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ANALYSIS OF STEADY NON ISOTHERMAL TWO DIMENSIONAL FLOW OF SECOND GRADE FLUID IN A CONSTRICTED ARTERY

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Abstract. Steady analytical solution of non-isothermal, second grade fluid through an artery having constriction of cosine shape in two dimension is presented. The governing equations are transformed into stream function formulation which are solved analytically with the help of regular perturbation technique. The solutions thus obtained are presented graphically in terms of streamlines, wall shear stress, separation points, pressure gradient and temperature distribution. It is observed that an increase in height of constriction (ϵ) gives rise in wall shear stress, pressure gradient and temperature, whereas critical Reynolds number (R_e) decreases. Further an increase in second grade parameter (α) increases the temperature, pressure gradient, velocity and wall shear stress while critical R_e decreases. Its worthy to mention that the present results are compared with the already published results which ensures good agreement.

Keywords: Second grade fluid, heat transfer, wall shear stress, pressure distribution
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1. Introduction

The number and complexity of arterial plaques increase with age and with systemic risk factors but the rate of progression of individual plaques is variable. There is a complex and dynamic interaction between mechanical wall stress and atherosclerotic lesions. The blood flows in the closed circuit from the heart to arteries, arterioles, capillaries, venules, veins and then back to the heart and kept in continuous motion within the cardiovascular system. The natural flow of blood depends upon the pumping of the heart, this pumping

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of heart produces the oscillatory flow of blood in the arteries. The flow in the capillaries is characterized by low Reynolds number and large flexible particles. An in growth of tissue into the artery not only causes an increased resistance to flow, but it may also reduce the blood flow through the artery, of course the tissue may continue to grow until the artery is completely occluded.

Many authors studied the flow characteristic of blood by considering blood as a Newtonian as well as non-Newtonian fluid. Womersley [1] studied the flow of blood in an elastic artery considering the blood as a Newtonian fluid under simple harmonic pressure gradient and studied the frequency influence on flow rate. Newman et al. [2] presented a model to examine the oscillatory blood flow through the rigid artery numerically with mild constriction. Although blood is non-Newtonian at low shear rates Merrill [3], it can be treated as an incompressible Newtonian fluid at the flow rates encountered in the larger arteries where constriction commonly occur. Some authors studied the blood flow considering the blood as non-Newtonian fluid [4-10].

Constriction in the artery disturbs the flow of blood and becomes the cause for the diseases in the arteries and the hydrodynamic factor can play a significant role in the development and progression of these diseases. We are not interested in the actual cause of this constriction in the artery, but these constrictions effect the flow of blood in the arteries. One of the earliest paper in which the flow characteristics of blood in a constriction is studied is Young [11], who considered the flow in a mildly constricted artery based on a highly simplified linear model. Young [11] has suggested that once a lesion has developed there may be a coupling effect between its further development and the changed flow characteristics. The work of Young [11] was extended by Forrester and Young [12] to discuss the effect of flow separation on a mild constriction. Lee and Fung [13] solved the problem of flow of blood through constricted artery numerically. Morgan and Young [14] extend and modify the work of Forrester and Young [12] which is applicable to both the mild and severe constriction for Reynolds numbers below transition.

The goal of the present investigation was to predict analytically when and where separation of flow occurs for the constriction of given geometry along with heat transfer analysis. This analysis is concerning with the oscillatory blood flow through the locally constricted artery by using perturbation technique considering δ as a small parameter. The solution is applicable to both mild and severe constriction for Reynolds number below transition. The general approach is an extension and modification of the work by K. Haldar [15] and makes use of both the integral-momentum and integral-energy equation. Chow and Soda [16] presents the analytical solution for Newtonian fluid in an axisymmetric artery valid for the case where the spread of roughness is large compared with mean radius of the artery. Chow, Soda and Dean [17] analyze the steady laminar flow of Newtonian fluid for different physical quantities by considering the sinusoidal wall variation. Analytical solutions are obtained by considering the blood as a non-Newtonian fluid. At the end graphical results are presented which shows the effect of second grade parameters on wall shear stress, points of separation and reattachment and on temperature distribution. Fox and Hugh [18] investigate that in the arterial system static zones occur, which are due to separation of the main flow from the walls of the arteries.

2. Governing Equations

It is assumed that the blood behaves like a homogeneous, incompressible, non-isothermal and Non-Newtonian fluid of second grade. The governing equations are continuity, conservation of momentum and conservation of energy defined as follows

$$\widetilde{\nabla} \cdot \widetilde{\mathbf{V}} = 0, \tag{1}$$

$$\rho \frac{d\widetilde{\mathbf{V}}}{dt} = -\widetilde{\nabla}\widetilde{p} + div\widetilde{\tau} + \rho\widetilde{b}, \tag{2}$$

and

$$\rho c_p \frac{d\widetilde{T}}{dt} = \kappa \widetilde{\nabla}^2 \widetilde{T} + \phi, \tag{3}$$

where $\widetilde{\mathbf{V}}$ is the velocity vector, ρ the constant density, $\widetilde{p}$ the dynamic pressure, $\widetilde{b}$ the body force per unit mass and $\widetilde{\tau}$ the extra stress tensor, c_p, κ are respectively the specific heat and thermal conductivity, d/dt is the material time derivative defined as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \widetilde{\mathbf{V}} \cdot \widetilde{\nabla}. \tag{4}$$

and $\widetilde{T}$ is the temperature, ϕ the dissipation function defined as $\phi = \widetilde{\tau} \cdot \widetilde{\nabla}\widetilde{\mathbf{V}}$ and

$$\widetilde{\nabla}^2 = \frac{\partial^2}{\partial \widetilde{r}^2} + \frac{1}{\widetilde{r}} \frac{\partial}{\partial \widetilde{r}} + \frac{\partial^2}{\partial \widetilde{z}^2}, \tag{5}$$

is Laplacian.
The constitutive equation for the extra stress tensor $\widetilde{\tau}$ for second grade fluid is given as

$$\widetilde{\tau} = \mu \widetilde{\mathbf{A}}_1 + \alpha_1 \widetilde{\mathbf{A}}_2 + \alpha_2 \widetilde{\mathbf{A}}_1^2, \tag{6}$$

in which, μ is the coefficient of dynamic viscosity and α_1, α_2 are normal stress moduli for second grade fluid. The Rivlin-Ericksen tensors $\widetilde{\mathbf{A}}_1$ and $\widetilde{\mathbf{A}}_2$ are defined as

$$\widetilde{\mathbf{A}}_1 = \left(\widetilde{\nabla}\widetilde{\mathbf{V}}\right) + \left(\widetilde{\nabla}\widetilde{\mathbf{V}}\right)^*, \tag{7}$$

and

$$\widetilde{\mathbf{A}}_2 = \frac{d}{dt}\widetilde{\mathbf{A}}_1 + \widetilde{\mathbf{A}}_1 \left(\widetilde{\nabla}\widetilde{\mathbf{V}}\right) + \left(\widetilde{\nabla}\widetilde{\mathbf{V}}\right)^* \widetilde{\mathbf{A}}_1, \tag{8}$$

where superscript * stands for the transpose of the tensor. The velocity vector for steady axisymmetric flow in cylindrical coordinates is

$$\widetilde{\mathbf{V}} = \left(\widetilde{u}\left(\widetilde{r},\widetilde{z}\right),0,\widetilde{w}\left(\widetilde{r},\widetilde{z}\right)\right). \tag{9}$$

If we substitute (6) in (2) and making use of (7)-(8), we obtain the momentum equation in the absence of body forces of the form as [19]

$$\rho\frac{d\widetilde{\mathbf{V}}}{dt} = -grad\widetilde{p}+\mu\widetilde{\nabla}^2\widetilde{\mathbf{V}}+\left(\alpha_1+\alpha_2\right)\nabla\cdot\widetilde{\mathbf{A}}_1^2+\alpha_1\left[\widetilde{\nabla}^2\left(\widetilde{\nabla}\times\widetilde{\mathbf{V}}\right)\times\widetilde{\mathbf{V}}+grad\left\{\left(\widetilde{\mathbf{V}}\cdot\widetilde{\nabla}^2\widetilde{\mathbf{V}}\right)+\frac{1}{4}|\widetilde{\mathbf{A}}_1^2|\right\}\right], \tag{10}$$

where $|\widetilde{\mathbf{A}}_1^2|$ denote the norm of matrix and is given by

$$|\widetilde{\mathbf{A}}_1^2| = 4\left(\frac{\partial\widetilde{u}}{\partial\widetilde{r}}\right)^2+4\left(\frac{\partial\widetilde{w}}{\partial\widetilde{z}}\right)^2+4\left(\frac{\widetilde{u}}{\widetilde{r}}\right)^2+2\left(\frac{\partial\widetilde{u}}{\partial\widetilde{z}}+\frac{\partial\widetilde{w}}{\partial\widetilde{r}}\right)^2. \tag{11}$$

Component form of equations (1) and (10) by making use of (9) and (11), along with energy equation in dimensional form are

$$\frac{\partial\widetilde{u}}{\partial\widetilde{r}}+\frac{\widetilde{u}}{\widetilde{r}}+\frac{\partial\widetilde{w}}{\partial\widetilde{z}} = 0, \tag{12}$$

$$\frac{\partial\widetilde{h}}{\partial\widetilde{r}}-\rho\widetilde{w}\widetilde{\Omega} = -\mu\frac{\partial\widetilde{\Omega}}{\partial\widetilde{z}}+\frac{(\alpha_1+\alpha_2)}{\widetilde{r}}\left\{\widetilde{\Omega}^2+2\frac{\partial}{\partial\widetilde{z}}\left(\widetilde{u}\widetilde{\Omega}\right)\right\}-\alpha_1\widetilde{w}\left(\widetilde{\nabla}^2\widetilde{\Omega}-\frac{\widetilde{\Omega}}{\widetilde{r}^2}\right), \tag{13}$$

$$\frac{\partial\widetilde{h}}{\partial\widetilde{z}}+\rho\widetilde{u}\widetilde{\Omega} = \mu\left(\frac{\partial\widetilde{\Omega}}{\partial\widetilde{r}}+\frac{\widetilde{\Omega}}{\widetilde{r}}\right)-2\frac{(\alpha_1+\alpha_2)}{\widetilde{r}}\frac{\partial}{\partial\widetilde{r}}\left(\widetilde{u}\widetilde{\Omega}\right)+\alpha_1\widetilde{u}\left(\widetilde{\nabla}^2\widetilde{\Omega}-\frac{\widetilde{\Omega}}{\widetilde{r}^2}\right), \tag{14}$$

and equation (3) takes the form

$$\begin{aligned}
&\rho c_p\left(\widetilde{u}\frac{\partial}{\partial\widetilde{r}}+\widetilde{w}\frac{\partial}{\partial\widetilde{z}}\right)\widetilde{T} = \kappa\widetilde{\nabla}^2\widetilde{T}+\left\{2\mu+\alpha_1\left(\widetilde{u}\frac{\partial}{\partial\widetilde{r}}+\widetilde{w}\frac{\partial}{\partial\widetilde{z}}\right)\right\}\\
&\left\{\left(\frac{\partial\widetilde{u}}{\partial\widetilde{r}}\right)^2+\left(\frac{\widetilde{u}}{\widetilde{r}}\right)^2+\left(\frac{\partial\widetilde{w}}{\partial\widetilde{z}}\right)^2+\frac{1}{2}\left(\frac{\partial\widetilde{u}}{\partial\widetilde{z}}+\frac{\partial\widetilde{w}}{\partial\widetilde{r}}\right)^2\right\}\\
&+4(\alpha_1+\alpha_2)\left\{\left(\frac{\partial\widetilde{u}}{\partial\widetilde{r}}\right)^3+\left(\frac{\widetilde{u}}{\widetilde{r}}\right)^3+\left(\frac{\partial\widetilde{w}}{\partial\widetilde{z}}\right)^3-\frac{3\widetilde{u}}{4\widetilde{r}}\left(\frac{\partial\widetilde{u}}{\partial\widetilde{z}}+\frac{\partial\widetilde{w}}{\partial\widetilde{r}}\right)^2\right\},
\end{aligned} \tag{15}$$

where the modified pressure $\widetilde{h}$ and $\widetilde{\Omega}$ are given as follows

$$\widetilde{h} = \frac{\rho}{2}\left(\widetilde{u}^2+\widetilde{w}^2\right)+p-\alpha_1\left\{\widetilde{w}\left(\frac{\partial}{\partial\widetilde{r}}+\frac{1}{\widetilde{r}}\right)-\widetilde{u}\frac{\partial}{\partial\widetilde{z}}\right\}\widetilde{\Omega}-\frac{1}{4}\left(3\alpha_1+2\alpha_2\right)|\widetilde{A}_1^2| \quad \text{and} \quad \widetilde{\Omega} = \frac{\partial\widetilde{w}}{\partial\widetilde{r}}-\frac{\partial\widetilde{u}}{\partial\widetilde{z}}. \tag{16}$$

Eliminating modified pressure $\widetilde{h}$ from the equations (13) and (14), the compatibility equation is obtained in dimensional form as follows

$$\rho\left(\widetilde{u}\frac{\partial}{\partial\widetilde{r}}+\widetilde{w}\frac{\partial}{\partial\widetilde{z}}-\frac{\widetilde{u}}{\widetilde{r}}\right)\widetilde{\Omega}=\left\{\mu+\alpha_1\left(\widetilde{u}\frac{\partial}{\partial\widetilde{r}}+\widetilde{w}\frac{\partial}{\partial\widetilde{z}}-\frac{\widetilde{u}}{\widetilde{r}}\right)\right\}\left(\widetilde{\nabla}^2\widetilde{\Omega}-\frac{\widetilde{\Omega}}{\widetilde{r}^2}\right)$$
$$-\frac{2(\alpha_1+\alpha_2)}{\widetilde{r}}\left[\left(\frac{\partial^2}{\partial\widetilde{r}^2}-\frac{1}{\widetilde{r}}\frac{\partial}{\partial\widetilde{r}}+\frac{\partial^2}{\partial\widetilde{z}^2}\right)(\widetilde{u}\widetilde{\Omega})+\frac{1}{2}\frac{\partial}{\partial\widetilde{z}}(\widetilde{\Omega}^2)\right]. \tag{17}$$

Now the boundary conditions for current problem on velocity components according to the geometry are

$$\begin{aligned}&\widetilde{u}=\widetilde{w}=0, \quad at \quad \widetilde{r}=R(\widetilde{z})\\&\frac{\partial\widetilde{w}}{\partial\widetilde{r}}=0, \quad at \quad \widetilde{r}=0\\&\widetilde{Q}=\int_0^{R(\widetilde{z})}\widetilde{r}\widetilde{w}d\widetilde{r}=\frac{1}{2}u_0R_o^2,\end{aligned} \tag{18}$$

and on temperature in dimensional form are

$$\begin{aligned}&\widetilde{T}=T_1 \quad at \quad \widetilde{r}=R(\widetilde{z})\\&\frac{\partial\widetilde{T}}{\partial\widetilde{r}}=0 \quad at \quad \widetilde{r}=0.\end{aligned} \tag{19}$$

3. Problem Formulation:

It is assumed that the blood behaves like a homogeneous, non-isothermal, incompressible, non-Newtonian fluid of second grade and the flow field is independent of time. At the inlet and outlet sections of the artery, the flow is assumed to be the Poiseuille or fully developed flow. Consider the blood flow in an artery with symmetric constriction of cosine shape as [15] having radius of unobstructed region is R_o and $R(z)$ is the variable radius of the constricted region, z-axis is assumed to be the axial axis of the artery and r-axis normal to it as

$$\begin{aligned}R(\widetilde{z})&=R_o-\frac{\lambda}{2}\left(1+\cos\left(\frac{4\pi\widetilde{z}}{l_o}\right)\right), \quad -\frac{l_o}{4}\le\widetilde{z}\le\frac{l_o}{4},\\&=R_o \quad \text{otherwise},\end{aligned} \tag{20}$$

where λ is the maximum height of constriction and $l_o/2$ is the length of constricted region. Introducing the dimensionless quantities of the form

$$u=\frac{\widetilde{u}}{u_o}, \quad w=\frac{\widetilde{w}}{u_o}, \quad r=\frac{\widetilde{r}}{R_o}, \quad z=\frac{\widetilde{z}}{l_o}, \quad h=\frac{R_o^2}{\mu u_o l_o}\widetilde{h}, \quad \theta=\frac{\widetilde{T}-T_o}{T_1-T_o}, \tag{21}$$

where $\widetilde{*}$ denotes the dimensional variables, μ is dynamic viscosity, h is modified pressure, u_o is the characteristic velocity, T_1 and T_o represents the temperature

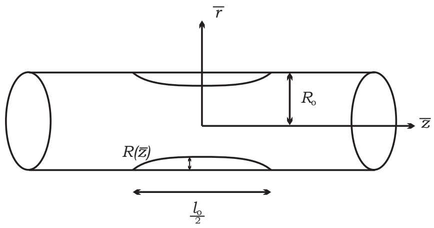


Fig.1 Geometry of the constricted tube.

on the boundary and blood respectively.
Dimensionless form of the boundary profile by using (21) is

$$\begin{aligned} f(z) &= 1 - \frac{\epsilon}{2}(1+\cos 4\pi z), \quad -\frac{1}{4} < z < \frac{1}{4}, \\ &= 1 \quad \text{otherwise}, \end{aligned} \tag{22}$$

where $\epsilon = \lambda/R_o$ is dimensionless height of constriction and $f = R/R_o$. The governing equations for second grade fluid in two dimensions are highly non-linear in two variables i,e. u and w. Introducing the stream functions of the form

$$u = \frac{\delta}{r}\frac{\partial\psi}{\partial z}, \quad w = -\frac{1}{r}\frac{\partial\psi}{\partial r}, \tag{23}$$

which satisfy the continuity equation identically and component form of the momentum equation in terms of stream function takes the form

$$\begin{aligned} \frac{\partial h}{\partial r} - \frac{\delta R_e}{r^2}\frac{\partial\psi}{\partial r}\left(E^2\psi\right) =& \frac{\delta^2}{r}\frac{\partial\left(E^2\psi\right)}{\partial z} + \frac{\delta(\alpha+\beta)}{r^3}\left(\left(E^2\psi\right)^2 - 2\delta^2\frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial z}E^2\psi\right)\right) \\ & - \frac{\alpha\delta}{r}\frac{\partial\psi}{\partial r}\left(\nabla^2 - \frac{1}{r^2}\right)\frac{E^2\psi}{r}, \end{aligned} \tag{24}$$

$$\begin{aligned} \frac{\partial h}{\partial z} - \frac{R_e\delta}{r^2}\frac{\partial\psi}{\partial z}\left(E^2\psi\right) =& -\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\left(\frac{1}{r}E^2\psi\right) + 2\frac{(\alpha+\beta)}{r}\frac{\partial}{\partial r}\left(\frac{\delta}{r^2}\frac{\partial\psi}{\partial z}\left(E^2\psi\right)\right) \\ & - \frac{\alpha\delta}{r}\frac{\partial\psi}{\partial z}\left(\nabla^2 - \frac{1}{r^2}\right)\left(\frac{1}{r}E^2\psi\right), \end{aligned} \tag{25}$$

where the modified pressure h is defined as

$$h = \frac{R_e\delta}{2}\left(u^2+w^2\right) + p - \alpha\delta\left\{w\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) - \delta u\frac{\partial}{\partial z}\right\}\Omega - \frac{\delta}{4}\left(3\alpha+2\beta\right)|A_1^2|, \tag{26}$$

and $|\mathbf{A}_1^2|$ is given by

$$|\mathbf{A}_1^2| = 4\left(\frac{\partial}{\partial r}\left(\frac{\delta}{r}\frac{\partial\psi}{\partial z}\right)\right)^2 + 4\left(\frac{\partial}{\partial z}\left(-\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right)^2 + 4\left(\frac{\delta}{r^2}\frac{\partial\psi}{\partial z}\right)^2 + 2\left(\frac{\partial}{\partial z}\left(\frac{\delta}{r}\frac{\partial\psi}{\partial z}\right) + \frac{\partial}{\partial r}\left(-\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right)^2. \tag{27}$$

Eliminating modified pressure h from (24) and (25) by cross differentiation to obtain the compatibility equation in terms of stream function of the form

$$R_e\delta\frac{\partial\left(\psi,\frac{E^2\psi}{r^2}\right)}{\partial(r,z)} + \frac{1}{r}E^4\psi = \alpha\delta\frac{\partial\left(\psi,\frac{E^4}{r^2}\psi\right)}{\partial(r,z)} + \frac{2\delta(\alpha+\beta)}{r}\left\{E^2\left(\frac{\partial\psi}{\partial z}\frac{E^2\psi}{r^2}\right) - \frac{E^2\psi}{r^2}\frac{\partial}{\partial z}\left(\frac{E^2\psi}{r^2}\right)\right\}, \tag{28}$$

and energy equation in terms of stream function becomes

$$\begin{aligned}
\frac{P_e\delta}{r}\frac{\partial(\psi,\theta)}{\partial(z,r)} &= \nabla^2\theta + B_r\left\{2 + \delta\frac{\alpha}{r}\left(\frac{\partial\psi}{\partial z}\frac{\partial}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial}{\partial z}\right)\right\} \\
&\left\{\delta^2\left(\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)\right)^2 + \left(\frac{1}{r^2}\frac{\partial\psi}{\partial z}\right)^2 + \left(\frac{1}{r}\frac{\partial^2\psi}{\partial r\partial z}\right)^2\right) + \frac{1}{2}\left(\frac{\delta^2}{r}\frac{\partial^2\psi}{\partial z^2} - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right)^2\right\} \\
&+ 4B_r(\alpha+\beta)\left\{\delta^3\left(\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)\right)^3 + \left(\frac{1}{r^2}\frac{\partial\psi}{\partial z}\right)^3 + \left(\frac{1}{r}\frac{\partial^2\psi}{\partial r\partial z}\right)^3\right)\right. \\
&\left. + \frac{3\delta}{4r^2}\frac{\partial\psi}{\partial z}\left(\frac{\delta^2}{r}\frac{\partial^2\psi}{\partial z^2} - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right)^2\right\}.
\end{aligned} \tag{29}$$

Boundary conditions in terms of stream functions reduces as

$$\begin{aligned}
&-\frac{1}{r}\frac{\partial\psi}{\partial r} = 0, \quad \psi = -\frac{1}{2}, \quad \theta = 1 \quad \text{at} \quad r = f, \\
&-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) = 0, \quad \psi = 0, \quad \frac{\partial\theta}{\partial r} = 0 \quad \text{at} \quad r = 0,
\end{aligned} \tag{30}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \delta^2\frac{\partial^2}{\partial z^2}, \quad \Omega = \frac{\partial w}{\partial r} - \delta\frac{\partial u}{\partial z}, \quad E^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \delta^2\frac{\partial^2}{\partial z^2}, \tag{31}$$

and

$$\alpha = \frac{\alpha_1 u_o}{\mu R_o}, \quad \beta = \frac{\alpha_2 u_o}{\mu R_o}, \quad \delta = \frac{R_o}{l_o}, \quad R_e = \frac{u_o R_o}{\nu}, \quad B_r = \frac{\mu u_o^2}{\kappa(T_1 - T_o)}, \quad P_e = \frac{\rho c_p R_o u_o}{\kappa}. \tag{32}$$

Now our task is to find the solution in terms of stream function (ψ) and theta (θ). Once the stream function is obtained, one can easily find the expressions for velocity components.

4. Solution:

Regular perturbation technique is applied on ψ and θ to solve the compatibility equation (28) and energy equation (29) along with boundary conditions (30) by considering δ as the small parameter of the form

$$\begin{aligned} \psi &= \psi_o + \delta\psi_1 + \delta^2\psi_2 + \cdots, \\ \theta &= \theta_o + \delta\theta_1 + \delta^2\theta_2 + \cdots. \end{aligned} \tag{33}$$

We obtained the system of equations after substituting equation (33) into equations (28)-(30) and equating the coefficients of δ^0, δ and δ^2. The systems obtained from these equations are named as zeroth, first and second order system. From the first systems, we have

$$\psi_o = \frac{\eta^2}{2}\left(\eta^2 - 2\right), \quad \text{where} \quad \eta = \frac{r}{f}, \tag{34}$$

which is zeroth order solution and similar as [15,16].
The zeroth order temperature is obtained by making use of ψ_o and corresponding boundary conditions on temperature as follows

$$\theta_o = 1 - \frac{B_r}{f^4}\left(\eta^4 - 1\right). \tag{35}$$

We observed that zeroth order temperature is independent of second grade parameters and depends upon the ratio of heat production by dissipation to heat transport by conduction. The first order solution is found by using ψ_o along with the boundary conditions as follows

$$\psi_1 = \frac{R_e f' \eta^2}{36 f}\left(\eta^6 - 6\eta^4 + 9\eta^2 - 4\right), \tag{36}$$

it is observed that the first order solution is similar as viscous fluid [15,16] and independent of second grade parameter. First order temperature is obtained by using ψ_o, ψ_1, θ_o and boundary conditions on temperature for first order as follows

$$\begin{aligned} \theta_1 = &- \frac{B_r f'(\eta^2 - 1)}{72 f^7}\left[48(2\alpha + \beta)\left(4\eta^4 - 5\eta^2 - 5\right) + f^2\left\{4R_e\left(3\eta^6 - 13\eta^4 + 5\eta^2 + 5\right)\right.\right. \\ &\left.\left. + P_e\left(9\eta^6 - 7\eta^4 - 43\eta^2 + 101\right)\right\}\right]. \end{aligned} \tag{37}$$

One can recover the first order temperature for viscous solution easily by substituting $\alpha = \beta = 0$. First order temperature also depends upon heat transport by convection to conduction. Similarly second order solution is obtained by integrating and making use of ψ_o, ψ_1 subject to the boundary conditions as

$$\psi_2 = \frac{\eta^2(\eta^2-1)^2}{21600f^4}\left[f'^2\left\{-120R_e\left\{\alpha(31\eta^4-138\eta^2+233)+10\beta(\eta^4-4\eta^2+3)\right\}+R_e^2f^2\left(38\eta^6 -314\eta^4+759\eta^2-818\right)-18000f^4\right\}+3ff''\left\{40R_e\alpha(3\eta^4-14\eta^2+29)-R_e^2f^2\left(2\eta^6 -16\eta^4+41\eta^2-52\right)+1200f^4\right\}\right]. \tag{38}$$

The second order temperature is found by using the expressions for ψ_o, ψ_1, ψ_2 and θ_o, θ_1 along with the boundary conditions of the form

$$\theta_2 = \frac{B_r}{6350400f^8}\left[49f'^2(1-\eta^2)\left\{36\left\{21P_e(2\alpha+\beta)\left(64\eta^8-216\eta^6+139\eta^4+639\eta^2-1361\right) + 2R_e\left\{\alpha\left(912\eta^8-5538\eta^6+2064\eta^4+22\eta^2+22\right)+\beta\left(728\eta^8-4447\eta^6+6553\eta^4-2047\eta^2 -2047\right)\right\}\right\}+5f^2\left\{225P_e^2\left(\eta^{10}-3\eta^8-8\eta^6+72\eta^4-173\eta^2+231\right)+5P_eR_e\left(88\eta^{10}-632\eta^8 +1069\eta^6+141\eta^4-2055\eta^2+2697\right)+4R_e^2\left(156\eta^{10}-1284\eta^8+3441\eta^6-3697\eta^4+815\eta^2 +815\ \right)\right\}-43200f^4\left(2\eta^4+18\eta^2-33\right)\right\}-13824f^4f'^3\left(1125\eta^{10}-1284\eta^8+3441\eta^6-3697\eta^4 +815\eta^2-292\right)+49f''(\eta^2-1)\left\{36\left\{3P_e(2\alpha+\beta)\left(64\eta^8-261\eta^6+139\eta^4+639\eta^2-1361\right) +2R_e\left\{2\alpha\left(8\eta^8-67\eta^6-267\eta^4+73\eta^2+73\right)+3\beta\left(8\eta^8-67\eta^6+133\eta^4-67\eta^2-67\right)\right\}\right\} + f^2\left\{225P_e^2\left(\eta^{10}-3\eta^8-8\eta^6+72\eta^4-173\eta^2+231\right)+12R_e^2\left(20\eta^{10}-172\eta^8+503\eta^6 -697\eta^4+173\eta^2+173\right)+P_eR_e\left(200\eta^{10}-1672\eta^8+3953\eta^6-1647\eta^4-6147\eta^2+11853\right)\right\} -14400f^4(10\eta^4-2\eta^2-11)\right\}+4608f'f''f^5\left(675\eta^7-1764\eta^5+1225\eta^3-136\right)\right], \tag{39}$$

which is general solution for second order temperature, we can recover the viscous second order solution by setting $\alpha = \beta = 0$. Now we can easily find the velocity components u, v and temperature distribution θ by using (33) and (23).

5. Pressure Distribution:

In this section the modified pressure and pressure are calculated by applying the perturbation technique of the form

$$\begin{aligned} h &= h_o + \delta h_1 + \delta^2 h_2 + \cdots, \\ p &= p_o + \delta p_1 + \delta^2 p_2 + \cdots. \end{aligned} \tag{40}$$

Using equation (40) in equations (24)-(26) along with perturb form of ψ. The different orders of pressure are obtained by the relation:

$$(*) = \int_0^z \frac{\partial(*)}{\partial z}dz + \int_0^r \frac{\partial(*)}{\partial r}dr. \tag{41}$$

Equating the coefficients of δ^0, δ, δ^2 on both sides of (24)- (26). The zeroth order pressure gives the solution as follows

$$\begin{aligned} h_o = p_o =& \frac{1}{12\pi}\left[\frac{3}{(\epsilon-1)^{\frac{7}{2}}}\left(5\epsilon^3-18\epsilon^2+24\epsilon-16\right)\tanh^-1\left(\frac{\tan 2\pi z}{\sqrt{\epsilon-1}}\right)\right.\\ &\left.+\frac{\epsilon\sin 4\pi z}{2f^3(\epsilon-1)^3}\left\{8(\epsilon-1)^2+10f(\epsilon-1)(\epsilon-2)+f^2\left(15\epsilon^2-44(\epsilon-1)\right)\right\}\right]. \end{aligned} \tag{42}$$

It is observed that the second grade parameters α, β are absent from zeroth order modified pressure and the zeroth order pressure. Solutions for first order modified pressure and pressure are obtained by applying (41) of the form

$$\begin{aligned} h_1 =& \frac{1}{3f^6}\left\{16(\alpha+\beta)(3\eta^2-2)+3R_ef^2(4\eta^4-8\eta^2+1)\right\}\\ &+\frac{1}{3(\epsilon-1)^8}\left\{8(\alpha+\beta)\left(2(\epsilon-1)^2-3\eta^2f^2\right)-3R_e\left((\epsilon-1)^4-4\eta^2f^2(\epsilon-1)^2+2\eta^4f^4\right)\right\}. \end{aligned} \tag{43}$$

$$\begin{aligned} p_1 =& \frac{1}{f^6}\left\{8\alpha\left(7\eta^2-\frac{8}{3}\right)+16\beta\left(2\eta^2-\frac{1}{3}\right)-3R_ef^2\right\}\\ &-\frac{1}{3(\epsilon-1)^8}\left\{3R_e\left(2\eta^4f^4-4(\epsilon-1)^2\eta^2f^2+(\epsilon-1)^4\right)+8(\alpha+\beta)\left(3\eta^2f^2-2(\epsilon-1)^2\right)\right\}, \end{aligned} \tag{44}$$

it is found that the first order pressure for viscous fluid is obtained by setting $\alpha=\beta=0$. Similarly second order modified pressure and pressure in integral form are given as follows

$$\begin{aligned} h_2 =& \frac{2\eta^2 f'}{9f^7}\left\{-4R_e\left(\alpha(10\eta^4-27\eta^2+27)-\beta(2\eta^4-9\eta^2+9)\right)+R_e^2f^2(2\eta^6-11\eta^4+18\eta^2-11)\right.\\ &\left.-36f^4\right\}+\frac{1}{270}\int_0^z\frac{1}{f^8}\left[f'^2\left\{48R_e\left(\alpha(650\eta^6-1485\eta^4+1215\eta^2-182)-5\beta\left(26\eta^6-99\eta^4\right.\right.\right.\right.\\ &\left.\left.+81\eta^2-14)\right)-R_e^2f^2(1560\eta^8-7260\eta^6+9720\eta^4-4620\eta^2+479)+360f^4(30\eta^2-11)\right\}\\ &+3ff''\left\{-16R_e\left(\alpha(50\eta^6-135\eta^4+135\eta^2-26)-5\beta(2\eta^6-9\eta^4+9\eta^2-2)\right)\right.\\ &\left.\left.+R_e^2f^2(40\eta^8-220\eta^6+360\eta^4-220\eta^2+29)-120f^4(6\eta^2-1)\right\}\right]dz, \end{aligned} \tag{45}$$

$$
\begin{aligned}
p_2 =& \frac{2f'}{9f^7}\left\{4R_e\left(\alpha(28\eta^6-90\eta^4+54\eta^2-11)+\beta\eta^2(14\eta^4-45\eta^2+27)\right)\right.\\
&\left.-R_e^2f^2(2\eta^8-11\eta^6+18\eta^4-11\eta^2+4)-36\eta^2f^4\right\}+\frac{1}{270}\int_0^z\frac{1}{f^8}\\
&\left[f'^2\left\{48R_e\left(\alpha(650\eta^6-1485\eta^4+1215\eta^2-182)-5\beta(26\eta^6-99\eta^4+81\eta^2-14)\right)\right.\right.\\
&\left.-R_e^2f^2(1560\eta^8-7260\eta^6+9720\eta^4-4620Y^2+479)+360f^4(30\eta^2-11)\right\}\\
&+3ff''\left\{-16R_e\left(\alpha(50\eta^6-135\eta^4+135Y^2-26)-5\beta(2\eta^6-9\eta^4+9\eta^2-2)\right)\right.\\
&\left.\left.+R_e^2f^2(40\eta^8-220\eta^6+360\eta^4-220\eta^2+29)-120f^4(6\eta^2-1)\right\}\right]dz.
\end{aligned}
\tag{46}
$$

Setting $\alpha=\beta=0$ gives second order pressure for viscous fluid.
Wall shear stress for the second grade fluid is obtained from the component of extra shear stress as follows

$$
\begin{aligned}
\tau_\omega =& -\frac{4}{f^3}+\frac{2\delta f'}{3f^6}(R_ef^2-48\alpha)+\frac{\delta^2}{f^7}\left\{f'^2\left(\frac{244R_e\alpha}{15}+\frac{67R_e^2f^2}{540}+\frac{8f^4}{3}\right)\right.\\
&\left.-ff''\left(\frac{4R_e\alpha}{5}+\frac{R_e^2f^2}{36}+\frac{4f^4}{3}\right)\right\}.
\end{aligned}
\tag{47}
$$

The points of separation and reattachment at the wall are calculated by setting $\tau_\omega=0$, which gives us quadratic equation in terms of R_e. The solution for R_e is

$$
R_e=\frac{4}{\delta f^2G}\left[F\pm\sqrt{F^2-f^2G(135f^4+4320\alpha\delta ff'-90\delta^2f^4f'^2+45\delta^2f^5f'')}\right], \tag{48}
$$

where

$$
\begin{aligned}
F&=45f^3f'+1098\alpha\delta f'^2-54\alpha\delta ff'',\\
G&=15ff''-67f'^2.
\end{aligned}
\tag{49}
$$

Equation (48) gives the critical value of R_e which gives the separation and reattachment points.

6. Graphical Discussions :

In this section solutions are presented graphically for stream lines, wall shear stress, zero wall shear stress, temperature distribution and pressure gradient. Analysis are presented numerically for second grade parameters (α,β), height of constriction (ϵ), Reynolds number (R_e), Brinkman number (B_r) and Pecket number (P_e). In figure 2 behavior of stream lines are shown for zeroth order 2(a), first order 2(b), second order 2(c) and up to second order 2(d) respectively, for $R_e=12,\epsilon=0.2,\delta=0.1,\alpha=0.04,\beta=0.02$. In these figures $z-axis$ lies in the horizontal direction and $r-axis$ is perpendicular to it. The

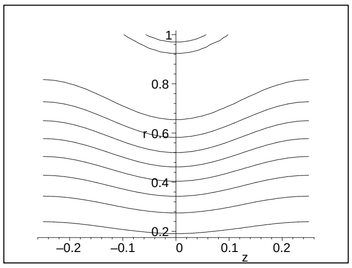


Fig.2(a)Zeroth order stream lines.

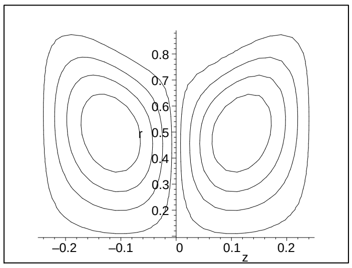


Fig. 2(b) First order stream lines.

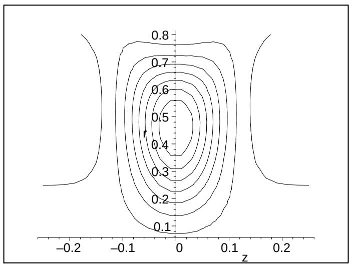


Fig.2(c) Second order stream lines.

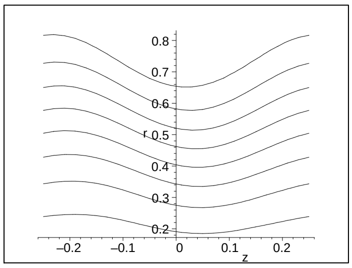
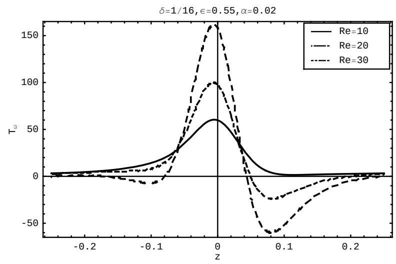


Fig. 2(d) Up to second order stream lines.

zeroth order solution corresponds to laminar flow, first order solution induces the clockwise and counterclockwise rotational motion in the converging and diverging regions, which is prediction of separation and reattachment points. Figure 2(c) shows the stream lines for second order solution, which also shows the rotational motion and indicates the presences of separation and reattachment points. Figure 2(d) presents the stream line solution up to second order. By setting $\alpha = 0$, stream lines presents Newtonian behavior which are similar to [17]. The distribution of wall shear stress for R_e is presented in figure 3. It is found that as we increases the R_e wall shear stress becomes negative in the converging region and then increases near the throat of the constricted region and becomes negative in the diverging region. The negative shearing in the converging and diverging sections of the artery predicts the reverse flow and indicates the points of separation in the upstream region and reattachment in the downstream region of the artery.

Figure 4 predicts the effect of ϵ on wall shear stress. The straight line indicates that there is no constriction and the flow is Poiseuille flow. It is observed that as we increases the ϵ, wall shear stress increases near the throat of the artery and becomes negative in the diverging section of the artery, which predicts the point of reattachment. The separation point was considered to be the point nearest the throat where a reversed flow along the wall of the artery could be observed. The point farthest down stream from the throat where back flow occurs is defined as the reattachment point. Figure 5 presents the



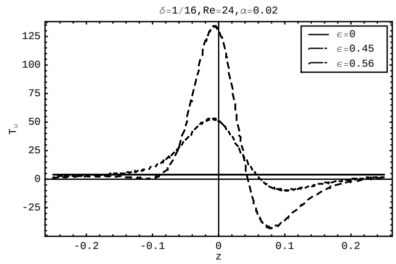
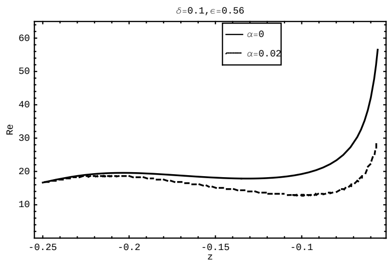


Fig.3 Effect of R_e on wall shear stress. Fig.4 Effect of ϵ on wall shear stress.



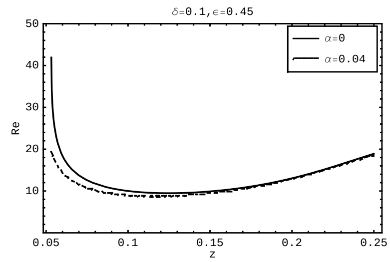
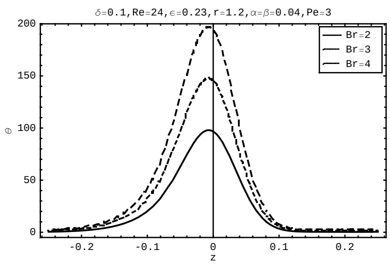


Fig.5 separation point for α in converging region. Fig.6 Reattachment point for α in diverging region.

distribution of separation point in the converging section of the artery for the different values of α along with the fixed values of δ and ϵ. It is noted that the separation point lie to the right of the minimum points, actually the purpose for the zero wall shear stress is to find the critical Reynolds number where the separation occurs. It is observed from figure 5 that the critical Reynolds number decreases as ϵ increases.

In figure 6 zero wall shear stress is plotted for α in diverging section of the artery. The purpose is to determine the critical value of R_e at which reattachment occurred in the diverging region. It is noted that as the critical Reynolds number reached the reattachment occurs in the diverging region. It is found form figure 6 that as α increases the critical Reynolds number decreases.

To study the behavior of the temperature distribution numerically for different values of α, ϵ , B_r, P_e, calculations are carried out through graphs. Figure 7 shows the behavior of temperature for distinct values of B_r. It is found that as the B_r number increases, the temperature rises high over the constriction for the fixed values of the other parameters. Figure 8 predicts the effect of ϵ over the pressure gradient for fixed values of R_e, δ and second grade parameters. It is observed that as ϵ increases pressure gradient increases. The straight line indicates the absence of constriction in the artery.



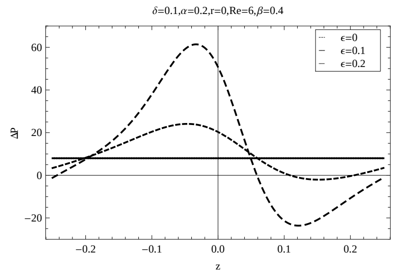


Fig.7 Temperature distribution for B_r. Fig.8 Pressure distribution for ϵ

7. Conclusion:

In this article consideration has been given to the second grade steady state flow of blood through an artery of infinite length with heat transfer having constriction of length $l_o/2$. Underlying problem is solved with the help of approximate analytical technique, namely, regular perturbation technique. In the current investigation second grade fluid is analyzed for flow pattern, pressure gradient, wall shear stress, separation point, temperature distribution and draw the graphs for each. We note that by setting $\alpha = \beta = 0$, the results obtained are similar to viscous fluid. It is also observed that the general pattern of streamlines is same as [16 - 17], wall shear stress is similar as [14 - 15] and separation and reattachment points identical with [15]. It is noted that:

- As we increase R_e, wall shear stress and pressure gradient increases.
- Increase in ϵ increases the wall shear stress, pressure gradient and temperature.
- Critical R_e decreases as the ϵ increases.
- Increase in α increases the temperature, pressure gradient, velocity and wall shear stress.
- Temperature increases with the increase in B_r and P_e.
- Critical R_e decreases in the converging and diverging regions with the increase in second grade parameter α.

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