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EXACT WIENER INDICES OF THE STRONG PRODUCT OF GRAPHS

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Abstract. The *Wiener index,* denoted by $W(G)$, of a connected graph G is the sum of all pairwise distances of vertices of the graph, that is, $W(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} d(u,v)$. In this paper, we obtain the Wiener index of the strong product of a path and a cycle and strong product of two cycles.

Key words: strong product, Wiener index.
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1. Introduction

The *strong product* of graphs G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H) = \{(u,v) : u \in V(G), v \in V(H)\}$, where (u,x) is adjacent to (v,y) whenever (i) $u = v$ and $xy \in E(H)$, or (ii) $uv \in E(G)$ and $x = y$, or (iii) $uv \in E(G)$ and $xy \in E(H)$. Let G and H be graphs with vertex sets $V(G) = \{x_1, x_2, \ldots, x_m\}$ and $V(H) = \{y_1, y_2, \ldots, y_n\}$. Then $V(G \boxtimes H) = V(G) \times V(H)$ and for our convenience, we write $V(G \boxtimes H) = \bigcup_{i=1}^{m} X_i$, where $X_i = \{x_i\} \times V(H)$; we may also write $V(G \boxtimes H) = \bigcup_{j=1}^{n} Y_j$, where $Y_j = V(G) \times \{y_j\}$. We shall denote the vertices of X_i by $\{x_{i,j} \,|1 \leq j \leq n\}$ and the vertices of Y_j by $\{x_{i,j}|1 \leq i \leq m\}$, where $x_{i,j}$ stands for the vertex (x_i, y_j). We shall call $X_i, 1 \leq i \leq m$, the i^{th} *layer* of $G \boxtimes H$ and $Y_j, 1 \leq j \leq n$, the j^{th} *column* of $G \boxtimes H$, see Fig.1. For terms not defined here see [1] or [8].

The *Cartesian product,* $G \,\square\, H$, of graphs G and H has the vertex set $V(G \,\square\, H) = V(G) \times V(H)$ and $(u,x)(v,y)$ is an edge of $G \,\square\, H$ if $u = v$ and $xy \in E(H)$ or, $uv \in E(G)$ and $x = y$. For two simple graphs G and H their *tensor product,* denoted by $G \times H$, has vertex set $V(G) \times V(H)$ in which (g_1, h_1) and (g_2, h_2) are adjacent whenever g_1g_2 is an edge in G and h_1h_2 is an edge in H. Note that if G and H are connected graphs, then $G \times H$ is

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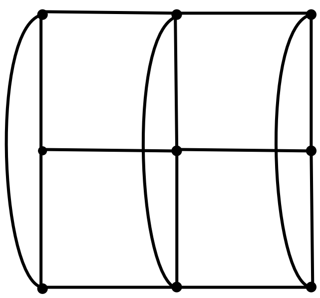
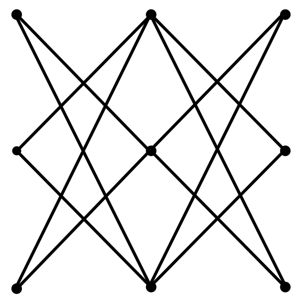
connected only if at least one of the graph is nonbipartite. One can observe that $G \boxtimes H = (G \times H) \oplus (G \square H)$, where $\oplus$ denotes the edge disjoint union of graphs, see Fig.1.



$C_3 \square P_3$



$C_3 \times P_3$

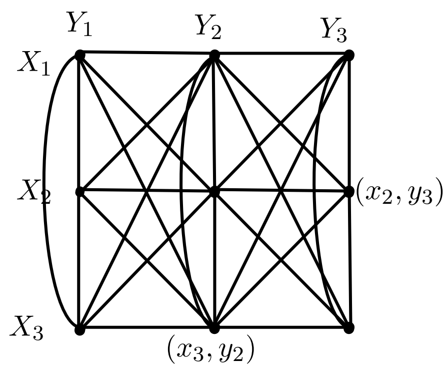


Fig.1 .$C_3 \boxtimes P_3$

Graph theory successfully provides the chemists with a variety of very useful tools, namely, different topological indices. A *topological index* of a graph is a parameter related to the graph; it does not depend on labeling or pictorial representation of the graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds [6]. Several types of such indices exist, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index.

The *Wiener index,* $W(G)$, of a connected graph G, is defined by $W(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} d_G(u,v)$, where $d_G(u,v)$ denotes the distance between the distinct vertices u and v in graph G. The Wiener index has important applications in chemistry. The graphical invariant $W(G)$ has been studied by many researchers under different names such as distance, transmissions, total status and sum of all distances; see [4],[5], [10] and [11]. The chemist Harold Wiener was the first to point out in 1947 that $W(G)$ is well correlated with certain physico-chemical properties of the organic compound.

Besides applications in chemistry, there are many situations in communication, facility location, cryptology, etc., that are effectively modeled by a connected graph G satisfying certain restrictions. Because of cost restraints one is often interested in finding a spanning tree of G that is optimal with respect to one or more properties. Average distance between vertices is frequently one of these properties. Finding a spanning tree T of G that has minimum Wiener index is proved to be important see, [9]. For recent results on Wiener index, see [2, 15, 12, 14, 13]. Wiener indices of Cartesian product and tensor product of graphs are given in [7, 13]. In this paper, we obtain the exact Wiener index of $P_r \boxtimes C_s$ and $C_r \boxtimes C_s$.

2. Wiener index of strong product of paths and cycles

We quote the following lemma which is not difficult to prove.

Lemma 1. [3]

(1) $W(P_n) = \binom{n+1}{3},\ n \geq 2$.
(2) $W(C_{2n}) = n^3$.
(3) $W(C_{2n+1}) = \frac{n(n+1)(2n+1)}{2}$.

Proof of Lemma 1 is given in [3]. The following lemma is very useful to find a shortest path from any pair of vertices of $G \boxtimes H$.

Lemma 2. [2] *Let G and H be two graphs and $(u,x),(v,y) \in V(G \boxtimes H)$. Then $d_{G\boxtimes H}((u,x),(v,y)) = max\{d_G(u,v), d_H(x,y)\}$.*

Theorem 3. *The Wiener index of $P_{2m+1} \boxtimes C_{2n+1}$ is* $\frac{(2m+1)(2n+1)}{6}\big\{2m(2m^2+3m+1) + 3n(n+1)(2m+1)\big\}$.

Proof. Let S_{kj} denote the sum of the distances from $x_{k,j}$ to all other vertices of $G = P_{2m+1} \boxtimes C_{2n+1}$, that is, $\sum\limits_{v \neq x_{k,j} \in V(G)} d_G(x_{k,j}, v)$. Since there is an automorphism of G which maps $x_{k,i}$ to $x_{k,j},\ i \neq j,\ S_{kj} = S_{ki}$. Hence, instead of computing S_{rs} for every pair r and s, it is enough to compute S_{k1} for $k = 1, 2, \ldots, 2m+1$, and then multiply each S_{k1} with number of columns of G to compute $\sum\limits_{u,v \in V(G)} d_G(u,v)$.

For the computation of S_{k1}, for a fixed k, we partition the layers of G into three sets $\{X_1, X_2, \ldots, X_{k-1}\},\ \{X_k\}$ and $\{X_{k+1}, X_{k+2}, \ldots, X_{2m+1}\}$ (Note that when $k = 1$ or $2m+1$, the partition consists of only two sets, namely, $\{X_1\},\ \{X_2, X_3, \ldots, X_{2m+1}\}$ and $\{X_1, X_2, \ldots, X_{2m}\},\ \{X_{2m+1}\}$, respectively) and we find the distances from $x_{k,1}$ to all the vertices in the layers in the partition separately, that is,

$$\sum_{v\in V(G)} d_G(x_{k,1},v) = \sum_{\substack{v\in X_i\\ 1\leq i\leq k-1}} d_G(x_{k,1},v) + \sum_{v\in X_k} d_G(x_{k,1},v) + \sum_{\substack{v\in X_i\\ k+1\leq i\leq 2m+1}} d_G(x_{k,1},v) \quad (1)$$

We divide the proof into three parts **(A)**, **(B)** and **(C)**. In **(A)**, we find the distances from $x_{k,1}$ to all the vertices of $\bigcup_{i=1}^{k-1} X_i$; in **(B)**, we find the distances from $x_{k,1}$ to all the vertices of X_k; in **(C)**, we find the distances from $x_{k,1}$ to all the vertices of $\bigcup_{i=k+1}^{2m+1} X_i$.

(A): Here we find the sum of the distances from $x_{k,1}$ to all the vertices of $\bigcup_{i=1}^{k-1} X_i$. For this, first we compute $\sum_{v\in X_i} d_G(x_{k,1},v)$, for a single layer X_i, $1 \leq i \leq k-1$.

$$\begin{aligned}\sum_{v\in X_i} d_G(x_{k,1},v) &= (k-i) + 2\{\underbrace{(k-i)+\ldots+(k-i)}_{(k-i)\ times} + (k-i+1) + (k-i+2) \\ &\quad + \ldots + (n-1) + n\} \qquad (2)\end{aligned}$$

Explanations for the terms appearing in (2) are as follows:
For $1 \leq j \leq i$, $d_G(x_{k,1}, x_{i,j}) = k-i$ and the respective shortest paths are similar to the one shown in Fig.2. The distances from $x_{k,1}$ to the vertices $x_{i,i+1}, x_{i,i+2}, \ldots, x_{i,n+1}$ are $k-i+1, k-i+2, \ldots, n$, respectively, and the corresponding shortest paths are similar to the one shown in Fig.3. The multiplication factor 2 appears in the sum of (2), except for one term, because $d_G(x_{k,1}, x_{i,j}) = d_G(x_{k,1}, x_{i,2n-j+3})$, $2 \leq j \leq n+1$. Summing the terms of (2), that is, the summing the distances from $x_{k,1}$ to all the vertices of X_i gives,

$$\sum_{v\in X_i} d_G(x_{k,1},v) = (k-i)^2 + n(n+1) \quad (3)$$

Hence,

$$\begin{aligned}\sum_{\substack{v\in X_i\\ 1\leq i\leq k-1}} d_G(x_{k,1},v) &= \sum_{1\leq i\leq k-1} \{(k-i)^2 + n(n+1)\} \\ &= \frac{k(k-1)(2k-1)}{6} + n(n+1)(k-1) \qquad (4)\end{aligned}$$

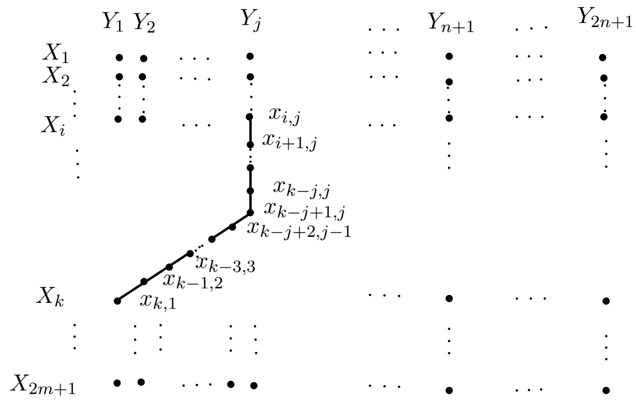


Fig.2 Vertices of $P_{2m+1} \boxtimes C_{2n+1}$

(B): Next we find the sum of the distances from $u = x_{k,1}$ to all other vertices of the single layer X_k.

$$\sum_{v=x_{k,j}\in X_k} d_G(x_{k,1}, v) = 2\{1+2+\ldots+n\} \tag{5}$$

Explanations for the terms appearing in (5) are described below: $d_G(x_{k,1}, x_{k,j}) = j-1,\ j = 2, 3, 4, \ldots, n+1$. The multiplication factor 2 appears in the Equation (5) because $d_G(x_{k,1}, x_{k,j}) = d_G(x_{k,1}, x_{k,2n-j+3})$, for $2 \le j \le n+1$. Hence

$$\begin{aligned}\sum_{v=x_{k,j}\in X_k} d_G(x_{k,1}, v) &= 2(1+2+\ldots+n)\\ &= n(n+1)\end{aligned} \tag{6}$$

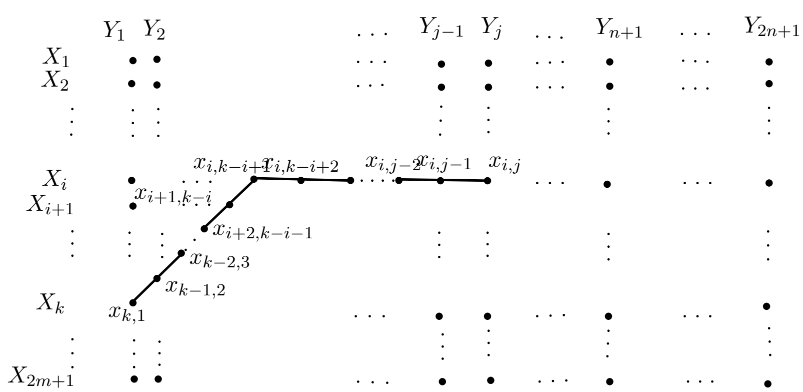


Fig.3 Vertices of $P_{2m+1} \boxtimes C_{2n+1}$

(C): Finally, we find the sum of the distances from $u = x_{k,1}$ to all the vertices of $\bigcup_{i=k+1}^{2m+1} X_i$. For this, it is enough to replace $k-i$ by $i-k$, in the argument given in **(A)**. Hence

$$\sum_{v\in X_i} d_G(x_{k,1}, v) = (i-k)^2 + n(n+1) \tag{7}$$

Hence,

$$\begin{aligned}\sum_{\substack{v\in X_i\\ k+1\le i\le 2m+1}} d_G(x_{k,1}, v) &= \sum_{k+1\le i\le 2m+1} \big\{(i-k)^2 + n(n+1)\big\}\\ &= \Big\{\frac{(2m-k+1)(2m-k+2)(4m-2k+3)}{6}\\ &\quad +n(n+1)(2m-k+1)\Big\} \qquad (8)\end{aligned}$$

From (4), (6) and (8), we have

$$\begin{aligned}\sum_{v\in V(G)} d_G(x_{k,1}, v) &= \frac{k(k-1)(2k-1)}{6} + \frac{(2m-k+1)(2m-k+2)(4m-2k+3)}{6}\\ &\quad +n(n+1)(k-1) + n(n+1) + n(n+1)(2m-k+1) \qquad (9)\end{aligned}$$

Eqn (9) gives the sum of the distances from $x_{k,1}$ to all other vertices of G. Summing the equation (9) over $k = 1, 2, \ldots, 2m+1$, we get

$$\begin{aligned}\sum_{k=1}^{2m+1}\sum_{v\in V(G)} d_G(x_{k,1}, v) &= \sum_{k=1}^{2m+1}\Big\{\frac{k(k-1)(2k-1)}{6} + n(n+1)(k-1) + n(n+1) +\\ &\quad \frac{(2m-k+1)(2m-k+2)(4m-2k+3)}{6} + n(n+1)(2m-k+1)\Big\}\\ &= \frac{m}{3}(m+1)(2m+1)^2 + n(n+1)(2m+1)^2\\ &\quad +\frac{m}{3}(2m+1)(2m^2+3m+1)\\ &= \frac{2m(2m+1)(2m^2+3m+1)}{3} + n(n+1)(2m+1)^2 \qquad (10)\end{aligned}$$

As there is an automorphism of G which maps $x_{k,1}$ to $x_{k,j}$ the sum of the distances from $x_{k,1}$ to all the vertices of G is same as the sum of the distances from $x_{k,j},\ 2 \le j \le 2n+1$, to all the vertices of G; consequently we have

$$\begin{aligned}\sum_{u,v\in V(G)} d_G(u, v) &= (2n+1)\Big(\sum_{k=1}^{2m+1}\sum_{v\in V(G)} d_G(x_{k,1}, v)\Big)\\ &= (2n+1)\Big\{\frac{2m(2m+1)(2m^2+3m+1)}{3} + n(n+1)(2m+1)^2\Big\},\\ &\quad \text{using equation (10)}\\ &= \frac{(2m+1)(2n+1)}{3}\big\{2m(2m^2+3m+1) + 3n(n+1)(2m+1)\big\} \qquad (11)\end{aligned}$$

Hence,

$$\begin{aligned} W(P_{2m+1} \boxtimes C_{2n+1}) &= \frac{1}{2}\Big(\sum_{u,v \in V(G)} d_G(u,v)\Big) \\ &= \frac{(2m+1)(2n+1)}{6}\big\{2m(2m^2+3m+1)+3n(n+1)(2m+1)\big\}, \\ & \quad \text{using equation (11).} \end{aligned}$$

□

In the following theorem, we compute the Wiener index of the strong product of an even length path and an odd cycle.

Theorem 4. *The Wiener index of* $P_{2m} \boxtimes C_{2n+1}$ *is* $\frac{m^2(2n+1)}{3}\big\{(4m^2-1)+6n(n+1)\big\}$.

Proof. As in the proof of the previous theorem, we consider **(A)**,**(B)** and **(C)**. **(A)** and **(B)** are the same as in Theorem 2.3 and hence, for **(A)** and **(B)** we consider the sum as in the proof of the Theorem 2.3. Also, here in **(C)**, the summation varies from $k+1$ to $2m$ instead of $k+1$ to $2m+1$ in eqn (8). Hence we consider the sum corresponding to **(C)** of the Theorem 2.3. Therefore,

$$\sum_{v \in X_i} d_G(x_{k,1}, v) = (i-k)^2 + n(n+1) \tag{12}$$

Hence,

$$\begin{aligned} \sum_{\substack{v \in X_i \\ k+1 \le i \le 2m}} d_G(x_{k,1}, v) &= \sum_{k+1 \le i \le 2m} \big\{(i-k)^2 + n(n+1)\big\} \\ &= \frac{(2m-k)(2m-k+1)(4m-2k+1)}{6} + n(n+1)(2m-k) \end{aligned} \tag{13}$$

By the observation made at the beginning of the proof, we can use (4) and (6) here and hence using (4), (6) and (13), we have

$$\begin{aligned} \sum_{v \in V(G)} d_G(x_{k,1}, v) &= \frac{k(k-1)(2k-1)}{6} + n(n+1)(k-1) + n(n+1) \\ &\quad + \frac{(2m-k)(2m-k+1)(4m-2k+1)}{6} + n(n+1)(2m-k) \end{aligned} \tag{14}$$

Summing the equation (14) over $k = 1, 2, \ldots, 2m$, we get

$$\begin{aligned} \sum_{k=1}^{2m} \sum_{v \in V(G)} d_G(x_{k,1}, v) &= \sum_{k=1}^{2m} \Big\{ \frac{k(k-1)(2k-1)}{6} + \frac{(2m-k)(2m-k+1)(4m-2k+1)}{6} \Big\} \\ &\quad + \sum_{k=1}^{2m} \big\{ n(n+1)(k-1) + n(n+1) + n(n+1)(2m-k) \big\} \\ &= \frac{m^2}{3}(2m+1)(2m-1) + \frac{m^2}{3}(2m+1)(2m-1) + 4m^2 n(n+1) \\ &= \frac{2m^2}{3}\big\{(2m-1)(2m+1) + 6n(n+1)\big\} \end{aligned} \tag{15}$$

As there is an automorphism of G which maps $x_{k,1}$ to $x_{k,j}$ the sum of the distances from $x_{k,1}$ to all the vertices of G is same as the sum of the distances from $x_{k,j},\ 2 \leq j \leq 2n+1$, to all the vertices of G and hence we have

$$\begin{aligned}\sum_{u,v\in V(G)} d_G(u,v) &= (2n+1)\big(\sum_{k=1}^{2m}\sum_{v\in V(G)} d_G(x_{k,1},v)\big)\\ &= (2n+1)\big\{\frac{2m^2}{3}\big((2m-1)(2m+1)+6n(n+1)\big)\big\},\\ &\quad \text{using equation (15)}\\ &= \frac{2m^2(2n+1)}{3}\big\{(2m-1)(2m+1)+6n(n+1)\big\} \qquad (16)\end{aligned}$$

Hence,

$$\begin{aligned}W(P_{2m}\boxtimes C_{2n+1}) &= \frac{1}{2}\big(\sum_{u,v\in V(G)} d_G(u,v)\big)\\ &= \frac{m^2(2n+1)}{3}\big\{(4m^2-1)+6n(n+1)\big\},\ \text{using equation (15).}\end{aligned}$$

□

Theorem 5. *The Wiener index of* $P_{2m}\boxtimes C_{2n}$ *is* $\frac{2m^2n}{3}\big\{4m^2+6n^2-1\big\}$.

Proof. As in the proof of the previous theorem, we divide the proof of the theorem into three parts **(A)**, **(B)** and **(C)**. In **(A)**, we find the distances from $x_{k,1}$ to all the vertices of $\bigcup_{i=1}^{k-1} X_i$; in **(B)**, we find the distances from $x_{k,1}$ to all the vertices of X_k; in **(C)**, we find the distances from $x_{k,1}$ to all the vertices of $\bigcup_{i=k+1}^{2m} X_i$.

(A): Initially we find the sum of the distances from $x_{k,1}$ to all the vertices of $\bigcup_{i=1}^{k-1} X_i$. For this, first we compute $\sum_{v\in X_i} d_G(x_{k,1},v)$, for a single layer $X_i,\ 1 \leq i \leq k-1$.

$$\begin{aligned}\sum_{v\in X_i} d_G(x_{k,1},v) &= (k-i)+2\big\{\underbrace{(k-i)+\ldots+(k-i)}_{(k-i)\ times}+(k-i+1)+(k-i+2)\\ &\quad +\ldots+(n-1)\big\}+n \qquad (17)\end{aligned}$$

Explanations for the terms appearing in (17) are as follows:
$d_G(x_{k,1},x_{i,j}) = k-i,\ j = 1,2,3,\ldots,i$ and the respective shortest paths are similar to the one shown in Fig.2. The distances from $x_{k,1}$ to the vertices $x_{i,i+1},x_{i,i+2},\ldots,x_{i,n+1}$ are $k-i+1,k-i+2,\ldots,n$, respectively, and the corresponding shortest paths are similar to the one shown in Fig.3. The multiplication factor 2 appears in the sum (17), expect for two terms, because

$d_G(x_{k,1}, x_{i,j}) = d_G(x_{k,1}, x_{i,2n-j+2}),\ 2 \le j \le n$. Summing the terms of (17), that is, the summing the distances from $x_{k,1}$ to all the vertices of X_i gives,

$$\sum_{v \in X_i} d_G(x_{k,1}, v) = (k-i)^2 + n^2 \tag{18}$$

Hence,

$$\begin{aligned} \sum_{\substack{v \in X_i \\ 1 \le i \le k-1}} d_G(x_{k,1}, v) &= \sum_{1 \le i \le k-1} \{(k-i)^2 + n^2\} \\ &= \frac{k(k-1)(2k-1)}{6} + n^2(k-1) \end{aligned} \tag{19}$$

(B): Next we find the sum of the distances from $u = x_{k,1}$ to all other vertices of X_k.

$$\sum_{v = x_{k,j} \in X_k} d_G(x_{k,1}, v) = 2\{1 + 2 + \ldots + (n-1)\} + n \tag{20}$$

Explanations for the terms appearing in (20) are described below:
$d_G(x_{k,1}, x_{k,j}) = j - 1,\ j = 2, 3, \ldots, n$. The multiplication factor 2 appears in the Equation (20) because $d_G(x_{k,1}, x_{k,j}) = d_G(x_{k,1}, x_{k,2n-j+2}),\ 2 \le j \le n$. Further $d_G(x_{k,1}, x_{k,n+1}) = n$. Hence

$$\begin{aligned} \sum_{v = x_{k,j} \in X_k} d_G(x_{k,1}, v) &= 2\big(1 + 2 + \ldots + (n-1)\big) + n \\ &= n^2 \end{aligned} \tag{21}$$

(C): Finally, we find the sum of the distances from $u = x_{k,1}$ to all the vertices of $\bigcup_{i=k+1}^{2m} X_i$. For this, it is enough to replace $k - i$ by $i - k$, in the argument given in **(A)**. Hence

$$\sum_{v \in X_i} d_G(x_{k,1}, v) = (i-k)^2 + n^2 \tag{22}$$

Hence,

$$\begin{aligned} \sum_{\substack{v \in X_i \\ k+1 \le i \le 2m}} d_G(x_{k,1}, v) &= \sum_{k+1 \le i \le 2m} \big\{(i-k)^2 + n^2\big\} \\ &= \frac{(2m-k)(2m-k+1)(4m-2k+1)}{6} + n^2(2m-k) \end{aligned} \tag{23}$$

From (19), (21) and (23), we get,

$$\begin{aligned} \sum_{v \in V(G)} d_G(x_{k,1}, v) &= \frac{k(k-1)(2k-1)}{6} + n^2(k-1) + n^2 + n^2(2m-k) \\ &\quad + \frac{(2m-k)(2m-k+1)(4m-2k+1)}{6} \end{aligned} \tag{24}$$

Summing the equation (24) over $k = 1, 2, \ldots, 2m$, we have

$$\begin{aligned}\sum_{k=1}^{2m}\sum_{v\in V(G)} d_G(x_{k,1}, v) &= \sum_{k=1}^{2m}\Big\{\frac{k(k-1)(2k-1)}{6} + n^2(k-1) + n^2 + n^2(2m-k) \\ &\quad + \frac{(2m-k)(2m-k+1)(4m-2k+1)}{6}\Big\} \\ &= \frac{m^2}{3}(2m+1)(2m-1) + 4m^2n^2 + \frac{m^2}{3}(2m-1)(2m+1) \\ &= \frac{2m^2(2m-1)(2m+1)}{3} + 4m^2n^2 \qquad (25)\end{aligned}$$

As there is an automorphism of G which maps $x_{k,1}$ to $x_{k,j}$ the sum of the distances from $x_{k,1}$ to all the vertices of G is same as the sum of the distances from $x_{k,j},\ 2 \le j \le 2n$, to all the vertices of G; consequently, we have

$$\begin{aligned}\sum_{u,v\in V(G)} d_G(u, v) &= 2n\Big(\sum_{k=1}^{2m}\sum_{v\in V(G)} d_G(x_{k,1}, v)\Big) \\ &= 2n\Big\{\frac{2m^2(2m-1)(2m+1)}{3} + 4m^2n^2\Big\}, \text{by equation } (25) \\ &= \frac{4m^2n}{3}\big\{4m^2 + 6n^2 - 1\big\} \qquad (26)\end{aligned}$$

Hence,

$$\begin{aligned}W(P_{2m} \boxtimes C_{2n}) &= \frac{1}{2}\sum_{u,v\in V(G)} d_G(u, v) \\ &= \frac{2m^2n}{3}\big\{4m^2 + 6n^2 - 1\big\}, \text{ by equation (26).}\end{aligned}$$

□

In the next theorem, we compute the Wiener index of the strong product of an odd length path and an even cycle.

Theorem 6. *The Wiener index of* $P_{2m+1} \boxtimes C_{2n}$ *is* $\frac{n(2m+1)}{6}\big\{2m(2m^2 + 3m + 1) + 3n^2(2m+1)\big\}$.

Proof. As in the proof of the previous theorem, we consider **(A)**,**(B)** and **(C)**. **(A)** and **(B)** are the same as in Theorem 2.5 and hence, for **(A)** and **(B)** we consider the sum as in the proof of Theorem 2.5. Also, here in **(C)**, the summation varies from $k+1$ to $2m+1$ instead of $k+1$ to $2m$ in eqn (23). Hence we consider the sum corresponding to **(C)** of the above theorem. Therefore,

$$\sum_{v\in X_i} d_G(x_{k,1}, v) = (i-k)^2 + n^2 \qquad (27)$$

Hence,

$$\sum_{\substack{v\in X_i\\ k+1\le i\le 2m+1}} d_G(x_{k,1},v) = \sum_{k+1\le i\le 2m+1}\big\{(i-k)^2+n(n+1)\big\}$$
$$= \frac{(2m-k+1)(2m-k+2)(4m-2k+3)}{6} + n^2(2m-k+1) \quad (28)$$

By the observation made at the beginning of the proof, we can use (19), (21) here and hence using (19), (21) and (28), we have

$$\sum_{v\in V(G)} d_G(x_{k,1},v) = \frac{k(k-1)(2k-1)}{6} + n^2(k-1) + n^2 + n^2(2m-k+1) + \frac{(2m-k+1)(2m-k+2)(4m-2k+3)}{6} \quad (29)$$

Summing the equation (29) over $k=1,2,\ldots,2m+1$, we get

$$\sum_{k=1}^{2m+1}\sum_{v\in V(G)} d_G(x_{k,1},v) = \sum_{k=1}^{2m+1}\{\frac{k(k-1)(2k-1)}{6} + \frac{(2m-k)(2m-k+1)(4m-2k+1)}{6}\} + \sum_{k=1}^{2m+1}\big\{n^2(k-1)+n^2+n^2(2m-k+1)\big\}$$
$$= \frac{m^2}{3}(2m+1)(2m-1) + \frac{2m(2m+1)(2m^2+3m+1)}{3} + n^2(2m+1)^2$$
$$= \frac{2m(2m+1)(2m^2+3m+1)}{3} + n^2(2m+1)^2 \quad (30)$$

As there is an automorphism of G which maps $x_{k,1}$ to $x_{k,j}$ the sum of the distances from $x_{k,1}$ to all the vertices of G is same as the sum of the distances from $x_{k,j}$, $2\le j\le 2n$, to all the vertices of G and hence we have

$$\sum_{u,v\in V(G)} d_G(u,v) = 2n\big(\sum_{k=1}^{2m+1}\sum_{v\in V(G)} d_G(x_{k,1},v)\big)$$
$$= 2n\big\{\frac{2m(2m+1)(2m^2+3m+1)}{3} + n^2(2m+1)^2\big\} \quad (31)$$

Hence,

$$W(P_{2m+1}\boxtimes C_{2n}) = \frac{1}{2}\big(\sum_{u,v\in V(G)} d_G(u,v)\big)$$
$$= \frac{n(2m+1)}{6}\big\{2m(2m^2+3m+1)+3n^2(2m+1)\big\}.$$

□

3. The Weiner Index of strong product of cycles

In this section, we find the Wiener index of strong product of two cycles. A graph G is *vertex transitive* if for every pair $u, v \in V(G)$ there is an automorphism that maps u to v.

Theorem 7. *The Wiener index of the graph $C_{2m+1} \boxtimes C_{2n+1}$ is* $\frac{(2m+1)^2(2n+1)}{6}$ $\big\{m(m+1) + 3n(n+1)\big\}$.

Proof. As $G = C_{2m+1} \boxtimes C_{2n+1}$ is vertex transitive, it is enough to find the distances from $x_{1,1}$ to all other vertices of G. We compute the sum of the distances from $u = x_{1,1}$ to all other vertices of G.

$$\sum_{u,v \in V(G)} d_G(u,v) = \sum_{v \in X_1} d_G(u,v) + 2\Big(\sum_{v \in X_i, 2 \le i \le m+1} d_G(u,v)\Big), \tag{32}$$

where X_i denotes the vertices of the i^{th} layer of G, the multiplication factor 2 in one of the terms in (1) appears as the distances from u to all the vertices of the layer X_i is same as the distances from $u = x_{1,1}$ to all the vertices of $X_{2m-i+3},\ 2 \le i \le m+1$; this is true because the length of a shortest path that descends to a vertex from $u = x_{1,1}$ to a vertex of $X_i,\ 2 \le i \le m+1$, is same as the length of a shortest path that goes from $x_{1,1}$ to a vertex of X_{2m+1} and then ascending to a vertex in X_{2m-i+3} are the same. We shall calculate the sum of the terms of (32), separately.

(A): First we calculate the sum of the distances from $u = x_{1,1}$ to all other vertices of X_1.

$$\sum_{v \in X_1} d_G(u,v) = 2\big(1 + 2 + \ldots + n\big) \tag{33}$$

since $d_G(u, x_{1,j}) = j-1$, for $j = 2, 3, \ldots, n+1$, since the path traverses through j rows with its origin and terminus at X_1.

$$\sum_{v \in X_1} d_G(u,v) = n(n+1) \tag{34}$$

(B): Next we shall calculate the sum of the distances from $u = x_{1,1}$ to the vertices of $X_i,\ 2 \le i \le m+1$. For this, we compute $\sum\limits_{v \in X_i} d_G(u,v)$, for the single layer X_i.

$$\sum_{v \in X_i} d_G(u,v) = (i-1) + 2\big\{\underbrace{(i-1) + \cdots + (i-1)}_{(i-1)\ times} + i + (i+1) + (i+2) + \ldots + n\big\} \tag{35}$$

Explanations for the terms in (35) are as follows:
$d_G(u, x_{i,j}) = i-1,\ j = 1, 2, 3, \ldots, i$, see Fig.4. The distances from u to the vertices $x_{i,i+1}, x_{i,i+2}, x_{i,i+3}, \ldots, x_{i,n+1}$ are $i, i+1, i+2, \ldots, n$, respectively, see Fig.5. The multiplication factor 2 appears in all the terms except the first

term of the sum (35) because $d_G(u, x_{i,j}) = d_G(u, x_{i,2n-j+3})$, $2 \leq j \leq n+1$, due to the "symmetry" of the graph.

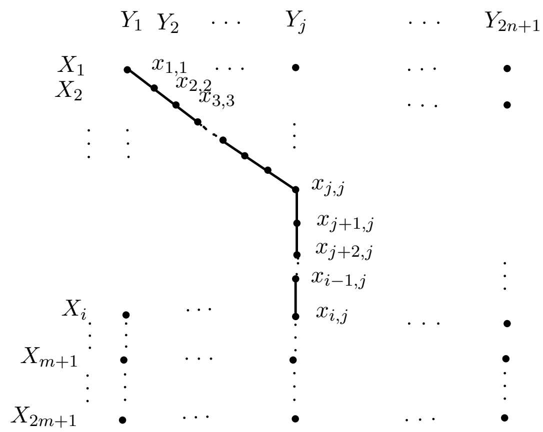


Fig.4 Vertices of $C_r \boxtimes C_s$

The summation of the terms of (35) gives

$$\sum_{v \in X_i} d_G(u,v) = (i-1)^2 + n(n+1) \tag{36}$$

Hence,

$$\begin{aligned} \sum_{2 \leq i \leq m+1} d_G(u,v) &= \sum_{2 \leq i \leq m+1} \left\{ (i-1)^2 + n(n+1) \right\} \\ &= \frac{m(m+1)(2m+1)}{6} + mn(n+1) \end{aligned} \tag{37}$$

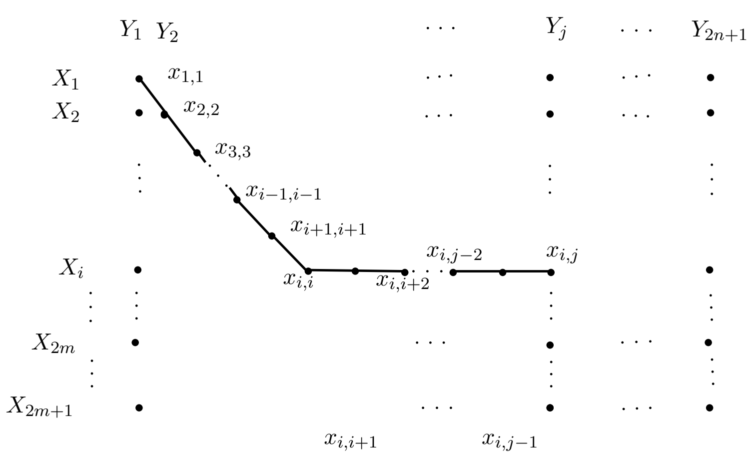


Fig.5 Vertices of $C_r \boxtimes C_s$

Substituting the values obtained in (34) and (37) in (32), we have

$$\begin{aligned}\sum_{u,v\in V(G)} d_G(u,v) &= \sum_{v\in X_1} d_G(u,v) + 2\big(\sum_{v\in X_i,\ 2\le i\le m+1} d_G(u,v)\big)\\ &= n(n+1) + 2\big\{\frac{m(m+1)(2m+1)}{6} + mn(n+1)\big\}\\ &= \frac{2m+1}{3}\{m(m+1)+3n(n+1)\} \qquad (38)\end{aligned}$$

As the graph G is vertex transitive, the sum of the distances from $u = x_{1,1}$ to all other vertices of G is same as the sum of the distances from $x_{i,j}$ to all other vertices of G, for all $i,\ j,\ 1 \le i \le 2m+1,\ 1 \le j \le 2n+1$. Hence

$$\begin{aligned}W(G) &= \frac{|V(G)|}{2}\big(\sum_{u,v\in V(G)} d_G(u,v)\big)\\ &= \frac{(2m+1)(2n+1)}{2}\big\{\frac{2m+1}{3}\big(m(m+1)+3n(n+1)\big)\big\}\\ &= \frac{(2m+1)^2(2n+1)}{6}\big\{m(m+1)+3n(n+1)\big\}.\end{aligned}$$

□

Theorem 8. *The Wiener index of the graph* $C_{2m} \boxtimes C_{2n+1}$ *is* $\frac{m^2(2n+1)}{3}\big(2m^2 + 6n(n+1)+1\big)$.

Proof. As in the proof of the previous theorem, we consider **(A)** and **(B)**. **(A)** is the same as in the previous theorem and hence, for **(A)** we consider the sum as in the proof of the previous theorem. Also, in **(B)**, the summation varies from 2 to m instead of 2 to $m+1$. Hence we consider the sum corresponding to **(B)** of the above theorem. Therefore,

$$\begin{aligned}\sum_{2\le i\le m} d_G(u,v) &= \sum_{2\le i\le m} \big((i-1)^2 + n(n+1)\big)\\ &= \frac{m(m-1)(2m-1)}{6} + (m-1)n(n+1) \qquad (39)\end{aligned}$$

(C): we calculate the sum of the distances from $u = x_{1,1}$ to all other vertices of X_{m+1}.

$$\sum_{v\in X_{m+1}} d_G(u,v) = m^2 + n(n+1) \qquad (40)$$

From (3), (39) and (40), we have

$$\begin{aligned}\sum_{u,v\in V(G)} d_G(u,v) &= \sum_{v\in X_1} d_G(u,v) + 2\big(\sum_{v\in X_i,\ 2\le i\le m} d_G(u,v)\big) + \sum_{v\in X_{m+1}} d_G(u,v)\\ &= n(n+1) + \frac{(m-1)m(2m-1)}{6} + (m-1)n(n+1) + m^2 + n(n+1)\\ &= \frac{m}{3}\big(2m^2 + 1 + 6n(n+1)\big) \qquad (41)\end{aligned}$$

As the graph G is vertex transitive, the sum of the distances from $u = x_{1,1}$ to all other vertices of G is same as the sum of the distances from $x_{i,j}$ to all other vertices of G, for all $i,\ j$, $1 \leq i \leq 2m,\ 1 \leq j \leq 2n+1$. Hence

$$\begin{aligned} W(G) &= \frac{|V(G)|}{2}\Big(\sum_{u,v \in V(G)} d_G(u,v)\Big) \\ &= \frac{2m(2n+1)}{2}\Big\{\frac{m}{3}\big(2m^2+1+6n(n+1)\big)\Big\} \\ &= \frac{m^2(2n+1)}{3}\big\{2m^2+1+6n(n+1)\big\}. \end{aligned}$$

□

The proof of the following theorem uses similar arguments as in Theorems 3.1 and 3.2 and hence it is left to the reader.

Theorem 9. $W(C_{2m} \boxtimes C_{2n}) = \frac{2m^2n}{3}(2m^2+6n^2+1)$.

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References

[1] R. Balakrishnan and K. Ranganathan: *A Text Book of Graph Theory*, Springer-Verlag, New York (2000).

[2] R. Balakrishanan, N. Sridharan, and K. Viswanathan Iyer: *Wiener index of graphs with more than one cut vertex*, Applied Mathematics Letters, 21(2008), 922-927.

[3] Bruce E. Sagan, Yeong-Nan Yeh, Ping Zhang: *The Wiener polynomial of a graph* (Manuscript).

[4] M. Fischermann, A. Hoffmann, D. Rautenbach, L. Volkmann: *Wiener index versus maximum degree in trees*, Discrete Appl. Math. 122(2002), 127-137.

[5] I. Gutman, S. Klavžar, B. Mohar (Eds.): *Fiftieth anniversary of the Wiener index*, Discrete Appl. Math. 80(1997), 1-113.

[6] I. Gutman, O.E. Polansky: *Mathematical concepts in organic chemistry*, Springer-verlag, Berlin 1986.

[7] I. Gutman, Y. Yeh: *On the sum of all distances in composite graphs*, Discrete Math. 135(1994),359-365.

[8] W. Imrich and S. Klavžar: *Product graphs: Structure and Recognition*, John Wiley, New York (2000).

[9] Hu, T. C.: *Optimum communication spanning trees*, SIAM J. Comput. 3(1974), 188-195.

[10] F. Jelen, E. Triesch: *Superdominance order and distance of trees with bounded maximum degree*, Discrete Appl. Math. 125(2003), 225-233.

[11] S. Klavžar, I. Gutman: *Wiener number of vertex-weighted graphs and a chemical application*, Discrete Appl. Math. 125(2003), 225-233.

[12] K. Pattabiraman, P. Paulraja: *Wiener index of the tensor product of a path and a cycle*, Discuss. Math. Graph Theory 31 (2011), 737 - 751.

[13] K. Pattabiraman, P. Paulraja: *On some topological indices of the tensor products of graphs*, Discrete Appl. Math. 160 (2012), 267-279.

[14] K. Pattabiraman, P. Paulraja: *Wiener and PI indices of the strong product of graphs*, Discuss. Math. Graph Thoery 32 (2012), 749-769.

[15] J. Rada: *Variation of the Wiener index under tree transformations*, Discrete Appl. Math. 148(2005), 135-146.