

WEIGHT CHARACTERIZATION OF THE BOUNDEDNESS FOR THE RIEMANN-LIOUVILLE DISCRETE TRANSFORM

ALEXANDER MESKHI^{1,2}, GHULAM MURTAZA³

ABSTRACT. We establish necessary and sufficient conditions on a weight sequence $\{v_j\}_{j=1}^{\infty}$ governing the boundedness for the Riemann-Liouville discrete transform I_{α} from $l^p(\mathbb{N})$ to $l_{v_j}^q(\mathbb{N})$ (trace inequality), where $0 < \alpha < 1$. The derived conditions are of D. Adams or Maz'ya-Verbitsky (pointwise) type.

Key words: Riemann-Liouville discrete transform with product kernels, discrete Hardy operator, discrete potentials, weighted inequality, trace inequality.

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1. INTRODUCTION

Our aim in this paper is to characterize a weight sequence $\{v_j\}_{j=1}^{\infty}$ for which the operator

$$\{I_{\alpha}\beta_k\}_n = \sum_{k=1}^n \frac{\beta_k}{(n-k+1)^{1-\alpha}}, \quad n \in \mathbb{N},$$

maps boundedly from $l^p(\mathbb{N})$ to the weighted space $l_{v_j}^q(\mathbb{N})$, where $1 < p \leq q < \infty$ and $0 < \alpha < 1$. If $p < q$, then we derive necessary and sufficient condition of D. Adams [1] type, while in the diagonal case $p = q$ we establish Maz'ya-Verbitsky [4] type criteria guaranteeing the trace inequality for I_{α} .

¹Department of Mathematical Analysis, I. Javakhishvili Tbilisi State University, Tbilisi, Georgia,

²Abdus Salam School of Mathematical Sciences, GC University, 68-B New Muslim Town, Lahore, Pakistan. Email: meskhi@rmi.ge,

³Department of Applied Sciences, National Textile University, Faisalabad, Pakistan. Email: gmnizami@gmail.com.

Let $1 < p < \infty$. Suppose that $\{v_k\}_{k=1}^{\infty}$ is a sequence of positive numbers (weight sequence). Let $l_{v_k}^p(\mathbb{N})$ be the class of all sequences $\{\beta_k\}_{k=1}^{\infty}$ for which

$$\|\beta_k\|_{l_{v_k}^p(\mathbb{N})} := \left(\sum_{k=1}^{\infty} |\beta_k|^p v_k \right)^{1/p} < \infty.$$

If $v_k \equiv 1$, then we denote $l_{v_k}^p(\mathbb{N})$ by $l^p(\mathbb{N})$.

Further, for an a.e. positive function (weight) v on $\mathbb{R}_+ := (0, \infty)$, we denote by $L_v^p(\mathbb{R}_+)$ the class of all measurable functions f on $\mathbb{R}_+$ for which

$$\|f\|_{L_v^p(\mathbb{R}_+)} := \left(\int_{\mathbb{R}_+} |f(x)|^p v(x) dx \right)^{1/p} < \infty.$$

If $v \equiv 1$, then we denote $L_v^p(\mathbb{R}_+)$ by $L^p(\mathbb{R}_+)$.

Continuous analog of the operator I_{α} is the Riemann-Liouville transform defined on $\mathbb{R}_+$ given by the formula

$$R_{\alpha}f(x) = \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad 0 < \alpha < 1.$$

The $L^p \rightarrow L_v^q$ characterization of R_{α} was studied in the papers [5], [3] (we refer also [7] and the monograph [2], Ch.1). The statements derived by these authors read as follows:

Theorem A ([5], [7]). *Let $1 < p \leq q < \infty$, $1/p < \alpha < 1$. Then the following conditions are equivalent:*

- (i) R_{α} is bounded from $L^p(\mathbb{R}_+)$ into $L_v^q(\mathbb{R}_+)$;
- (ii)

$$B \equiv \sup_{t>0} \left(\int_t^{\infty} \frac{v(x)}{x^{(1-\alpha)q}} dx \right)^{1/q} t^{1/p'} < \infty;$$

- (iii)

$$B_1 \equiv \sup_{k \in \mathbb{Z}} \left(\int_{2^k}^{2^{k+1}} v(x) dx \right)^{1/q} 2^{k(\alpha-1/p)q} < \infty.$$

For the case $0 < \alpha < 1/p$, there are known the following statements:

Theorem B ([3]). *Let $1 < p < \infty$ and let $0 < \alpha < \frac{1}{p}$. Then the inequality*

$$\int_{\mathbb{R}_+} |R_{\alpha}f(x)|^p v(x) dx \leq c_0 \int_{\mathbb{R}_+} |f(x)|^p dx, \quad f \in L^p(\mathbb{R}_+),$$

holds if and only if $W_{\alpha}v \in L_{loc}^{p'}(\mathbb{R}_+)$ and

$$W_{\alpha}[W_{\alpha}v]^{p'}(x) \leq cW_{\alpha}v(x) \quad \text{a.e.,}$$

where

$$W_{\alpha}g(t) = \int_t^{\infty} \frac{g(\tau)}{(\tau-t)^{1-\alpha}} d\tau.$$

Theorem C ([2], p. 131.) *Let $1 < p < q < \infty$, $0 < \alpha < 1/p$. Then the following statements are equivalent:*

(i) *There exists a positive constant c such that for all $f \in L^p(\mathbb{R}_+)$,*

$$\|R_\alpha f\|_{L_v^q(\mathbb{R}_+)} \leq c \|f\|_{L^p(\mathbb{R}_+)};$$

(ii)

$$\sup_{0 \leq h \leq a} (\nu[a; a+h])^{1/q} h^{\alpha-1/p} < \infty;$$

In the paper [3] the authors applied Theorem B to prove the existence of a positive solution for certain non-linear Volterra integral equation.

In the discrete case the following statement holds (see [6], [7]):

Theorem D. *Let $1 \leq p \leq q < \infty$ and let $1/p < \alpha < 1$. Then the following conditions are equivalent:*

(i) *The operator I_α is bounded from $l^p(\mathbb{N})$ to $l_{v_j}^q(\mathbb{N})$;*

(ii)

$$\sup_{k \in \mathbb{N}} \left(\sum_{m=k}^{\infty} \frac{v_m}{m^{(1-\alpha)q}} \right)^{1/q} k^{1/p'} < \infty;$$

(iii)

$$\sup_{k \in \mathbb{Z}_+} \left(\sum_{m=2^k}^{2^{k+1}} v_m \right)^{1/q} 2^{k(\alpha p - 1)} < \infty.$$

Our purpose in this paper is to derive the results similar to Theorems B and C in the discrete case and to derive criteria on a weight sequence $\{v_j\}_j$ guaranteeing the boundedness of I_α from $l^p(\mathbb{N})$ to $l_{v_k}^q(\mathbb{N})$ in the case when $0 < \alpha < 1/p$. As we shall see in this case conditions on $\{v_j\}_{j \in \mathbb{N}}$ are different to the criteria in Theorem D.

Throughout the paper the symbol $\mathbb{N}$ means the set of natural numbers; $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$; $p' := \frac{p}{p-1}$, where $1 < p < \infty$; $\mathbb{R}_+ := [0, \infty)$; the characteristic sequence $\chi_{\{i: a \leq i \leq b\}}$ (a and b are positive integers) is defined in the usual way:

$$\chi_{\{i: a \leq i \leq b\}} = \begin{cases} 1 & a \leq i \leq b; \\ 0 & \text{elsewhere.} \end{cases}$$

The operator formal adjoint to I'_α is given by the formula:

$$\{I'_\alpha \beta_k\}_n = \sum_{k=n}^{\infty} \frac{\beta_k}{(k-n+1)^{1-\alpha}}, \quad n \in \mathbb{N}.$$

Finally we point out that constants (often different constants in the same series of inequalities) will generally be denoted by c .

2. THE MAIN RESULTS

Now we formulate the main results of this paper.

Theorem 2.1. *[Adams type characterization] Let $1 < p < q < \infty$ and $0 < \alpha < 1/p$. Then I_α is bounded from $l^p(\mathbb{N})$ to $l_{v_k}^q(\mathbb{N})$ if and only if*

$$B := \sup_{m,j \in \mathbb{N}; j \leq m} \left(\sum_{k=m}^{m+j} v_k \right)^{1/q} j^{\alpha-1/p} < \infty.$$

Theorem 2.2. *[Maz'ya-Verbitsky type characterization] Let $1 < p < \infty$ and let $0 < \alpha < 1/p$. Then the inequality*

$$\sum_{i=1}^{\infty} \left(I_\alpha g_j \right)_i^p v_i \leq c \sum_{i=1}^{\infty} g_i^p \quad (1)$$

holds for all non-negative sequences $\{g_i\}_i$ if and only if $\{I'_\alpha v_i\}_i < \infty$ for all $i \in \mathbb{N}$ and there is a positive constant c such that

$$\left\{ I'_\alpha [I'_\alpha v_j]^{p'} \right\}_i \leq c \left\{ I'_\alpha v_j \right\}_i \quad (2)$$

for all $i \in \mathbb{N}$.

3. PROOF OF THE MAIN RESULTS

Let $(X, \mathcal{U}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces with ν being σ -finite. Suppose that $k(x, y)$ is a non-negative real-valued $\mathcal{U} \times \mathcal{B}$ -measurable function and that

$$Kf(y) = \int_X k(x, y) f(x) d\mu(x)$$

is the kernel operator.

Denote:

$$e_\lambda(x) := \{y \in Y : k(x, y) > \lambda\}, \quad e_\lambda(y) := \{x \in X : k(x, y) > \lambda\},$$

where λ is a positive number;

$$M_r(\mu)(y) := \sup_{\lambda > 0} \lambda^r \mu(e_\lambda(y)); \quad M_s(\nu)(x) := \sup_{\lambda > 0} \lambda^s \nu(e_\lambda(x)),$$

where r and s are real numbers.

To prove Theorem 2.1 we use the following statement which is a corollary of part (ii) of Theorem A in [1].

Theorem E. *Suppose that $1 < p < q < \infty$, $\frac{s}{q} = \frac{r}{p} + 1 - r$, where $r > 0$. If $M_r(\mu)(y) \leq A < \infty$ for all $y \in Y$; $M_s(\nu)(x) \leq B < \infty$ for all $x \in X$, then the operator K is bounded from $L^p(X, \mu)$ to $L^q(Y, \nu)$, where $L^p(X, \mu)$ $L^q(Y, \nu)$ are Lebesgue spaces defined with respect to the measures μ and ν respectively.*

Proof of Theorem 2.1. Sufficiency. Suppose that $X = Y = \mathbb{N}$, μ is the counting measure on $\mathbb{N}$ and that $d\nu(n) = v_n d\mu(n)$, where $\{v_n\}_{n=1}^{\infty}$ is the weight sequence. In our case the kernel operator is given by

$$\{I_\alpha\{g_m\}\}_n = \sum_{m=1}^{\infty} k(m, n)g_m, \quad n \in \mathbb{N},$$

where

$$k(m, n) = \chi_{\{m \in \mathbb{N}: 1 \leq m \leq n\}} (n - m + 1)^{\alpha-1}.$$

Let $r = \frac{1}{1-\alpha}$ and let $\frac{s}{q} = \frac{r}{p} + 1 - r$. That is $s = \frac{q(\alpha-1/p)}{\alpha-1} > 0$. We have

$$\begin{aligned} \sup_{n \in \mathbb{N}} M_r(\mu)(n) &= \sup_{\lambda \leq 1, n \in \mathbb{N}} \lambda^r \mu\{m \in \mathbb{N} : m \leq n; (n - m + 1)^{\alpha-1} > \lambda\} \\ &= \sup_{\lambda \geq 1, n \in \mathbb{N}} \lambda^{r(\alpha-1)} \mu\{m \in \mathbb{N} : m \leq n; n - m + 1 < \lambda\} \\ &\leq \sup_{k, n \in \mathbb{N}} k^{-1} \sum_{m=\max\{n-k, 1\}}^n 1 \leq c. \end{aligned}$$

Further,

$$\begin{aligned} \sup_{m \in \mathbb{N}} M_s(\nu)(m) &= \sup_{\lambda \leq 1, m \in \mathbb{N}} \lambda^s \nu\{n \in \mathbb{N} : m \leq n; (n - m + 1)^{\alpha-1} > \lambda\} \\ &= \sup_{\lambda \geq 1, m \in \mathbb{N}} \lambda^{s(\alpha-1)} \nu\{n \in \mathbb{N} : m \leq n; n - m + 1 < \lambda\} \\ &\leq \sup_{k, m \in \mathbb{N}} k^{s(\alpha-1)} \sum_{n=m}^{m+k} v_n \leq c \sup_{n \leq j; n, j \in \mathbb{N}} S_{n,j}, \end{aligned}$$

where

$$S_{n,j} := (j - n + 1)^{q(\alpha-1/p)} \sum_{m=n}^j v_m.$$

Further, let m, n be positive integers satisfying the condition $1 \leq m \leq n$. Then there exists a non-negative integer k_0 such that $2^{k_0} \leq n - m + 1 \leq 2^{k_0+1}$. Therefore by using the fact that $0 < \alpha < 1/p$, we obtain that

$$\begin{aligned} S_{m,n} &= \left(\sum_{k=m}^n v_k \right) (n - m + 1)^{(\alpha-1/p)q} \\ &\leq \sum_{l=1}^{k_0+2} \left(\sum_{k=m+2^{l-1}}^{m+2^l} v_k \right) (2^{k_0+1})^{(\alpha-1/p)q} \\ &\leq \sum_{l=1}^{k_0+2} \left(\sum_{k=m+2^{l-1}}^{m+2^l} v_k \right) 2^{l(\alpha-1/p)q} 2^{l(1/p-\alpha)q} (2^{k_0+1})^{(\alpha-1/p)q} \leq cB^q. \end{aligned}$$

Necessity: Let

$$\beta^{(m,j)} = \begin{cases} 1 & \text{if } m-j < k \leq m; \\ 0 & \text{otherwise,} \end{cases}$$

where m, j are positive integers such that $1 \leq j \leq m$. Then we have

$$\begin{aligned} \left(\sum_{n=1}^{\infty} v_n \left(\sum_{k=1}^n \frac{(\beta^{(m,j)})_k}{(n-k+1)^{1-\alpha}} \right)^q \right)^{1/q} &\geq \left(\sum_{n=m}^{m+j} v_n \left(\sum_{k=m-j}^m \frac{1}{(n-k+1)^{1-\alpha}} \right)^q \right)^{1/q} \\ &\geq c \left(\sum_{n=m}^{m+j} v_n \right)^{1/q} j^{\alpha}. \end{aligned}$$

Therefore, by the boundedness of I_{α} we conclude that

$$\left(\sum_{n=m}^{m+j} v_n \right)^{1/q} j^{\alpha-1/p} \leq c, \quad 1 \leq j \leq m.$$

□

To prove Theorem 2.2 we need to prove some auxiliary statements.

Proposition 3.1. *Let $1 < p < \infty$, and let $0 < \alpha < 1/p$. If I_{α} is bounded from $l^p(\mathbb{N})$ to $l^p_{v_i}(\mathbb{N})$ then there exist a positive constant c such that*

$$\sum_{i=m}^{m+h} v_i \leq c h^{1-\alpha p} \quad (3)$$

holds for all positive integers m and h .

Proof. First we suppose that $h \leq m$. For the sequence $g^{(m,h)} = \chi_{\{k: m-h < k \leq m\}}$ we have

$$\begin{aligned} \left(\sum_{i=1}^{\infty} v_i \left(\sum_{j=1}^i \frac{(g^{(m,h)})_j}{(i-j+1)^{1-\alpha}} \right)^p \right)^{1/p} &\geq \left(\sum_{i=m}^{m+h} v_i \left(\sum_{j=m-h+1}^m (i-j+1)^{\alpha-1} \right)^p \right)^{1/p} \\ &\geq c \left(\sum_{i=m}^{m+h} v_i \right)^{1/p} h^{\alpha}. \end{aligned}$$

Therefore, by the boundedness of I_{α} we get

$$\left(\sum_{i=m}^{m+h} v_i \right)^{1/p} \leq c h^{1/p-\alpha}.$$

Hence (3) holds for all positive integers m and h satisfying $h \leq m$. Now let $m < h$. Then there exist a positive integer k such that $2^k \leq h \leq 2^{k+1}$.

Therefore taking into account the condition $0 < \alpha < 1/p$ we obtain

$$\begin{aligned}
\sum_{i=m}^{m+h} v_i &\leq \sum_{i=1}^k \left(\sum_{j=m+2^{i-1}}^{m+2^i} v_j \right) \\
&= \sum_{i=1}^k \left[\left(\sum_{j=m+2^{i-1}}^{m+2^i} v_j \right) 2^{i(1-\alpha p)} 2^{i(\alpha p-1)} \right] \\
&\leq c \sum_{i=1}^k 2^{i(1-\alpha p)} \leq c 2^{k(1-\alpha p)} \leq c h^{1-\alpha p}.
\end{aligned}$$

□

Proof of necessity of Theorem 2.2. Let us first show that, from (1) it follows that $\{I'_\alpha v_j\}_k < \infty$ for all $k \in \mathbb{N}$. By the duality arguments (1) is equivalent to the inequality

$$\sum_{i=1}^{\infty} \left(I'_\alpha g_j \right)_i^{p'} \leq c \sum_{i=1}^{\infty} g_i^{p'} v_i^{1-p'}. \quad (4)$$

Let $v_i^{(1)} = v_i \chi_{\{i: m \leq i < m+2h\}}$ and $v_i^{(2)} = v_i \chi_{\{i: 1 \leq i < m \text{ or } i \geq m+2h\}}$, where m and h are positive integers.

Note that for $k \geq m+2h-1$ and $m \leq i \leq m+h$, we have $k-m+1 \leq 2(k-i+1)$. Further, by using (3), we come to the estimates:

$$\begin{aligned}
\{I'_\alpha v_j^{(2)}\}_i &\leq \sum_{k=m+2h-1}^{\infty} v_k (k-i+1)^{\alpha-1} \\
&\leq c \sum_{k=m+h}^{\infty} v_k (k-m+1)^{\alpha-1} \\
&\leq c \sum_{k=m+h}^{\infty} v_k \left(\sum_{j=k-m+1}^{\infty} j^{\alpha-2} \right) \\
&= c \sum_{j=h+1}^{\infty} j^{\alpha-2} \left(\sum_{k=m}^{j+m-1} v_k \right) \\
&\leq c \sum_{j=h+1}^{\infty} j^{\alpha-2} \cdot j^{1-\alpha p} < \infty.
\end{aligned}$$

Therefore $(I'_\alpha v_j^{(2)})_i < \infty$. The fact that $(I'_\alpha v_j^{(1)})_i < \infty$ is obvious. Thus, $(I'_\alpha v_j)_i < \infty$ for every $i \in \mathbb{N}$ because m and h are taken arbitrarily.

Now we prove that (1) yields (2). For this we need some lemmas.

Lemma 3.2. *Let $0 < \alpha < 1$. Then there are positive constants $c_\alpha^{(1)}$ and $c_\alpha^{(2)}$ depending only on α such that for all $m \in \mathbb{N}$ the inequality*

$$(I'_\alpha \beta_s)_m \leq c_\alpha^{(1)} \sum_{j=1}^{\infty} j^{\alpha-2} \left(\sum_{k=m}^{m+j-1} \beta_k \right) \leq c_\alpha^{(2)} (I'_\alpha \beta_s)_m$$

holds, where $\beta_m \geq 0$.

Proof. The proof follows easily if we observe that there are positive constants $b_\alpha^{(1)}$ and $b_\alpha^{(2)}$ independent of k and m such that

$$\sum_{j=k-m+1}^{\infty} j^{\alpha-2} \leq b_\alpha^{(1)} (k-m+1)^{\alpha-1} \leq b_\alpha^{(2)} \sum_{j=k-m+1}^{\infty} j^{\alpha-2}.$$

It remains to change the order of summation. □

Corollary 3.3. *Let $0 < \alpha < 1$, $\beta_m \geq 0$. Then there are positive constants $c_\alpha^{(1)}$ and $c_\alpha^{(2)}$ depending only on α such that for all $m \in \mathbb{N}$ the inequality*

$$\left\{ I'_\alpha [I'_\alpha \beta_s]^{p'} \right\}_m \leq c_\alpha^{(1)} \sum_{j=1}^{\infty} j^{\alpha-2} \left(\sum_{k=m}^{m+j-1} \{I'_\alpha \beta_s\}_k^{p'} \right) \leq c_\alpha^{(2)} \left\{ I'_\alpha [I'_\alpha \beta_s]^{p'} \right\}_m$$

holds.

Let $\{v_i^{(1)}\}$ and $\{v_i^{(2)}\}$ be defined as above. Then by using (4) we have that

$$\sum_{i=m}^{m+h} \left(I'_\alpha v_j^{(1)} \right)_i^{p'} \leq c \sum_{i=m}^{m+h} v_i.$$

Thus, by Corollary 3.3 we conclude that

$$\left\{ I'_\alpha [I'_\alpha v_k^{(1)}]^{p'} \right\}_i \leq c \sum_{j=1}^{\infty} j^{\alpha-2} \left(\sum_{k=i}^{i+2(j-1)} v_k \right) \leq c \left\{ I'_\alpha v_s \right\}_i.$$

For the estimate of $\left\{ I'_\alpha [I'_\alpha v_k^{(2)}]^{p'} \right\}_i$, we need some auxiliary statements.

Lemma 3.4. *Let $0 < \alpha < 1$. Then there is a positive constant c such that for all natural numbers m, k and j with $m \leq k \leq m+j-1$, the inequality*

$$\left\{ I'_\alpha v_s^{(2)} \right\}_k \leq c \sum_{s=j}^{\infty} s^{\alpha-2} \left(\sum_{t=m}^{m+s-1} v_t \right)$$

holds, where $v_s^{(2)} = v_s \chi_{\{s: 1 \leq s < m \text{ or } s \geq m+2j\}}$.

Proof. Using the arguments of the proof of Lemma 3.2 and the fact that

$$\left(I'_\alpha v_s^{(2)} \right)_k = \sum_{s=m+2j}^{\infty} v_s (s-k+1)^{\alpha-1}$$

we have

$$\begin{aligned}
\left(I'_\alpha v_s^{(2)}\right)_k &\leq c \sum_{s=m+2j}^{\infty} v_s (s-m+1)^{\alpha-1} \\
&\leq c \sum_{s=m+2j}^{\infty} v_s \sum_{t=s-m+1}^{\infty} t^{\alpha-2} \\
&\leq c \sum_{t=j}^{\infty} t^{\alpha-2} \left(\sum_{s=m}^{m+t-1} v_s \right).
\end{aligned}$$

□

Lemma 3.5. *Let $0 < \alpha < 1$. Then there is a positive constant c such that*

$$\left\{ I'_\alpha [I'_\alpha v_i^{(2)}]^{p'} \right\}_m \leq c \sum_{t=1}^{\infty} t^{\alpha-1} \left(\sum_{s=t}^{\infty} s^{\alpha-2} \left(\sum_{j=m}^{m+s-1} v_j \right) \right)^{p'}$$

Proof. Using Lemma 3.4 in Corollary 3.3 we have that

$$\begin{aligned}
\left\{ I'_\alpha [I'_\alpha v_i^{(2)}]^{p'} \right\}_m &\leq c \sum_{t=1}^{\infty} t^{\alpha-2} \left(\sum_{k=m}^{m+t-1} \{I'_\alpha v_k\}^{p'} \right) \\
&\leq c \sum_{t=1}^{\infty} t^{\alpha-2} \sum_{k=m}^{m+t-1} \left(\sum_{s=t}^{\infty} s^{\alpha-2} \sum_{\epsilon=m}^{m+s-1} v_\epsilon \right)^{p'} \\
&\quad \text{(the inner sum does not depend on k)} \\
&= c \sum_{t=1}^{\infty} t^{\alpha-2} \left(\sum_{s=t}^{\infty} s^{\alpha-2} \sum_{\epsilon=m}^{m+s-1} v_\epsilon \right)^{p'} \left(\sum_{k=m}^{m+t-1} 1 \right) \\
&= c \sum_{t=1}^{\infty} t^{\alpha-2} \left(\sum_{s=t}^{\infty} s^{\alpha-2} \sum_{\epsilon=m}^{m+s-1} v_\epsilon \right)^{p'}.
\end{aligned}$$

□

Lemma 3.6. *Let $0 < \alpha < 1$. Then there is a positive constant c such that*

$$\left\{ I'_\alpha [I'_\alpha v_i^{(2)}]^{p'} \right\}_m \leq c \sum_{t=1}^{\infty} t^\alpha \left(\sum_{s=t}^{\infty} s^{\alpha-2} \sum_{\epsilon=m}^{m+s-1} v_\epsilon \right)^{p'-1} \left(t^{\alpha-2} \sum_{j=m}^{m+t-1} v_j \right).$$

Proof. We will deduce the discrete case to the continuous case. Let $v(x) = v_j$, $j \leq x < j+1$. Then $\int_j^{j+1} v(x) dx = v_j$. Hence, by using lemmas proved above, the Lebesgue differentiation theorem and integration by parts, we

find that

$$\begin{aligned}
\left\{ I'_\alpha [I'_\alpha v_i^{(2)}]^{p'} \right\}_m &\leq c \sum_{n=1}^{\infty} n^{\alpha-1} \left(\sum_{j=n}^{\infty} j^{\alpha-2} \left(\sum_{k=m}^{m+2j} v_k \right) \right)^{p'} \\
&\leq c \sum_{n=1}^{\infty} \int_n^{n+1} x^{\alpha-1} \left(\sum_{i=2n}^{\infty} \int_i^{i+1} y^{\alpha-2} \left(\sum_{k=m}^{m+y} v_k \right) dy \right)^{p'} dx \\
&\leq c \int_1^{\infty} x^{\alpha-1} \left(\int_x^{\infty} y^{\alpha-2} \left(\sum_{k=m}^{m+y} v_k \right) dy \right)^{p'} dx \\
&= c \left[\frac{x^\alpha}{\alpha} \left(\int_x^{\infty} \dots \right)^{p'} \Big|_1^{\infty} + \int_1^{\infty} x^\alpha \left(\int_x^{\infty} \dots \right)^{p'-1} x^{\alpha-2} \left(\sum_{k=m}^{m+x} v_k \right) dx \right] \\
&= c \left[-\frac{1}{\alpha} \left(\int_1^{\infty} \dots \right)^{p'} + \int_1^{\infty} x^\alpha \left(\int_x^{\infty} \dots \right)^{p'-1} x^{\alpha-2} \left(\sum_{k=m}^{m+x} v_k \right) dx \right] \\
&\leq c \int_1^{\infty} x^\alpha \left(\int_x^{\infty} \dots \right)^{p'-1} x^{\alpha-2} \left(\sum_{k=m}^{m+x} v_k \right) dx \\
&= c \sum_{n=1}^{\infty} \int_n^{n+1} x^\alpha \left(\int_x^{\infty} \dots \right)^{p'-1} x^{\alpha-2} \left(\sum_{k=m}^{m+n+1} v_k \right) dx \\
&\leq c \sum_{n=1}^{\infty} n^\alpha \left(\int_n^{\infty} \dots \right)^{p'-1} n^{\alpha-2} \left(\sum_{k=m}^{m+n+1} v_k \right) \\
&\leq c \sum_{n=1}^{\infty} n^\alpha \left(\sum_{k=n}^{\infty} \int_k^{k+1} k^{\alpha-2} \left(\sum_{i=m}^{m+k+1} v_i \right) dy \right)^{p'-1} n^{\alpha-2} \left(\sum_{k=m}^{m+n+1} v_k \right) \\
&= c \sum_{n=1}^{\infty} n^\alpha \left(\sum_{k=n}^{\infty} k^{\alpha-2} \left(\sum_{i=m}^{m+k+1} v_i \right) \right)^{p'-1} n^{\alpha-2} \left(\sum_{k=m}^{m+n+1} v_k \right).
\end{aligned}$$

□

Now necessity of Theorem 2.2 follows easily because we know that the trace inequality implies (see Proposition 3.1)

$$\sum_{k=m}^{m+j} v_k \leq c j^{1-\alpha p},$$

where the positive constant c is independent of positive integers m and j . Indeed, by using this inequality in Lemma 3.6 we have that

$$\begin{aligned}
\left\{ I'_\alpha [I'_\alpha v_j^{(2)}]^{p'} \right\}_m &\leq c \sum_{n=1}^{\infty} n^\alpha \left(\sum_{k=n}^{\infty} k^{\alpha-2} (k+2)^{1-\alpha p} \right)^{p'-1} \left(n^{\alpha-2} \sum_{k=m}^{m+n+1} v_k \right) \\
&\leq c \sum_{n=1}^{\infty} n^{\alpha-2} \sum_{k=m}^{m+n+1} v_k \\
&\leq c \sum_{n=1}^{\infty} (3n)^{\alpha-2} \sum_{k=m}^{m+n+1} v_k \\
&\leq c \sum_{j=3}^{\infty} [3(j-2)]^{\alpha-2} \sum_{k=m}^{m+j-1} v_k \\
&\leq c \sum_{j=3}^{\infty} j^{\alpha-2} \sum_{k=m}^{m+j-1} v_k \\
&\leq c \left\{ I'_\alpha v_j \right\}_m.
\end{aligned}$$

In the last inequality we used Lemma 3.2, in particular, the estimate from below.

Necessity of Theorem 2.2 is proved.

Now we prove *sufficiency* of Theorem 2.2. We will need some auxiliary statements.

Lemma 3.7. *Let $1 < p < \infty$ and $0 < \alpha < 1$. Then there exists a positive constant c such that for all non-negative sequences $\{g_i\}_{i \in \mathbb{Z}}$ and all $i \in \mathbb{N}$, the following inequality holds*

$$\{I_\alpha g_k\}_i^p \leq c \{I_\alpha [I_\alpha g_k]_j^{p-1} g_m\}_i, \quad (5)$$

Proof. First we assume that $\{V_\alpha g_j\}_i := \{I_\alpha [I_\alpha g_k]_j^{p-1} g_j\}_i$ and

$$\{V_\alpha g_j\}_i \leq \{I_\alpha g_j\}_i^p;$$

otherwise (5) is obvious for $c = 1$. Now let us assume that $1 < p \leq 2$. Then we have

$$\begin{aligned}
\{I_\alpha g_k\}_i^p &= \sum_{k=1}^i (i-k+1)^{\alpha-1} g_k \left(\sum_{j=1}^i (i-j+1)^{\alpha-1} g_j \right)^{p-1} \\
&\leq \sum_{k=1}^i (i-k+1)^{\alpha-1} g_k \left(\sum_{j=1}^k (i-j+1)^{\alpha-1} g_j \right)^{p-1} \\
&\quad + \sum_{k=1}^i (i-k+1)^{\alpha-1} g_k \left(\sum_{j=k}^i (i-j+1)^{\alpha-1} g_j \right)^{p-1} \\
&\equiv I_i^{(1)} + I_i^{(2)}.
\end{aligned}$$

It is obvious that if $j \leq k \leq i$, then $k-j+1 \leq i-j+1$. Consequently,

$$I_i^{(1)} \leq \sum_{k=1}^i (i-k+1)^{\alpha-1} g_k \left(\sum_{j=1}^k (k-j+1)^{\alpha-1} g_j \right)^{p-1} = \{V_\alpha g_k\}_i.$$

Now we use Hölder's inequality with respect to the exponents $\frac{1}{p-1}$, $\frac{1}{2-p}$ and measure $d\mu(k) = (i-k+1)^{\alpha-1} g_k d\mu_c(k)$ (here μ_c is the counting measure). We have

$$\begin{aligned}
I_i^{(2)} &\leq \left(\sum_{k=1}^i (i-k+1)^{\alpha-1} g_k \right)^{2-p} \left(\sum_{k=1}^i \left(\sum_{j=k}^i (i-j+1)^{\alpha-1} g_j \right) (i-k+1)^{\alpha-1} g_k \right)^{p-1} \\
&= \{I_\alpha g_k\}_i^{2-p} (J_i)^{p-1},
\end{aligned}$$

where

$$J_i \equiv \sum_{k=1}^i \left(\sum_{j=k}^i (i-j+1)^{\alpha-1} g_j \right) (i-k+1)^{\alpha-1} g_k.$$

Using Fubini's theorem we find that

$$J_i = \sum_{j=1}^i (i-j+1)^{\alpha-1} g_j \left(\sum_{k=1}^j (i-k+1)^{\alpha-1} g_k \right).$$

Further, it is obvious that the following estimates

$$\begin{aligned}
\sum_{k=1}^j (i-k+1)^{\alpha-1} g_k &\leq \left(\sum_{k=1}^j (i-k+1)^{\alpha-1} g_k \right)^{p-1} \{I_\alpha g_k\}_i^{2-p} \\
&\leq \{I_\alpha g_k\}_j^{p-1} \{I_\alpha g_k\}_i^{2-p}
\end{aligned}$$

hold, where $j \leq i$. Taking into account the last estimate, we obtain

$$\begin{aligned}
J_i &\leq \left(\sum_{j=1}^i (i-j+1)^{\alpha-1} g_j \{I_\alpha g_k\}_j^{p-1} \right) \{I_\alpha g_k\}_i^{2-p} \\
&= \{V_\alpha g_k\}_i \{I_\alpha g_k\}_i^{2-p}.
\end{aligned}$$

Thus,

$$\begin{aligned} I_i^{(2)} &\leq \{I_\alpha g_k\}_i^{2-p} \{I_\alpha g_k\}_i^{(2-p)(p-1)} \{V_\alpha g_k\}_i^{p-1} \\ &= \{I_\alpha g_k\}_i^{p(2-p)} \{V_\alpha g_k\}_i^{p-1}. \end{aligned}$$

Combining the estimate for $I^{(1)}$ and $I^{(2)}$ we derive

$$\{I_\alpha g_k\}_i^p \leq \{V_\alpha g_k\}_i + \{I_\alpha g_k\}_i^{p(2-p)} \{V_\alpha g_k\}_i^{p-1}.$$

As we have assumed that $\{V_\alpha g_k\}_i \leq \{I_\alpha g_k\}_i^p$, we obtain

$$\{V_\alpha g_k\}_i = \{V_\alpha g_k\}_i^{2-p} \{V_\alpha g_k\}_i^{p-1} \leq \{V_\alpha g_k\}_i^{p-1} \{I_\alpha g_k\}_i^{p(2-p)}.$$

Hence

$$\begin{aligned} \{I_\alpha g_k\}_i^p &\leq \{V_\alpha g_k\}_i^{p-1} \{I_\alpha g_k\}_i^{p(2-p)} + \{V_\alpha g_k\}_i^{p-1} \{I_\alpha g_k\}_i^{p(2-p)} \\ &= 2\{V_\alpha g_k\}_i^{p-1} \{I_\alpha g_k\}_i^{p(2-p)}. \end{aligned}$$

Applying the fact $(I_\alpha g_j)_i < \infty$ we find that

$$\{I_\alpha g_k\}_i^p \leq 2^{\frac{1}{p-1}} \{V_\alpha g_k\}_i.$$

Now we deal with the case $p > 2$. Let us assume again that

$$\{V_\alpha g_j\}_i \leq \{I_\alpha g_j\}_i^p.$$

Since $p > 2$ we have

$$\begin{aligned} \{I_\alpha g_k\}_i^p &= \sum_{k=1}^i (i-k+1)^{\alpha-1} g_k \left(\sum_{j=1}^i (i-j+1)^{\alpha-1} g_j \right)^{p-1} \\ &\leq 2^{p-1} \sum_{k=1}^i (i-k+1)^{\alpha-1} g_k \left(\sum_{j=1}^k (i-j+1)^{\alpha-1} g_j \right)^{p-1} \\ &\quad + 2^{p-1} \sum_{k=1}^i (i-k+1)^{\alpha-1} g_k \left(\sum_{j=k}^i (i-j+1)^{\alpha-1} g_j \right)^{p-1} \\ &=: 2^{p-1} I_i^{(1)} + 2^{p-1} I_i^{(2)}. \end{aligned}$$

It is clear that if $j \leq k \leq i$, then $(i-j+1)^{\alpha-1} \leq (k-j+1)^{\alpha-1}$. Therefore like the case $1 < p \leq 2$ we have that $I_i^{(1)} \leq \{V_\alpha g_k\}_i$.

Now we estimate $I_i^{(2)}$. We obtain

$$\begin{aligned} \left(\sum_{j=k}^i (i-j+1)^{\alpha-1} g_j \right)^{p-1} &= \left(\sum_{j=k}^i (i-j+1)^{\alpha-1} g_j \right)^{p-2} \left(\sum_{j=k}^i (i-j+1)^{\alpha-1} g_j \right) \\ &\leq \left\{ I_\alpha g_j \right\}_i^{p-2} \sum_{j=k}^i (i-j+1)^{\alpha-1} g_j. \end{aligned}$$

Using Fubini's theorem and the last estimate we have

$$\begin{aligned}
I_i^{(2)} &\leq \left\{ I_\alpha g_j \right\}_i^{p-2} \sum_{k=1}^i (i-k+1)^{\alpha-1} g_k \sum_{j=k}^i (i-j+1)^{\alpha-1} g_j \\
&= \left\{ I_\alpha g_j \right\}_i^{p-2} \sum_{j=1}^i (i-j+1)^{\alpha-1} g_j \sum_{k=1}^j (i-k+1)^{\alpha-1} g_k \\
&\leq \left\{ I_\alpha g_j \right\}_i^{p-2} \sum_{j=1}^i (i-j+1)^{\alpha-1} g_j \sum_{k=1}^j (j-k+1)^{\alpha-1} g_k.
\end{aligned}$$

Due to Hölder's inequality with respect to the exponents $\{p-1, \frac{p-1}{p-2}\}$ and the measure $d\mu(j) = (i-j+1)^{\alpha-1} g_j d\mu_c(j)$ (μ_c is the counting measure) we derive

$$\begin{aligned}
\sum_{j=1}^i (i-j+1)^{\alpha-1} g_j \sum_{k=1}^j (j-k+1)^{\alpha-1} g_k &\leq \left(\sum_{j=1}^i (i-j+1)^{\alpha-1} g_j \right)^{\frac{p-2}{p-1}} \\
&\quad \times \left(\sum_{j=1}^i \left(\sum_{k=1}^j (j-k+1)^{\alpha-1} g_k \right)^{p-1} (i-j+1)^{\alpha-1} g_j \right)^{\frac{1}{p-1}} \\
&= \{I_\alpha g_j\}_i^{\frac{p-2}{p-1}} \{V_\alpha g_j\}_i^{\frac{1}{p-1}}.
\end{aligned}$$

Combining these estimates we obtain

$$\{I_\alpha g_j\}_i^p \leq 2^{p-1} \{V_\alpha g_j\}_i + 2^{p-1} \{I_\alpha g_j\}_i^{\frac{p(p-2)}{p-1}} \{V_\alpha g_j\}_i^{\frac{1}{p-1}}.$$

By virtue of the inequality $\{V_\alpha g_i\}_i \leq \{I_\alpha g_j\}_i^p$ it follows that

$$\{V_\alpha g_j\}_i = \{V_\alpha g_j\}_i^{\frac{1}{p-1}} \{V_\alpha g_j\}_i^{\frac{p-2}{p-1}} \leq \{V_\alpha g_j\}_i^{\frac{1}{p-1}} \{I_\alpha g_j\}_i^{\frac{p(p-2)}{p-1}}.$$

Hence

$$\begin{aligned}
\{I_\alpha g_j\}_i^p &\leq 2^{p-1} \left(\{V_\alpha g_j\}_i^{\frac{1}{p-1}} \{I_\alpha g_j\}_i^{\frac{p(p-2)}{p-1}} + \{V_\alpha g_j\}_i^{\frac{1}{p-1}} \{I_\alpha g_j\}_i^{\frac{p(p-2)}{p-1}} \right) \\
&= 2^p \{V_\alpha g_j\}_i^{\frac{1}{p-1}} \{I_\alpha g_j\}_i^{\frac{p(p-2)}{p-1}}.
\end{aligned}$$

Further, from the last estimate we conclude that

$$\{I_\alpha g_j\}_i^p \leq 2^{p(p-1)} \{V_\alpha g_j\}_i,$$

where $2 < p < \infty$. □

Lemma 3.8. *Let $1 < p < \infty$, $0 < \alpha < 1$ and v_i be a sequence of positive numbers on $\mathbb{N}$. Let there exist a constant $c > 0$ such that the inequality*

$$\|I_\alpha \{g_i\}\|_{l_{v_i^{(1)}}^p(\mathbb{N})}^p \leq c_1 \|g_i\|_{l^p(\mathbb{N})}, \quad \{v_s^{(1)}\}_i = \{I'_\alpha v_s\}_i^{p'}$$

holds for all sequences $g_i \in l^p(\mathbb{N})$. Then

$$\|I_\alpha\{g_i\}\|_{l^p_{v_i}(\mathbb{N})} \leq c_2 \|g_i\|_{l^p(\mathbb{N})}, \quad g_i \in l^p(\mathbb{N}),$$

where $c_2 = c_1^{1/p'} c^{1/p}$.

Proof. Let $g_i \geq 0$. Using Lemma 3.7, Fubini's theorem and Hölder's inequality we derive

$$\begin{aligned} \sum_{k=1}^{\infty} \{I_\alpha g_s\}_k^p v_k &\leq c \sum_{k=1}^{\infty} \sum_{i=1}^k \{I_\alpha g_j\}_i^{p-1} g_i (k-i+1)^{\alpha-1} v_k \\ &= c \sum_{i=1}^{\infty} \{I_\alpha g_j\}_i^{p-1} g_i \{I'_\alpha v_j\}_i \\ &\leq c \left(\sum_{i=1}^{\infty} g_i^p \right)^{1/p} \left(\sum_{i=1}^{\infty} \{I_\alpha g_j\}_i^p v_i^{(1)} \right)^{1/p'} \\ &= c \|g_i\|_{l^p(\mathbb{N})} \|I_\alpha g_i\|_{l^p_{v_i^{(1)}}(\mathbb{N})}^{p-1} \\ &\leq c_1^{p-1} c \|g_i\|_{l^p(\mathbb{N})} \|g_i\|_{l^p(\mathbb{N})}^{p-1} \\ &= c_1^{p-1} c \|g_i\|_{l^p(\mathbb{N})}^p \end{aligned}$$

Hence,

$$\|I_\alpha g_j\|_{l^p_{v_i}(\mathbb{N})} \leq c_1^{1/p'} c^{1/p} \|g_j\|_{l^p(\mathbb{N})}.$$

□

Lemma 3.9. Let $0 < \alpha < 1$ and $1 < p < \infty$. Suppose that $\{I'_\alpha v_s\}_i < \infty$ and

$$\left\{ I'_\alpha [I'_\alpha v_s]^{p'} \right\}_i \leq c \left\{ I'_\alpha v_i \right\}_i$$

for all $i \in \mathbb{N}$. Then we have

$$\|I_\alpha\{g_i\}\|_{l^p_{v_i^{(1)}}(\mathbb{N})} \leq c \|g_i\|_{l^p(\mathbb{N})}, \quad g_i \in l^p(\mathbb{N}), \quad (6)$$

where $\{v_s^{(1)}\}_i = \{I'_\alpha v_s\}_i^{p'}$.

Proof. Let $g_i \geq 0$ and let g_i be supported on the set $E_m := \{i : 1 \leq i \leq m\}$, where m is a natural number. Let $t_{i,j}^{(n)} = \chi_{\{j:1 \leq j \leq i\}} \min\{(i-j+1)^{\alpha-1}, n\}$, $n \in \mathbb{N}$. Then using Lemma 3.7 (which is true also for the kernel $t_{i,j}^{(n)}$), Fubini's theorem and Hölder's inequality we obtain the following chain

of inequalities:

$$\begin{aligned}
\sum_{i=1}^{\infty} \left(\sum_{j=1}^i t_{i,j}^{(n)} g_j \right)^p v_i^{(1)} &\leq c \sum_{i=1}^{\infty} \left(\sum_{j=1}^i t_{i,j}^{(n)} \left(\sum_{k=1}^j t_{j,k}^{(n)} g_k \right)^{p-1} g_j \right) v_i^{(1)} \\
&\leq c \sum_{j=1}^{\infty} g_j \left(\sum_{k=1}^j t_{j,k}^{(n)} g_k \right)^{p-1} \left(\sum_{i=j}^{\infty} t_{i,j}^{(n)} v_i^{(1)} \right) \\
&\leq c \|g_i\|_{l^p(\mathbb{N})} \left(\sum_{j=1}^m \left(\sum_{k=1}^j t_{j,k}^{(n)} g_k \right)^p \left\{ I'_{\alpha} [I'_{\alpha} v_s]^{p'} \right\}_j^{p'} \right)^{1/p'} \\
&\leq c \|g_i\|_{l^p(\mathbb{N})} \left(\sum_{j=1}^m \left(\sum_{k=1}^j t_{j,k}^{(n)} g_k \right)^p \left\{ I'_{\alpha} v_s \right\}_j^{p'} \right)^{1/p'}.
\end{aligned}$$

Since $\sum_{k=1}^j t_{j,k}^{(n)} g_k < \infty$ and $\{I'_{\alpha} v_s\}_j < \infty$ for all j , therefore we have that

$$\left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^i t_{i,j}^{(n)} g_j \right)^p v_i^{(1)} \right)^{1/p} \leq c \|g_i\|_{l^p(\mathbb{N})}.$$

Passing now by m and n to $+\infty$ we derive (6). $\square$

Combining these lemmas we have also sufficiency of Theorem 2.2. Theorem 2.2 is completely proved.

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REFERENCES

- [1] D. Adams. A trace inequality for generalized potentials. *Studia Math.* **48**(1973), 99–105.
- [2] D. E. Edmunds, V. Kokilashvili and A. Meskhi, *Bounded and compact integral operators*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.
- [3] V. Kokilashvili and A. Meskhi, On a trace inequality for one-sided potentials and applications to the solvability of nonlinear integral equations. *Georgian Math. J.* **8**(2001), No. 3. 521–536.
- [4] V. G. Maz'ya and I. E. Verbitsky, Capacitary inequalities for fractional integrals, with applications to partial differential equations and Sobolev multipliers. *Ark. Mat.* **33**(1995), 81–115.

- [5] A. Meskhi, Solution of some weight problems for the Riemann–Liouville and Weyl operators, *Georgian Math. J.* **5**(1998), No. 6, 565–574.
- [6] A. Meskhi, On the boundedness and compactness criteria for the Riemann–Liouville type discrete operator. *Proc. A. Razmadze Math. Inst.* **126**(2001), 117–119.
- [7] D. V. Prokhorov, On the boundedness of a class of integral operators. *J. London Math. Soc.* **61**(2000), No.2, 617–628.