $\mathbf{x}_j, \quad j = 1, \ldots, n$, are distinct scattered data of region R. Introduce

$$\begin{aligned} r_n(\mathbf{x}) &= \lambda\rho_n(\mathbf{x}) - \textstyle\int_R k(\mathbf{x}, \nu)\rho_n(\nu)d\nu - \psi(\mathbf{x}) \\ &= \textstyle\sum_{j=1}^n c_j\{\lambda\Phi(\mathbf{x} - \mathbf{x}_j) - \int_R k(\mathbf{x}, \nu)\Phi(\nu - \mathbf{x}_j)d\nu\} - \psi(\mathbf{x}), \end{aligned} \tag{18}$$

where $\nu = (\xi, \eta)$ and $\mathbf{x} = (x, y) \in R$.

Then for nodes $\mathbf{x}_j \in R, \quad j = 1, \ldots, n$, compose

$$r_n(\mathbf{x}_i) = 0, \quad i = 1, \ldots, n. \tag{19}$$

This leads to determining $\{c_1, \ldots, c_n\}$ as the solution of the linear system

$$\sum_{j=1}^{n} c_j\{\lambda\Phi(\mathbf{x}_i - \mathbf{x}_j) - \int_R k(\mathbf{x}_i, \nu)\Phi(\nu - \mathbf{x}_j)d\nu\} = \psi(\mathbf{x}_i) \tag{20}$$

for $i = 1, \ldots, n$. We note that

$$\mathcal{P}_n z = 0 \;\; if \;\; and \;\; only \;\; if \;\; z(\mathbf{x}_i) = 0, \quad i = 1, \ldots, n.$$

The condition (19) can now be rewritten as

$$\mathcal{P}_n r_n = 0,$$

or equivalently

$$\mathcal{P}_n(\lambda - \mathcal{K})\rho_n = \mathcal{P}_n\psi, \quad \rho_n \in \chi_n, \tag{21}$$

where $\mathcal{K}\rho = \displaystyle\int_R k(x, y, \xi, \eta)\rho(\xi, \eta)d\xi d\eta$, and $\chi_n = span\{\Phi(\mathbf{x} - \mathbf{x}_j)\}_{j=1,\ldots,n}$. We can see in [1] that the eq. (21) is equivalent to

$$(\lambda - \mathcal{P}_n\mathcal{K})\rho_n = \mathcal{P}_n\psi, \quad \rho_n \in \chi \tag{22}$$

where χ is a Banach space. For the error analysis we compare eq. (22) with the original equation

$$(\lambda - \mathcal{K})\rho = \psi \tag{23}$$

since both are defined on the space χ.

3. Error Analysis

In this section we give an error bound for approximate solution of (1) by suggested method of section 2.2 which comparable with those of section 2.1. First we give some definitions.

Definition 3. *The Fourier transform of $f \in \mathbf{L}_1(\mathbb{R}^s)$ is given by*

$$\widehat{f}(\omega) = 1/\sqrt{(2\pi)^s} \int_{\mathbb{R}^s} f(\mathbf{x})e^{-i\omega\mathbf{x}}d\mathbf{x}. \tag{24}$$

Now let Ω be a domain in $\mathbb{R}^s$, and $\mathbf{X} = \{x_1, \ldots, x_M\} \subseteq \Omega \subseteq \mathbb{R}^s$, and a *kernel* function

$$\mathbf{\Phi} : \Omega \times \Omega \to \mathbb{R}, \qquad \Omega \subseteq \mathbb{R}^s. \tag{25}$$

Consider a finite dimensional space

$$\mathcal{S}_{\mathbf{X},\Phi} := span\{\Phi(x,.) \ : \ x \in \mathbf{X}\} \tag{26}$$

of dimension at most M. The union of these spaces is

$$\mathcal{S}_{\Phi} := span\{\Phi(x,.) \ : \ x \in \Omega\}. \tag{27}$$

If we want to have a norm structure on the space (27) we can define:

Definition 4. *A function (25) on $\Omega \subseteq \mathbb{R}^s$ that generates an inner product of the form*

$$\langle \Phi(x,.), \Phi(y,.) \rangle_{\Phi} = \Phi(x,y) \quad for\ all \ \ x, y \in \Omega, \tag{28}$$

on the space $\mathcal{S}_{\Phi}$ will be called a reproducing kernel on Ω. Equation (28) turns $\mathcal{S}_{\Phi}$ into a pre-Hilbert space, and it allows to write

$$\langle f(.), \Phi(y,.) \rangle_{\Phi} := f(y) \quad for\ all \ \ y \in \Omega, \ \ f \in \mathcal{S}_{\Phi},$$

because the equation holds for $f_x(y) := \Phi(x,y)$ by eq. (28), therefore that is true for all functions in $\mathcal{S}_{\Phi}$. The closure $\mathcal{N}_{\Phi}$ of $\mathcal{S}_{\Phi}$ under the inner product $\langle ., . \rangle_{\Phi}$ will be a Hilbert space [17]. In fact

Definition 5. *If Φ is a reproducing kernel on $\Omega \in \mathbb{R}^s$, we call the space*

$$\mathcal{N}_{\Phi} := clos_{\langle .,. \rangle_{\Phi}} \mathcal{S}_{\Phi} := clos_{\langle .,. \rangle_{\Phi}} \ span \ \{\Phi(x,.) \ : \ x \in \Omega\},$$

the native space for Φ.

Definition 6. *we define*

$$\mathbf{W}_2^m(\mathbb{R}^s) = \{f \in \mathbf{L}_2(\mathbb{R}^s) \cap \mathbf{C}(\mathbb{R}^s) : \widehat{f}(.)(1 + \|.\|_2^2)^{m/2} \in \mathbf{L}_2(\mathbb{R}^s)\}. \tag{29}$$

With definition (6) if we use a compactly supported function $\Phi_{s,k}$ (as definition 2 no. 5 which is zero outside of [0,1]) for interpolating function f by eq. (16), we have the following error bound, [19]

$$\|f - \mathcal{P}_N f\|_{\mathbf{L}_2(\Omega)} \leq \mathbf{C} h^{2k+1+s} \|f\|_{\mathbf{W}_2^{2k+1+s}(\mathbb{R}^s)} \tag{30}$$

where f is assumed to lie in the subspace $\mathbf{W}_2^{2k+1+s}(\mathbb{R}^s)$ of $\mathcal{N}_{\Phi}(\mathbb{R}^s)$.

Theorem 3. *Assume $\mathcal{K} : \chi \to \chi$ is bounded, with χ a Banach space, and assume $\lambda - \mathcal{K} : \chi \to \chi$ is an onto and 1-1 map. Further assume*

$$\|\mathcal{K} - \mathcal{P}_n \mathcal{K}\| \to 0 \quad as \quad n \to \infty,$$

then for all sufficiently large n, say $n \geq N$, the operator $(\lambda - \mathcal{P}_n \mathcal{K})^{-1}$ exist as a bounded operator from χ to χ. Moreover, it is uniformly bounded:

$$sup_{n \geq N} \|(\lambda - \mathcal{P}_n \mathcal{K})^{-1}\| < \infty.$$

For the solution of (22) and (23),

$$\frac{|\lambda|}{\|(\lambda - \mathcal{P}_n\mathcal{K})\|}\|\rho - \mathcal{P}_n\rho\| \leq \|\rho - \rho_n\| \leq |\lambda|\|(\lambda - \mathcal{P}_n\mathcal{K})^{-1}\|\|\rho - \mathcal{P}_n\rho\|.$$

Proof: See [1].

Therefore we have obtained with assumption of preceding theorem and eq. (30) that

$$\|\rho - \rho_n\| = \mathcal{O}(h^{2k+1+s}).$$

As we observe, the method of scattered data interpolation for solving multivariable equation (1) is more simple than those of section 2.1 also with more better error bound.

4. Numerical Experiments

In our first example we use Gaussian functions i.e. $\Phi(\mathbf{x} - \mathbf{x}_j) = \exp(-\|\mathbf{x} - \mathbf{x}_j\|^2)$ in eq. (20).

1. Consider eq. (1) with the kernel function and the exact solution as bellow

$$k(x, y, \xi, \eta) = x\xi^2 + y^2\eta, \qquad \rho(x, y) = exp(xy),$$

$R = [0, 1] \times [0, 1]$, and the suitable right hand side.

2. In the second example we use Multiquadric functions i.e. $\Phi(\mathbf{x} - \mathbf{x}_j) = \sqrt{1 + \|\mathbf{x} - \mathbf{x}_j\|^2}$ in eq. (20).

Consider eq. (1) with the kernel function as example 1 and the exact solution as bellow

$$\rho(x, y) = 1 + x^2 + y^2,$$

$R = [0, 1] \times [0, 1]$, and the suitable right hand side.

In our third example we use Inverse Multi quadratic functions i.e. $\Phi(\mathbf{x} - \mathbf{x}_j) = 1/(1 + \|\mathbf{x} - \mathbf{x}_j\|^2)$ in eq. (20).

3. Consider eq. (1) with the kernel function and the exact solution as bellow

$$k(x, y, \xi, \eta) = \cos(x\xi)\cos(y\eta), \qquad \rho(x, y) = sin(x + y),$$

$R = [-1, 1] \times [-1, 1]$, and the suitable right hand side.

In the last example we use Gaussian functions with a parameter $\alpha = 0.5$ i.e. $\Phi(\mathbf{x} - \mathbf{x}_j) = \exp(-0.5\|\mathbf{x} - \mathbf{x}_j\|^2)$.

4. Consider eq. (1) with the kernel function and the exact solution as bellow

$$k(x, y, \xi, \eta) = 1, \qquad \rho(x, y) = x^2 - y^2,$$

$R = [-0.5, 0.5] \times [-0.5, 0.5]$, and the suitable right hand side.

The absolute error of $|\rho_n(x, y) - \rho(x, y)|$ in some points of function domain for our 4 examples are presented in table 1.

Table 1. Absolute error of exact and approximated solution of examples 1-4 in some points of functions domains

(x_i, y_j)	$ex.1$	$ex.2$	$ex.3$	$ex.4$
$(0.1, 0.2)$	$0.235e-6$	$0.524e-5$	$0.254e-5$	$0.433e-7$
$(0.2, 0.3)$	$0.564e-7$	$0.547e-6$	$0.548e-6$	$0.571e-7$
$(0.2, 0.5)$	$0.987e-7$	$0.657e-6$	$0.548e-6$	$0.224e-6$
$(0.7, 0.4)$	$0.654e-7$	$0.587e-6$	$0.654e-7$	–
$(0.8, 0.9)$	$0.874e-6$	$0.524e-5$	$0.554e-6$	–
$(0.5, 0.7)$	$0.547e-7$	$0.238e-6$	$0.554e-6$	–
$(-0.4, 0.1)$	–	–	$0.284e-5$	$0.226e-7$
$(-0.3, -0.1)$	–	–	$0.552e-6$	$0.332e-7$
$(-0.4, -0.2)$	–	–	$0.524e-6$	$0.245e-7$
$(-0.9, -0.7)$	–	–	$0.547e-5$	–

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