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NEW RECURRENCE RELATIONSHIPS BETWEEN ORTHOGONAL POLYNOMIALS WHICH LEAD TO NEW LANCZOS-TYPE ALGORITHMS

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Abstract. Lanczos methods for solving $A\mathbf{x} = \mathbf{b}$ consist in constructing a sequence of vectors $(\mathbf{x}_k), k = 1, ...$ such that $\mathbf{r}_k = \mathbf{b} - A\mathbf{x}_k = P_k(A)\mathbf{r}_0$, where P_k is the orthogonal polynomial of degree at most k with respect to the linear functional c defined as $c(\xi^i) = (\mathbf{y}, A^i\mathbf{r}_0)$. Let $P_k^{(1)}$ be the regular monic polynomial of degree k belonging to the family of formal orthogonal polynomials (FOP) with respect to $c^{(1)}$ defined as $c^{(1)}(\xi^i) = c(\xi^{i+1})$. All Lanczos-type algorithms are characterized by the choice of one or two recurrence relationships, one for P_k and one for $P_k^{(1)}$. We shall study some new recurrence relations involving these two polynomials and their possible combinations to obtain new Lanczos-type algorithms. We will show that some recurrence relations exist, but cannot be used to derive Lanczos-type algorithms, while others do not exist at all.

Key words: Lanczos algorithm, formal orthogonal polynomials, linear system, monic polynomials.
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1. Introduction

In 1950, C. Lanczos [26] proposed a method for transforming a matrix into an equivalent tridiagonal matrix. We know, by Cayley-Hamilton theorem, that the computation of the characteristic polynomial of a matrix and the solution of linear equations are equivalent, Lanczos, [27], in 1952 used his method for solving systems of Linear equations.

Since then, several Lanczos-type algorithms have been obtained and among them, the famous conjugate gradient algorithm of Hestenes and Stiefel, [24],

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when the matrix is Hermitian, and the bi-conjugate gradient algorithm of Fletcher, [20], for the general case. In the last three decades, Lanczos algorithms have evolved and different variants have been derived, [2, 3, 5, 7, 10, 11, 12, 13, 14, 22, 23, 28, 29, 30, 32, 35, 37].

Although Lanczos-type algorithms can be derived by using linear algebra techniques, the formal orthogonal polynomials (FOP) approach is perhaps the most common for deriving them. In fact, all recursive algorithms implementing the Lanczos method can be derived using the theory of FOP, [5].

A drawback of these algorithms is their inherent fragility due to the nonexistence of some orthogonal polynomials. This causes them to breakdown well before convergence. To avoid these breakdowns, variants that jump over the nonexisting polynomials have been developed; they are referred to as breakdown-free algorithms, [10, 11, 21, 23, 30, 37]. Note that it is not the purpose of this paper to discuss this breakdown issue although it will be highlighted when necessary.

Two types of recurrence relations are needed: one for $P_k(x)$ and one for $P_k^{(1)}(x)$, [2]. In [1, 2], recurrence relations for the computation of polynomials $P_k(x)$ are represented by A_i and those for polynomials $P_k^{(1)}(x)$, by B_j. Table 1 and Table 2 below give a comprehensive list.

C. Baheux and C. Brezinski have exploited some of the polynomial relations which involve few matrix-vector multiplications. In their work, the only relations that they studied were those where the degrees of the polynomials in the right and left hand sides of the relation differ by ONE or TWO at most. We are studying relations where the difference in degrees is TWO or THREE. For full details of these relations, see [16]. The following notation has been introduced in [1, 2]. We will adopt it here and extend the list accordingly.

TABLE 1. Computation of A_i and B_j from different polynomials [1]

Relation A_i	Computation of P_k from		Relation B_j	Computation of $P_k^{(1)}$ from	
A_1	P_{k-2}	$P_{k-2}^{(1)}$	B_1	P_{k-2}	$P_{k-2}^{(1)}$
A_2	P_{k-2}	$P_{k-1}^{(1)}$	B_2	P_{k-2}	$P_{k-1}^{(1)}$
A_3	P_{k-2}	$P_k^{(1)}$	B_3	P_{k-2}	P_k
A_4	P_{k-2}	P_{k-1}	B_4	P_{k-2}	P_{k-1}
A_5	$P_{k-2}^{(1)}$	P_{k-1}	B_5	$P_{k-2}^{(1)}$	$P_{k-1}^{(1)}$
A_6	$P_{k-2}^{(1)}$	$P_{k-1}^{(1)}$	B_6	$P_{k-2}^{(1)}$	$P_{k-1}^{(1)}$
A_7	$P_{k-2}^{(1)}$	$P_k^{(1)}$	B_7	$P_{k-2}^{(1)}$	P_k
A_8	$P_{k-1}^{(1)}$	P_{k-1}	B_8	$P_{k-1}^{(1)}$	P_{k-1}
A_9	P_{k-1}	$P_k^{(1)}$	B_9	P_{k-1}	P_k
A_{10}	$P_{k-1}^{(1)}$	$P_k^{(1)}$	B_{10}	$P_{k-1}^{(1)}$	P_k

TABLE 2. Polynomials used in the computation of new relations A_i and B_j

Relation A_i	Computation of P_k from		Relation B_j	Computation of $P_k^{(1)}$ from	
A_{11}	P_{k-3}	$P_{k-1}^{(1)}$	B_{11}	P_{k-3}	P_{k-1}
A_{12}	P_{k-2}	P_{k-3}	B_{12}	P_{k-2}	P_{k-3}
A_{13}	P_{k-2}	$P_{k-3}^{(1)}$	B_{13}	$P_{k-2}^{(1)}$	$P_{k-3}^{(1)}$
A_{14}	$P_{k-2}^{(1)}$	$P_{k-3}^{(1)}$	B_{14}	$P_{k-3}^{(1)}$	P_{k-1}
A_{15}	$P_{k-3}^{(1)}$	$P_{k-1}^{(1)}$	B_{15}	P_{k-2}	$P_{k-2}^{(1)}$
A_{16}	P_{k-2}	$P_{k-2}^{(1)}$	B_{16}	$P_{k-2}^{(1)}$	P_{k-1}
A_{17}	P_{k-2}	$P_{k-1}^{(1)}$	–	–	
A_{18}	$P_{k-1}^{(1)}$	$P_{k-2}^{(1)}$	–	–	
A_{19}	$P_{k-2}^{(1)}$	P_{k-1}	–	–	

The paper is organized as follow. Section 2, briefly recalls the Lanczos algorithm. Section 3, derives some of the possible recurrence relations A_i and B_j given in Table 2 and their combination to obtain Lanczos-type algorithms. It also discusses two recurrence relations which although exist and satisfy the normalization and orthogonality conditions, cannot be used for the computation of $\mathbf{r}_k$ and hence $\mathbf{x}_k$. Section 4 is the conclusion.

2. The Lanczos Algorithm

Consider a linear system in R^n with n unknowns

$$A\mathbf{x} = \mathbf{b}. \tag{1}$$

Lanczos method, [2, 11, 12, 27], for solving (1), consists in constructing a sequence of vectors $\mathbf{x}_k$ as follows.

- choose two arbitrary vectors $\mathbf{x}_0$ and $\mathbf{y} \neq 0$ in R^n,
- set $\mathbf{r}_0 = \mathbf{b} - A\mathbf{x}_0$,
- determine $\mathbf{x}_k$ such that

$\mathbf{x}_k - \mathrm{x}_0 \in E_k = span(\mathbf{r}_0, A\mathbf{r}_0, \ldots, A^{k-1}\mathbf{r}_0)$

$\mathbf{r}_k = \mathbf{b} - A\mathbf{x}_k \perp F_k = span(\mathbf{y}, A^T\mathbf{y}, \ldots, A^{T^{k-1}}\mathbf{y})$

where A^T is transpose of A.

$\mathbf{x}_k - \mathbf{x}_0$ can be written as

$$\mathbf{x}_k - \mathbf{x}_0 = -\alpha_1\mathbf{r}_0 - \cdots - \alpha_k A^{k-1}\mathbf{r}_0.$$

Multiplying both sides by A, adding and subtracting $\mathbf{b}$ and simplifying, we get

$$\mathbf{r}_k = \mathbf{r}_0 + \alpha_1 A\mathbf{r}_0 + \cdots + \alpha_k A^k\mathbf{r}_0$$

and the orthogonality condition above give

$$(A^{T^i}\mathbf{y}, \mathbf{r}_k) = 0 \text{ for i = 0,..., k-1},$$

which is a system of k linear equations in the k unknowns $\alpha_1, \ldots, \alpha_k$. This system is nonsingular only if $\mathbf{r}_0, A\mathbf{r}_0, \ldots, A^{k-1}\mathbf{r}_0$ and $\mathbf{y}, A^T\mathbf{y}, \ldots, A^{T^{k-1}}\mathbf{y}$ are linearly independent.

If we set

$$P_k(\xi) = 1 + \alpha_1\xi + \cdots + \alpha_k\xi^k$$

then we have

$$\mathbf{r}_k = P_k(A)\mathbf{r}_0.$$

Moreover, if we set

$$c_i = (\mathbf{y}, A^i\mathbf{r}_0) \text{ for i=0,1,...}$$

and we define the linear functional c on the space of polynomials by

$$c(\xi^i) = c_i \text{ for i=0,1,...}$$

then the preceding orthogonality conditions can be written as

$$c(\xi^i P_k) = 0 \text{ for i=0,...,} k-1.$$

These relations show that P_k is the polynomial of degree at most k belonging to the family of orthogonal polynomials with respect to c, [4]. This polynomial is defined apart from a multiplying factor which was chosen, in our case, such that $P_k(0) = 1$. With this normalization condition, P_k exists and is unique if and only if the following Hankel determinant

$$\mathbf{H}_k^{(1)} = \begin{vmatrix} c_1 & c_2 & \cdots & c_k \\ c_2 & c_3 & \cdots & c_{k+1} \\ \vdots & \vdots & & \vdots \\ c_k & c_{k+1} & \cdots & c_{2k-1} \end{vmatrix}$$

is different from zero.

Let us now consider the monic polynomial $P_k^{(1)}$ of degree k belonging to the FOP with respect to the functional $c^{(1)}$ defined by $c^{(1)}(\xi^i) = c(\xi^{i+1})$. $P_k^{(1)}$ exists and is unique, if and only if the Hankel determinant $\mathbf{H}_k^{(1)} \neq 0$, which is the same condition as for the existence and uniqueness of P_k.

A Lanczos-type algorithm consists in computing P_k recursively, then $\mathbf{r}_k$ and finally $\mathbf{x}_k$ such that $\mathbf{r}_k = \mathbf{b} - A\mathbf{x}_k$, without inverting A, which gives the solution of the system (1) in at most n steps, in exact arithmetic where n is the dimension of the system of linear equations, [5, 11].

3. Recursive Computation of P_k and $P_k^{(1)}$

The recursive computation of the polynomials P_k, needed in the Lanczos method, can be achieved in many ways. We can use the usual three-term recurrence relation, or the relation involving the polynomials of the form $P_k^{(1)}$. Such recurrence relations lead to all known algorithms for implementing Lanczos-type algorithms. For a unified presentation of all these methods based on the theory of FOP, see [2, 11, 16].

We need two recurrence relations, one for P_k and one for $P_k^{(1)}$. All Lanczos-type algorithms are characterized by the choice of these recurrence relationships. In the following we will discuss some of these recurrence relations for P_k, [16], and derive new recurrence relationships between adjacent orthogonal polynomials, [1, 2, 4, 5, 6, 12, 34], which can be used to design new Lanczos-type algorithms, as has been shown in [2, 11, 16]. Note that the term "Lanczos process" and "Lanczos-type algorithm" are used interchangeably throughout the paper.

3.1. **Relations A_i.** We will follow the notation explained in Section 1. First we will derive relations A_i ($i > 10$) for P_k which can be used to find $\mathbf{r}_k$ and then $\mathbf{x}_k$ without inverting A, the matrix of the system to be solved. We will only try to find the constant coefficients of recurrence relations which can be used for the implementation of Lanczos-type algorithms. If a recurrence relation exists but cannot be used for such an implementation, then there is no need to calculate its coefficients. The reason for that will be given. Note, however, that when a recurrence relation exists and can be computed, leading

therefore to a Lanczos-type algorithm, the algorithm may still break down for two reasons:
1. There is a loss of orthogonality as in most known Lanczos-type algorithms, [28, 33].
2. When coefficients involving determinants in their denominators, these determinants may be zero (or rather, in practice, just close to zero). This kind of breakdown is called ghost breakdown, [7, 8]. It may be cured by conditioning [31, 25, 36]. This is not considered here. Note that restarting and switching strategies, as presented in [16, 18, 19], can cure both types of breakdown.

In the following P_k stands for $P_k(x)$ and $P_k^{(1)}$ for $P_k^{(1)}(x)$. But, before we derive the relations A_i and B_j, we first define the notion of"orthogonal polynomials sequence", [15].

Definition 1. A sequence $\{P_n\}$ is called an orthogonal polynomial sequence with respect to the linear functional c if, for all nonnegative integers m and n,
(i) P_n is polynomial of degree n,
(ii) $c(x^m P_n) = 0$, for $m \neq n$,
(iii) $c(x^n P_n) \neq 0$.

3.1.1. *Relation* A_{11}. As explained earlier, we follow up from what is already known up to A_{10}, [1]. A_{11} is therefore the natural follow up relation. Consider the recurrence relationship

$$P_k(x) = (A_k x^3 + B_k x^2 + C_k x + D_k)P_{k-3}(x) + (E_k x + F_k)P_{k-1}^{(1)}(x), \qquad (2)$$

where P_k, $P_{k-1}^{(1)}$ and P_{k-3} are polynomials of degree k, $k-1$ and $k-3$ respectively.
Proposition 3.1: Relation of the form A_{11} does not exist.
Proof: Let us see if all coefficients of (2) can be identified. If x^i is a polynomial of exact degree i then

$$\forall i = 0, \ldots, k-1,\ c(x^i P_k) = 0. \longrightarrow (C_1)$$
$$\forall i = 0, \ldots, k-2,\ c^{(1)}(x^i P_{k-1}^{(1)}) = 0. \longrightarrow (C_2)$$
$$\forall i = 0, \ldots, k-4,\ c(x^i P_{k-3}) = 0. \longrightarrow (C_3)$$

where c and $c^{(1)}$ are defined respectively as follows.

$$c(x^i) = c_i. \longrightarrow (C_4)$$
$$c^{(1)}(x^i) = c(x^{i+1}). \longrightarrow (C_5)$$

Since $P_k(0) = 1$, $\forall k$, then for $x = 0$, equation (2) becomes

$$1 = D_k + F_k P_{k-1}^{(1)}(0). \qquad (3)$$

Multiplying both sides of equation (2) by x^i and applying c using the condition (C_5) where necessary, it can be written as

$$c(x^iP_k) = A_kc(x^{i+3}P_{k-3}) + B_kc(x^{i+2}P_{k-3}) + C_kc(x^{i+1}P_{k-3}) \quad (4)$$
$$+ D_kc(x^iP_{k-3}) + E_kc^{(1)}(x^iP^{(1)}_{k-1}) + F_kc(x^iP^{(1)}_{k-1}).$$

For $i = 0$, equation (4) gives $0 = F_kc(P^{(1)}_{k-1})$. Since $c(P^{(1)}_{k-1}) \neq 0$, [4, 9] then $F_k = 0$. Hence from (3), we have $D_k = 1$.
Equation (4) is always true for $i = 1, ..., k-7$ by (C_2) and (C_3).
For $i = k-6$, (4) becomes $A_kc(x^{k-3}P_{k-3}) = 0$, as all other terms vanish due to conditions (C_1), (C_2) and (C_3). But, according to condition (iii) of definition 1 in Section 3.1, $c(x^{k-3}P_{k-3}) \neq 0$, therefore, $A_k = 0$.
For $i = k-5$, (4) becomes $B_kc(x^{k-3}P_{k-3}) = 0$. Since $c(x^{k-3}P_{k-3}) \neq 0$, $B_k = 0$.
For $i = k-4$, (4) becomes $C_kc(x^{k-3}P_{k-3}) = 0$. Since $c(x^{k-3}P_{k-3}) \neq 0$, $C_k = 0$. Putting values of A_k, B_k, C_k, D_k and F_k in equation (4), we get

$$c(x^iP_k) = c(x^iP_{k-3}) + E_kc^{(1)}(x^iP^{(1)}_{k-1}).$$

For $i = k-3$, this equation becomes

$$c(x^{k-3}P_k) = c(x^{k-3}P_{k-3}) + E_kc^{(1)}(x^{k-3}P^{(1)}_{k-1}).$$

Using (C_1) and (C_2), both $c(x^{k-3}P_k) = 0$ and $c^{(1)}(x^{k-3}P^{(1)}_{k-1}) = 0$, therefore we get

$$c(x^{k-3}P_{k-3}) = 0,$$

which is impossible due to condition (iii) of definition 1 given in Section 3.1. Therefore, Proposition 3.1 holds.

3.1.2. *Relation* A_{12}. Consider the following recurrence relationship for $k \geq 3$,

$$P_k(x) = A_k[(x^2+B_kx+C_k)P_{k-2}(x)+(D_kx^3+E_kx^2+F_kx+G_k)P_{k-3}(x)], \quad (5)$$

This recurrence relation has been considered in [16, 17].

3.1.3. *Relation* A_{13}. Consider the following recurrence relationship

$$P_k(x) = A_k[(x^2+B_kx+C_k)P_{k-2}(x)+(D_kx^3+E_kx^2+F_kx+G_k)P^{(1)}_{k-3}(x)], \quad (6)$$

where P_k, P_{k-2} and $P^{(1)}_{k-3}$ are orthogonal polynomials of degree k, $k-2$ and $k-3$ respectively and A_k, B_k, C_k, D_k, E_k, F_k and G_k are constants to be determined using the normalization condition $P_k(0) = 1$ and the orthogonality condition (C_1).
Proposition 3.2: Relation of the form A_{13} exists.
Proof: We know that

$$\forall i = 0, \ldots, k-1,\ c^{(1)}(x^iP^{(1)}_k) = 0. \longrightarrow (C_6)$$

Since $\forall k, P_k(0) = 1$, equation (6) gives

$$1 = A_k[C_k + G_k P^{(1)}_{k-3}(0)]. \tag{7}$$

Multiplying both sides of (6) by x^i and then applying the linear functional c, we get

$$\begin{aligned} c(x^i P_k) &= A_k[c(x^{i+2}P_{k-2}) + B_k c(x^{i+1}P_{k-2}) + C_k c(x^i P_{k-2}) \\ &+ D_k c^{(1)}(x^{i+2}P^{(1)}_{k-3}) + E_k c^{(1)}(x^{i+1}P^{(1)}_{k-3}) + F_k c^{(1)}(x^i P^{(1)}_{k-3}) + G_k c(x^i P^{(1)}_{k-3})]. \end{aligned} \tag{8}$$

For $i = 0$, equation (8) becomes, $0 = G_k c(P^{(1)}_{k-3})$. Since $c(P^{(1)}_{k-3}) \neq 0$, this implies that $G_k = 0$. Therefore, from equation (7) we have $A_k = \frac{1}{C_k}$.
The orthogonality condition (C_1) is always true for $i = 1, \ldots, k-6$.
For $i = k-5$, we have $D_k c^{(1)}(x^{k-3}P^{(1)}_{k-3}) = 0$, which implies that $D_k = 0$.
For $i = k-4$, (8) becomes $c(x^{k-2}P_{k-2}) + E_k c^{(1)}(x^{k-3}P^{(1)}_{k-3}) = 0$,

$$E_k = -\frac{c(x^{k-2}P_{k-2})}{c^{(1)}(x^{k-3}P^{(1)}_{k-3})}.$$

For $i = k-3$, (8) gives

$$\begin{aligned} &B_k c(x^{k-2}P_{k-2}) + F_k c^{(1)}(x^{k-3}P^{(1)}_{k-3}) \\ &= -c(x^{k-1}P_{k-2}) - E_k c^{(1)}(x^{k-2}P^{(1)}_{k-3}). \end{aligned} \tag{9}$$

For $i = k-2$, (8) becomes

$$\begin{aligned} &B_k c(x^{k-1}P_{k-2}) + C_k c(x^{k-2}P_{k-2}) + F_k c^{(1)}(x^{k-2}P^{(1)}_{k-3}) \\ &= -c(x^k P_{k-2}) - E_k c^{(1)}(x^{k-1}P^{(1)}_{k-3}). \end{aligned} \tag{10}$$

For $i = k-1$, we get

$$\begin{aligned} &B_k c(x^k P_{k-2}) + C_k c(x^{k-1}P_{k-2}) + F_k c^{(1)}(x^{k-1}P^{(1)}_{k-3}) \\ &= -c(x^{k+1}P_{k-2}) - E_k c^{(1)}(x^k P^{(1)}_{k-3}). \end{aligned} \tag{11}$$

Let a_{11}, a_{21}, a_{31}, a_{12}, a_{22}, a_{32} and a_{13}, a_{23}, a_{33} be the coefficients of B_k, C_k and G_k in equations (9), (10) and (11) respectively and suppose b_1, b_2 and b_3 are the corresponding right hand side terms of these equations. Then we have
$a_{11} = c(x^{k-2}P_{k-2})$, $a_{12} = 0$, $a_{13} = c^{(1)}(x^{k-3}P^{(1)}_{k-3})$,
$a_{21} = c(x^{k-1}P_{k-2})$, $a_{22} = c(x^{k-2}P_{k-2})$, $a_{23} = c^{(1)}(x^{k-2}P^{(1)}_{k-3})$,
$a_{31} = c(x^k P_{k-2})$, $a_{32} = c(x^{k-1}P_{k-2})$, $a_{33} = c^{(1)}(x^{k-1}P^{(1)}_{k-3})$,
$b_1 = -c(x^{k-1}P_{k-2}) - E_k c^{(1)}(x^{k-2}P^{(1)}_{k-3})$, $b_2 = -c(x^k P_{k-2}) - E_k c^{(1)}(x^{k-1}P^{(1)}_{k-3})$,
$b_3 = -c(x^{k+1}P_{k-2}) - E_k c^{(1)}(x^k P^{(1)}_{k-3})$,

$$a_{11}B_k + 0C_k + a_{13}F_k = b_1, \tag{12}$$

$$a_{21}B_k + a_{22}C_k + a_{23}F_k = b_2, \tag{13}$$

$$a_{31}B_k + a_{32}C_k + a_{33}F_k = b_3. \tag{14}$$

If Δ_k represents the determinant of the coefficients matrix of the above system of equations then we have

$$\Delta_k = a_{11}(a_{22}a_{33} - a_{32}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}).$$

If $\Delta_k \neq 0$, then

$$B_k = \frac{[b_1(a_{22}a_{33} - a_{32}a_{23}) + a_{13}(b_2a_{32} - b_3a_{22})]}{\Delta_k},$$

$$F_k = \frac{b_1 - a_{11}B_k}{a_{13}},$$

$$C_k = \frac{b_2 - a_{21}B_k - a_{23}F_k}{a_{22}}$$

and

$$A_k = \frac{1}{C_k}.$$

Hence,

$$P_k(x) = A_k[(x^2 + B_kx + C_k)P_{k-2}(x) + (E_kx^2 + F_kx)P^{(1)}_{k-3}(x)]. \tag{15}$$

Hence, relation A_{13} exists. It therefore, can be used to implement the Lanczos process.

Remark 1: If $\Delta_k = 0$ then we cannot estimate coefficient B_k, which means relation A_{13} may not exit. However, it may exist, but for one or more of its coefficients which cannot be estimated for numerical reasons. This would be a case of ghost breakdown.

3.1.4. *Relation* A_{14}. Consider the following recurrence relationship

$$P_k(x) = A_k[(x^2+B_kx+C_k)P^{(1)}_{k-2}(x)+(D_kx^3+E_kx^2+F_kx+G_k)P^{(1)}_{k-3}(x)], \tag{16}$$

where P_k, $P^{(1)}_{k-2}$ and $P^{(1)}_{k-3}$ are orthogonal polynomials of degree k, $k-2$ and $k-3$ respectively and A_k, B_k, C_k, D_k, E_k, F_k and G_k are constants to be determined using the normalization condition $P_k(0) = 1$ and the orthogonality conditions (C_1) and (C_6).

Proposition 3.3: Relation of the form A_{14} exists.

Proof: Since $\forall k, P_k(0) = 1$, equation (16) gives

$$1 = A_k[C_kP^{(1)}_{k-2}(0) + G_kP^{(1)}_{k-3}(0)]. \tag{17}$$

Multiplying both sides of equation (16) by x^i and then applying the linear functional c and using condition (C_5), we get

$$\begin{aligned} &c(x^iP_k) = A_k[c^{(1)}(x^{i+1}P_{k-2}^{(1)}) + B_kc^{(1)}(x^iP_{k-2}^{(1)}) + C_kc(x^iP_{k-2}^{(1)}) \\ &+D_kc^{(1)}(x^{i+2}P_{k-3}^{(1)}) + E_kc^{(1)}(x^{i+1}P_{k-3}^{(1)}) + F_kc^{(1)}(x^iP_{k-3}^{(1)}) + G_kc(x^iP_{k-3}^{(1)})]. \end{aligned} \tag{18}$$

For $i = 0$, equation (18) becomes

$$C_kc(P_{k-2}^{(1)}) + G_kc(P_{k-3}^{(1)}) = 0. \tag{19}$$

The orthogonality condition for (18) is always true for $i = 1, \ldots, k-6$.
For $i = k-5$, we get $D_kc^{(1)}(x^{k-3}P_{k-3}^{(1)}) = 0$. Since $c^{(1)}(x^{k-3}P_{k-3}^{(1)}) \neq 0$, $D_k = 0$.
For $i = k-4$, $E_kc^{(1)}(x^{k-3}P_{k-3}^{(1)}) = 0$. Since $c^{(1)}(x^{k-3}P_{k-3}^{(1)}) \neq 0$, we have $E_k = 0$.
For $i = k-3$,

$$F_kc^{(1)}(x^{k-3}P_{k-3}^{(1)}) + G_kc(x^{k-3}P_{k-3}^{(1)}) = -c^{(1)}(x^{k-2}P_{k-2}^{(1)}). \tag{20}$$

For $i = k-2$,

$$\begin{aligned} &B_kc^{(1)}(x^{k-2}P_{k-2}^{(1)}) + C_kc(x^{k-2}P_{k-2}^{(1)}) + F_kc^{(1)}(x^{k-2}P_{k-3}^{(1)}) \\ &+G_kc(x^{k-2}P_{k-3}^{(1)}) = -c^{(1)}(x^{k-1}P_{k-2}^{(1)}). \end{aligned} \tag{21}$$

For $i = k-1$,

$$\begin{aligned} &B_kc^{(1)}(x^{k-1}P_{k-2}^{(1)}) + C_kc(x^{k-1}P_{k-2}^{(1)}) + F_kc^{(1)}(x^{k-1}P_{k-3}^{(1)}) \\ &+G_kc(x^{k-1}P_{k-3}^{(1)}) = -c^{(1)}(x^kP_{k-2}^{(1)}). \end{aligned} \tag{22}$$

The values of A_k, B_k, C_k, F_k and G_k can be obtained by solving equations (17), (19), (20), (21) and (22). Hence

$$P_k(x) = A_k[(x^2 + B_kx + C_k)P_{k-2}^{(1)}(x) + (F_kx + G_k)P_{k-3}^{(1)}(x)].$$

Now multiplying both sides of the above relation by $\mathbf{r}_0$, replacing x by A and using the relations $\mathbf{r}_k = P_k(A)\mathbf{r}_0$ and $\mathbf{z}_k = P_k^{(1)}(A)\mathbf{r}_0$, we get

$$\mathbf{r}_k = A_k[(A^2 + B_kA + C_kI)\mathbf{z}_{k-2} + (F_kA + G_kI)\mathbf{z}_{k-3}]. \tag{23}$$

Using $\mathbf{r}_k = \mathbf{b} - A\mathbf{x}_k$, we get from the last equation

$$A\mathbf{x}_k = \mathbf{b} - A_k[(A^2 + B_kA + C_kI)\mathbf{z}_{k-2} + (F_kA + G_kI)\mathbf{z}_{k-3}]. \tag{24}$$

From this relation it is clear that we cannot find $\mathbf{x}_k$ from $\mathbf{r}_k$ without inverting A. Hence, this recurrence relation exist as stipulated by Proposition 3.3. But, it is not desirable to implement a Lanczos-type algorithm.

For recurrence relations A_{15}, A_{16}, A_{17}, A_{18} and A_{19} and their corresponding coefficients, consult [16].

3.2. **Relations** B_j**.** Now we consider relations of the type B_j which have not been considered before, [1, 2], i.e B_j with $j > 10$. These relations, when they exist will be used in combination with relations A_i to derive further Lanczos-type algorithms as explained in [16].

3.2.1. *Relation* B_{11}. Consider the following recurrence relationship

$$P_k^{(1)}(x) = (A_k x^3 + B_k x^2 + C_k x + D_k)P_{k-3}(x) + (E_k x + F_k)P_{k-1}(x), \quad (25)$$

where $P_k^{(1)}(x)$, $P_{k-1}(x)$ and $P_{k-3}(x)$ are orthogonal polynomials of degree k, $k-1$ and $k-3$ respectively.
Proposition 3.4: Relation of the form B_{11} does not exist.
Proof: Let x^i be a polynomial of exact degree i then

$$\forall i = 0, \ldots, k-4,\ c(x^i P_{k-3}) = 0. \longrightarrow (C_{10})$$

Multiply both sides of equation (25) by x^i and applying $c^{(1)}$ and also using condition (C_5) where necessary, we get

$$\begin{aligned} c^{(1)}(x^i P_k^{(1)}) &= A_k c(x^{i+4}P_{k-3}) + B_k c(x^{i+3}P_{k-3}) + C_k c(x^{i+2}P_{k-3}) \\ &+ D_k c(x^{i+1}P_{k-3}) + E_k c(x^{i+2}P_{k-1}) + F_k c(x^{i+1}P_{k-1}). \end{aligned} \quad (26)$$

The relation (26) is always true for $i = 0, \ldots, k-8$.
For $i = k-7$, we have $A_k c(x^{k-3}P_{k-3}) = 0$. Which implies $A_k = 0$, because $c(x^{k-3}P_{k-3}) \neq 0$.

Similarly for $i = k-6$, $i = k-5$, $i = k-4$ and $i = k-3$, we get respectively $B_k = 0$, $C_k = 0$, $D_k = 0$ and $E_k = 0$. But $P_k^{(1)}(x)$ is a monic polynomial of degree k and we see that $A_k = 0$. Therefore $E_k = 1$. If $E_k = 0$ then $P_k^{(1)}(x)$ is no more of degree k as the degree of the $P_k^{(1)}$ depends on A_k and E_k and we know that $A_k = 0$ and if $E_k = 1$ then $c(x^{k-1}P_{k-1}) = 0$ which is also impossible. Similarly for $i = k-2$, we get $F_k = 0$ and if $E_k = 0$, then $P_k^{(1)} = 0$. Hence, relation B_{11} does not exist and, therefore, Proposition 3.4 holds.

Recurrence relation B_{12} has been explored in [16].

3.2.2. *Relation* B_{13}. Consider the following recurrence relationship

$$P_k^{(1)}(x) = (A_k x^3 + B_k x^2 + C_k x + D_k)P_{k-3}^{(1)}(x) + (E_k x^2 + F_k x + G_k)P_{k-2}^{(1)}(x), \quad (27)$$

where $P_k^{(1)}(x)$, $P_{k-2}^{(1)}(x)$ and $P_{k-3}^{(1)}(x)$ are orthogonal polynomials of degree k, $k-2$ and $k-3$ respectively.
Proposition 3.5: Relation of the form B_{13} exists.
Proof: Let x^i be a polynomial of exact degree i then

$$\forall i = 0, \ldots, k-3,\ c^{(1)}(x^i P_{k-2}^{(1)}) = 0. \longrightarrow (C_{11})$$

Multiply both sides of equation (27) by x^i and applying $c^{(1)}$, we get

$$\begin{aligned} &c^{(1)}(x^i P_k^{(1)}) = A_k c^{(1)}(x^{i+3} P_{k-3}^{(1)}) + B_k c^{(1)}(x^{i+2} P_{k-3}^{(1)}) + C_k c^{(1)}(x^{i+1} P_{k-3}^{(1)}) \\ &+D_k c^{(1)}(x^i P_{k-3}^{(1)}) + E_k c^{(1)}(x^{i+2} P_{k-2}^{(1)}) + F_k c^{(1)}(x^{i+1} P_{k-2}^{(1)}) + G_k c^{(1)}(x^i P_{k-2}^{(1)}). \end{aligned} \tag{28}$$

The orthogonality condition is always true for $i = 0, ..., k-7$.
For $i = k-6$, we get $A_k c^{(1)}(x^{k-3} P_{k-3}^{(1)}) = 0$, which implies that $A_k = 0$ as $c^{(1)}(x^{k-3} P_{k-3}^{(1)}) \neq 0$. But $P_k^{(1)}(x)$ is a monic polynomial of degree k. Therefore $E_k = 1$.
For $i = k-5$, we get $B_k c^{(1)}(x^{k-3} P_{k-3}^{(1)}) = 0$. Since $c^{(1)}(x^{k-3} P_{k-3}^{(1)}) \neq 0$, $B_k = 0$.
For $i = k-4$, we have

$$C_k = -\frac{c^{(1)}(x^{k-2} P_{k-2}^{(1)})}{c^{(1)}(x^{k-3} P_{k-3}^{(1)})}.$$

For $i = k-3$, we get

$$D_k c^{(1)}(x^{k-3} P_{k-3}^{(1)}) + F_k c^{(1)}(x^{k-2} P_{k-2}^{(1)}) = -c^{(1)}(x^{k-1} P_{k-2}^{(1)}) - C_k c^{(1)}(x^{k-2} P_{k-3}^{(1)}). \tag{29}$$

For $i = k-2$, (28) becomes

$$\begin{aligned} D_k c^{(1)}(x^{k-2} P_{k-3}^{(1)}) + F_k c^{(1)}(x^{k-1} P_{k-2}^{(1)}) + G_k c^{(1)}(x^{k-2} P_{k-2}^{(1)}) \\ = -c^{(1)}(x^k P_{k-2}^{(1)}) - C_k c^{(1)}(x^{k-1} P_{k-3}^{(1)}). \end{aligned} \tag{30}$$

For $i = k-1$, (28) gives

$$\begin{aligned} D_k c^{(1)}(x^{k-1} P_{k-3}^{(1)}) + F_k c^{(1)}(x^k P_{k-2}^{(1)}) + G_k c^{(1)}(x^{k-1} P_{k-2}^{(1)}) \\ = -c^{(1)}(x^{k+1} P_{k-2}^{(1)}) - C_k c^{(1)}(x^k P_{k-3}^{(1)}). \end{aligned} \tag{31}$$

Let $a'_{11} = c^{(1)}(x^{k-3} P_{k-3}^{(1)})$, using (C_5), $a'_{11} = c(x^{k-2} P_{k-3}^{(1)})$. By the same condition we can write,
$a'_{12} = c^{(1)}(x^{k-2} P_{k-2}^{(1)}) = c(x^{k-1} P_{k-2}^{(1)})$, $a'_{13} = 0$,
$a'_{21} = c^{(1)}(x^{k-2} P_{k-3}^{(1)}) = c(x^{k-1} P_{k-3}^{(1)})$, $a'_{22} = c^{(1)}(x^{k-1} P_{k-2}^{(1)}) = c(x^k P_{k-2}^{(1)})$,
$a'_{23} = c^{(1)}(x^{k-2} P_{k-2}^{(1)}) = a'_{12}$, $a'_{31} = c^{(1)}(x^{k-1} P_{k-3}^{(1)}) = c(x^k P_{k-3}^{(1)})$,
$a'_{32} = c^{(1)}(x^k P_{k-2}^{(1)}) = c(x^{k+1} P_{k-2}^{(1)})$, $a'_{33} = c^{(1)}(x^{k-1} P_{k-2}^{(1)}) = a'_{22}$,
$b'_1 = -c^{(1)}(x^{k-1} P_{k-2}^{(1)}) - C_k c^{(1)}(x^{k-2} P_{k-3}^{(1)}) = -a'_{22} - a'_{21} C_k$,
$b'_2 = -c^{(1)}(x^k P_{k-2}^{(1)}) - C_k c^{(1)}(x^{k-1} P_{k-3}^{(1)}) = -a'_{32} - a'_{31} C_k$,
$b'_3 = -c^{(1)}(x^{k+1} P_{k-2}^{(1)}) - C_k c^{(1)}(x^k P_{k-3}^{(1)})$.
Then equations (29), (30) and (31) become

$$a'_{11} D_k + a'_{12} F_k = b'_1, \tag{32}$$

$$a'_{21} D_k + a'_{22} F_k + a'_{23} G_k = b'_2 \tag{33}$$

and

$$a'_{31}D_k + a'_{32}F_k + a'_{33}G_k = b'_3. \tag{34}$$

If Δ'_k is the determinant of the coefficient matrix of the equations (32), (33) and (34) then

$$\Delta'_k = a'_{11}(a'_{22}a'_{33} - a'_{32}a'_{23}) - a'_{12}(a'_{21}a'_{33} - a'_{31}a'_{23}).$$

If $\Delta'_k \neq 0$, then

$$D_k = \frac{b'_1(a'_{22}a'_{33} - a'_{32}a'_{23}) - a'_{12}(b'_2a'_{33} - b'_3a'_{23})}{\Delta'_k},$$

$$F_k = \frac{b'_1 - a'_{11}D_k}{a'_{12}}$$

and

$$G_k = \frac{b'_2 - a'_{21}D_k - a'_{22}F_k}{a'_{23}}.$$

Hence, relation (27) can be written as

$$P_k^{(1)}(x) = (C_kx + D_k)P_{k-3}^{(1)}(x) + (x^2 + F_kx + G_k)P_{k-2}^{(1)}(x), \tag{35}$$

and, therefore, exists as stipulated in Proposition 3.5.
Remark 2: For the case where $\Delta'_k = 0$, please consult Remark 1 above.

For recurrence relations B_{14}, B_{15} and B_{16} and their corresponding coefficients, see [16].

4. Conclusion

In this paper, we looked in a systematic way at new recurrence relations between FOPs which have not been considered before. In particular, we have shown that relations A_{11}, A_{17}, B_{11} and B_{12} do not exist; relations A_{14}, A_{15}, A_{18} and B_{14} exist but are not suitable for implementing new Lanczostype algorithms; and relations A_{12}, A_{13}, A_{16}, A_{19}, B_{13}, B_{15} and B_{16} exist and can be used for the implementation of new Lanczos-type algorithms, [16]. Relation A_{12} is self-sufficient and leads to a new Lanczos-type algorithm on its own, [17], while the rest of the relations can lead to Lanczos-type algorithms when combined in A_i/B_j fashion. Possible combinations, which are studied in [16], are:

$$A_{13}/B_{13},\ A_{13}/B_{15},\ A_{13}/B_{16},$$
$$A_{16}/B_{13},\ A_{16}/B_{15},\ A_{16}/B_{16},$$
$$A_{19}/B_{13},\ A_{19}/B_{15},\ A_{19}/B_{16}.$$

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