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CONSTRUCTION OF OPTIMAL DERIVATIVE FREE ITERATIVE METHODS FOR NONLINEAR EQUATIONS USING LAGRANGE INTERPOLATION

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Abstract. In this paper, we present a general family of optimal derivative free iterative methods of arbitrary high order for solving nonlinear equations by using Lagrange interpolation. The special cases of this family with optimal order of convergence two, four, eight and sixteen are obtained. These methods do not need the Newton's or Steffensen's iterations in the first step of their iterative schemes. The advantage of the new schemes is that they are also extendable to the iterative methods with-memory. Numerical experiments and polynomiographs are presented to confirm the theoretical results and to compare the new iterative methods with other well known methods of similar kind.

Key words: Nonlinear equation, iterative methods, polynomiograph.
MSC: 65H04, 65H05.

1. Introduction

A large number of problems in different fields of engineering and science require to find the solution of a nonlinear equation. In this paper, we consider the problem of solving nonlinear equations numerically [1–3]. Multi-step iterative methods for this problem have been extensively studied in the last few decades as they are computationally efficient than the one-step methods such as the methods of Newton, Halley and Laguerre.

Following is the iteration of Newton's scheme to find a simple root α of a nonlinear equation $f(x) = 0$, where $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is a scalar function on an

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open interval D [3]:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n \geq 0. \tag{1}$$

Newton's method requires one functional and one derivative evaluation for one iteration. But this method is sensitive regarding the choice of initial approximation. It may not converge to real root if the initial approximation does not lie in the vicinity of the root and also if $f'(x)$ is zero in neighborhood of real root. Steffensen's iterative scheme is a well known modification of the Newton's method obtained by using the approximation

$$f'(x_n) \approx \frac{f(x_n) - f(u_n)}{x_n - u_n} = f[x_n, u_n], \tag{2}$$

in the Newton's scheme and is given as follows [4]:

$$x_{n+1} = x_n - \frac{f(x_n)}{f[u_n, x_n]}, n \geq 0, \tag{3}$$

where, $u_n = x_n + f(x_n)$. Both of the above schemes are quadratic in some neighborhood of the root α but Steffensen's method has an advantage that it does not need the evaluation of function's derivative which may be problematic and expensive to calculate for certain functions. There is a vast literature on optimal multi-step methods, which are developed by using the famous one-step Newton's method or the Steffensen's method at the first step.

To determine the efficiency of an iterative method, Ostrowski [2] defined the efficiency index as $q^{1/n}$, where q is the convergence order and n is the number of function evaluations per iterative step. Kung and Traub conjectured in [5] that the order of convergence of any multi-step method requiring $n+1$ function evaluations cannot exceed the bound 2^n. The methods that satisfy this bound are called optimal methods. For a background study of multi-step optimal methods for finding simple roots, one may consult the books of Traub and Petkovic et al. [3, 6].

In 2011, Geum and Kim [7] presented an optimal four-step iterative scheme given as follows:

$$\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, n \geq 0, \\
r_n &= y_n - K_1(u_n)\frac{f(y_n)}{f'(x_n)}, \\
s_n &= r_n - K_2(u_n, v_n, w_n)\frac{f(r_n)}{f'(x_n)}, \\
x_{n+1} &= s_n - K_3(u_n, v_n, w_n, t_n)\frac{f(s_n)}{f'(x_n)}
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
K_1(u_n) &= \frac{1+\rho u_n+(-9+5\rho/2)u_n^2}{1+(\rho-2)u_n+(-4+\rho/2)u_n^2},\\
K_2(u_n,v_n,w_n) &= \frac{1+2u_n+(2+\xi)w_n}{1-v_n+\xi w_n},\\
K_3(u_n,v_n,w_n,t_n) &= \frac{1+2u_n+(2+\xi)v_nw_n}{1-v_n-2w_n-t_n+2(1+\xi)v_nw_n}-\frac{1}{2}u_nw_n[6+12u_n\\
&\quad +u_n^2(24-11\rho)+u_n^3(11\rho^2-66\rho+136)+4\xi]\\
&\quad +(2u_n(\xi^2-2\xi-9)-4\xi-6)w_n^2.
\end{aligned}$$

In 2013, Cordero et al. [8] developed a family of optimal derivative free iterative methods by using polynomial interpolation given as follows:

$$\begin{aligned}
y_0 &= x_k, k\geq 0,\\
y_1 &= y_0+f(y_0),\\
x_{k+1}=y_{k+1} &= y_j-\frac{f(y_j)}{p_j'(y_j)}, j=1,2,...,n, \qquad (5)
\end{aligned}$$

where p_j is the polynomial that interpolates f in $y_0, y_1, ..., y_j$. A special case of their family as a three-step scheme is given as follows:

$$\begin{aligned}
y_0 &= x_k, k\geq 0,\\
y_1 &= y_0+f(y_0),\\
y_2 &= y_0-\frac{f(y_0)^2}{f(y_1)-f(y_0)},\\
y_3 &= y_2-\frac{f(y_2)}{\frac{y_1-y_2}{y_1-y_0}f[y_0,y_2]+\frac{y_0-y_2}{y_0-y_1}f[y_1,y_2]},\\
x_{n+1} &= y_4=y_3-\frac{f(y_3)}{p_3'(y_3)}, \qquad (6)
\end{aligned}$$

where,

$$\begin{aligned}
p_3'(y_3) &= \frac{(y_1-y_3)(y_2-y_3)}{(y_1-y_0)(y_2-y_0)}f[y_0,y_3]+\\
&\quad \frac{(y_0-y_3)(y_2-y_3)}{(y_0-y_1)(y_2-y_1)}f[y_1,y_3]+\\
&\quad \frac{(y_0-y_3)(y_1-y_3)}{(y_0-y_2)(y_1-y_2)}f[y_2,y_3].
\end{aligned}$$

In 2016, Nazeer et al. [9] proposed a generalized Newton-Raphson's method free from second derivative. Again in 2016, Kang et al. [10] presented an iterative method corresponding to Simpson's 1/3 rule with it's polynomiographs.

In 2017, Nazeer et al. [11] developed an iterative method of ninth-order for nonlinear equations along with it's polynomiographs.

In 2019, Saba et al. [12] presented a modified Abbasbandy's method free from second derivative for solving nonlinear equations. Also in 2019, Junjua et al. [13] presented a general family of derivative free root finding method based on inverse interpolation given as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f[z_n, x_n]}, z_n = x_n + f(x_n)^4, n \geq 0, \\ w_n &= y_n + h_2 f(x_n)^2, \\ t_n &= y_n + b_3 f(x_n)^2 - g_3 f(x_n)^3, \\ x_{n+1} &= y_n + b_4 f(x_n)^2 - g_4 f(x_n)^3 + g_5 f(x_n)^4, \end{aligned} \tag{7}$$

where

$$\begin{aligned} h_2 &= \frac{1}{[f(y_n) - f(x_n)]f[y_n, x_n]} - \frac{1}{[f(y_n) - f(x_n)]f[z_n, x_n]}, \\ g_3 &= \frac{1}{[f(y_n) - f(x_n)][f(y_n) - f(w_n)]f[y_n, x_n]} \\ &\quad - \frac{1}{[f(w_n) - f(x_n)][f(y_n) - f(w_n)]f[w_n, x_n]} \\ &\quad + \frac{1}{[f(w_n) - f(x_n)][f(y_n) - f(w_n)]f[z_n, x_n]} \\ &\quad - \frac{1}{[f(y_n) - f(x_n)][f(y_n) - f(w_n)]f[z_n, x_n]}, \\ b_3 &= \frac{1}{[f(y_n) - f(x_n)]f[y_n, x_n]} - \frac{1}{f[z_n, x_n][f(y_n) - f(x_n)]} \\ &\quad - g_3[f(y_n) - f(x_n)]. \end{aligned} \tag{8}$$

and

$$\begin{aligned} g_5 &= \frac{\frac{\varphi_t - \varphi_w}{[f(t_n) - f(w_n)]} - \frac{\varphi_y - \varphi_w}{[f(y_n) - f(w_n)]}}{[f(t_n) - f(y_n)]}, \\ g_4 &= \frac{\varphi_t - \varphi_w}{[f(t_n) - f(w_n)]} - g_5([f(t_n) - f(x_n)] + [f(w_n) - f(x_n)]), \\ b_4 &= \varphi_t - g_4[f(t_n) - f(x_n)] - g_5[f(t_n) - f(x_n)]^2 \end{aligned} \tag{9}$$

where

$$\begin{aligned}\varphi_t &= \frac{1}{f[t_n,x_n][f(t_n)-f(x_n)]}-\frac{1}{f[z_n,x_n][f(t_n)-f(x_n)]},\\ \varphi_w &= \frac{1}{f[w_n,x_n][f(w_n)-f(x_n)]}-\frac{1}{f[z_n,x_n][f(w_n)-f(x_n)]},\\ \varphi_y &= \frac{1}{f[y_n,x_n][f(y_n)-f(x_n)]}-\frac{1}{f[z_n,x_n][f(y_n)-f(x_n)]}.\end{aligned}$$

Motivated by the research going on in this direction and with a need to develop more optimal higher order methods, in this paper, we propose a new general family optimal derivative free iterative methods by using Lagrange interpolation with simple body structure for finding simple zeros of a univariate nonlinear function. In Section 2, the new family along with it's special cases of convergence order two, four, eighth and sixteen are presented. The convergence analysis of these methods is studied in Section 3. The advantage of the new schemes is that they are also extendable to the iterative methods with-memory [6]. In Section 4, polynomiographs of different iterative methods are presented. Section 5 includes the numerical results and comparisons of the proposed iterartive methods with the existing methods of similar kind. The concluding remaks are provided in Section 6.

2. Optimal higher order derivative free methods

In this section, we present a general family of optimal derivative free methods by applying Lagrange interpolation that satisfy the conjecture of Kung and Traub [5].

We define the following simple and general $n-$step family of optimal iterative methods.

$$\begin{aligned}w_0 &= x_k, k\geq 0,\\ w_1 &= w_0+\beta f(w_0),\\ &\vdots\\ x_{k+1} &= w_{n+1}=w_n-\frac{f(w_n)}{L_n'(w_n)}, n\geq 1,\end{aligned}\tag{10}$$

where, $\beta\in\mathbb{R}$ and $L_n'(w_n)$ is the derivative of nth degree Lagrange interpolating polynomial that interpolates f in $w_0, w_1, ..., w_n$. Some sepecial cases of the above scheme are described as follows.

For $n = 1$ in (10), we obtain the following one-step iterartive method:

$$\begin{aligned} w_0 &= x_k, k \geq 0, \\ w_1 &= w_0 + \beta f(w_0), \beta \in \mathbb{R} \\ x_{k+1} &= w_2 = w_1 - \frac{f(w_1)}{L_1'(w_1)}, \end{aligned} \tag{11}$$

where

$$L_1'(w_1) = \frac{f(w_1) - f(w_0)}{w_1 - w_0}. \tag{12}$$

It is to be remarked that, the above scheme is different from the Newton's (1) and Steffensen's (3) schemes.

For $n = 2$, we obtain the following two-step iterative scheme:

$$\begin{aligned} w_0 &= x_k, k \geq 0, \\ w_1 &= w_0 + \beta f(w_0), \\ w_2 &= w_1 - \frac{f(w_1)}{L_1'(w_1)}, \\ x_{k+1} &= w_3 = w_2 - \frac{f(w_2)}{L_2'(w_2)}, \end{aligned} \tag{13}$$

where where $L_1'(w_1)$ is given by (12) and

$$L_2'(w_2) = f[w_0, w_2] + f[w_1, w_2] - f[w_0, w_1]. \tag{14}$$

For $n = 3$, a new three-step iterative method is obtained as follows:

$$\begin{aligned} w_0 &= x_k, k \geq 0, \\ w_1 &= w_0 + \beta f(w_0), \\ w_2 &= w_1 - \frac{f(w_1)}{L_1'(w_1)}, \\ w_3 &= w_2 - \frac{f(w_2)}{L_2'(w_2)}, \\ x_{k+1} &= w_4 = w_3 - \frac{f(w_3)}{L_3'(w_3)}, \end{aligned} \tag{15}$$

where $L_1'(w_1)$ and $L_2'(w_2)$ are given by (12) and (14) respectively and

$$\begin{aligned} L_3'(w_3) &= f[w_3, w_1] + f[w_3, w_0] + f[w_3, w_2] - f[w_0, w_2] \\ &\quad - f[w_1, w_2] - f[w_0, w_1] + \frac{(w_0 - w_3) f(w_0)}{((w_0 - w_1)(w_0 - w_2))} \\ &\quad + \frac{(w_1 - w_3) f(w_1)}{((w_1 - w_0)(w_1 - w_2))} + \frac{(w_2 - w_3) f(w_2)}{((w_2 - w_1)(w_2 - w_0))}. \end{aligned} \tag{16}$$

For $n = 4$ in scheme (10), the following four step iterative method is obtained as under:

$$\begin{aligned} w_0 &= x_k, \quad k \geq 0, \\ w_1 &= w_0 + \beta f(w_0), \quad \beta \in \mathbb{R}, \\ w_2 &= w_1 - \frac{f(w_1)}{f[w_1, w_0]}, \\ w_3 &= w_2 - \frac{f(w_2)}{L'_2(w_2)}, \\ w_4 &= w_3 - \frac{f(w_3)}{L'_3(w_3)}, \\ x_{k+1} &= w_5 = w_4 - \frac{f(w_4)}{L'_4(w_4)}. \end{aligned} \tag{17}$$

where $L'_1(w_1)$, $L'_2(w_2)$ and $L'_3(w_3)$ are given by equations (12), (14) and (16) respectively and $L'_4(w_4)$ is given as follows:

$$\begin{aligned} L'_4(w_4) &= \frac{(w_4 - w_3)(w_4 - w_1)(w_4 - w_2)f(w_0)}{((w_0 - w_3)(w_0 - w_4)(w_0 - w_1)(w_0 - w_2))} \\ &+ \frac{(w_4 - w_3)(w_4 - w_0)(w_4 - w_2)f(w_1)}{((w_1 - w_2)(w_1 - w_4)(w_1 - w_0)(w_1 - w_3))} \\ &+ \frac{(w_4 - w_3)(w_4 - w_0)(w_4 - w_1)f(w_2)}{((w_2 - w_3)(w_2 - w_4)(w_2 - w_0)(w_2 - w_1))} \\ &+ \frac{(w_4 - w_0)(w_4 - w_1)(w_4 - w_2)f(w_3)}{((w_3 - w_4)(w_3 - w_0)(w_3 - w_1)(w_3 - w_2))} \\ &+ \frac{f(w_4)}{(w_4 - w_0)} + \frac{f(w_4)}{(w_4 - w_1)} + \frac{f(w_4)}{(w_4 - w_2)} + \frac{f(w_4)}{(w_4 - w_3)}. \end{aligned} \tag{18}$$

3. Analysis of convergence

Theorem 1. *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function in an open interval D and α be a simple root of f. If x_0 is close enough to α, then for all $\beta \in \mathbb{R}$, the iterative schemes defined by (11) and (13) are second and fourth order convergent respectively with the error equations given as follows respectively:*

$$e_{n,w_2} = c_2(1 + \beta c_1)e_n^2 + O(e_n^3), \tag{19}$$

and

$$e_{n,w_3} = c_2(1 + \beta c_1)^2(c_2^2 - c_3)e_n^4 + O(e_n^5). \tag{20}$$

where, $c_j = \frac{f^{(j)}(\alpha)}{j!f^j(\alpha)}, k \geq 2.$

Proof. Let $e_n = x_n - \alpha$ be the error at nth step. By using Taylor's series expansion of $f(x)$ about the roor α, we have:

$$f(x_n) = f'(\alpha)\left[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + O(e_n^5)\right],$$

Again by using Taylor's expansion, we get the error term of $w_1 = x_n + \beta f(x_n)$ as follows:

$$e_{n,w_1} = (1+\beta f'(\alpha))e_n + \beta f'(\alpha)(c_2e_n^2 + c_3e_n^3 + c_4e_n^4) + O(e_n^5),$$

where

$$e_{n,w_1} = w_1 - \alpha.$$

Similarly with the help of Taylor's series, the expression of $f(w_1)$ can be obtained. Hence, the error term of w_2 is as follows:

$$\begin{aligned} e_{n,w_2} \;=\;& c_2(1+\beta c_1)e_n^2 + (3\beta c_1c_3 - 2c_2^2\beta c_1 - \beta^2c_1^2c_2^2 + 2c_3 + c_3\beta^2c_1^2 - 2c_2^2)e_n^3 \\ & -7\beta^2c_1^2c_3c_2 - 2c_1^3c_2\beta^3c_3 + 3\beta^2c_1^2c_2^3 + 4c_4\beta^2c_1^2 + c_4\beta^3c_1^3 - 10c_1c_2\beta c_3 \\ & +3c_4 + 6\beta c_1c_4 + 5c_2^3\beta c_1 - 7c_2c_3 + 4c_2^3 + \beta^3c_1^3c_2^3)O(e_n^4) + O(e_n^5), \end{aligned} \tag{21}$$

where

$$e_{n,w_2} = w_2 - \alpha.$$

Now again with the help of Taylor's series expansions, we obtain the following error term of w_3:

$$e_{n,w_3} = c_2(1+\beta c_1)^2(c_2^2 - c_3)e_n^4 + O(e_n^5). \tag{22}$$

The above error relations (21) and (22) show that the iterative schemes (11) and (13) have second and fourth order convergence respectively. This completes the proof. □

Theorem 2. *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function in an open interval D and α be a simple root of f. If x_0 is close enough to α, then for all $\beta \in \mathbb{R}$, the iterative schemes defined by (15) and (17) have optimal order of convergence eight and sixteen respectively with the following error equations:*

$$e_{n,w_4} = c_2^2(1+\beta c_1)^4(-c_3 + c_2^2)(c_2^3 - c_2c_3 + c_4)e_n^8 + O(e_n^9) \tag{23}$$

and

$$e_{n,w_5} = c_2^4(-c_3+c_2^2)^2(1+\beta c_1)^8(c_2^3 - c_2c_3 + c_4)(-c_5 + c_2c_4 - c_2^2c_3 + c_2^4)e_n^{16} + O(e_n^{17}) \tag{24}$$

respectively, where $c_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)}, j \geq 2$ and $e_n = w_n - \alpha$.

Proof. Let $e_{n,w_4} = w_4 - \alpha$ and $e_{n,w_5} = w_5 - \alpha$. With the help of Taylor's series expansions, the proof is similar to the proof for Theorem 1. Hence it is skipped over. □

Remark 1. *From the above theorems, it is clear that the iterative schemes (11), (13), (15) and (17) are second, fourth, eighth and sixteenth order convergent requiring two, three, four and five function evaluations, respectively. Thus, the presented iterative schemes are optimal in the sense of hypothesis of Kung and Traub [5] with the efficiency indices* $1.414, 1.587, 1.681, 1.741$ *respectively. Hence, it is concluded that the general* $n-$*step iterative scheme (10) has optimal order of convergence* 2^n *and efficiency index* $2^{\frac{n}{n+1}}$ *as stated by the following theorem.*

Theorem 3. *Let* $f : D \subseteq \mathbb{R} \to \mathbb{R}$ *be a sufficiently differentiable function in an open interval* D *and* α *be a simple root of* f*. If* x_0 *is close enough to* α*, then for all* $\beta \in \mathbb{R}$*, the* $n-$*step iterative scheme defined by (10) has optimal order of convergence* 2^n *using the* $n+1$ *function evaluations.*

4. Numerical Results

We, now, check the prformance of the newly developed sixteenth order iterative method (TM16) (17) by comparing it with some well known sixteenth order methods using a number of nonlinear equations. We have employed multiprecision arithmetic with 4000 significant decimal digits in the programming package of Maple 16 to obtain a high accuracy and avoid the loss of significant digits. We compare the convergence behavior of our method (TM16) with the the sixteenth order schemes of Cordero et al. [8] (CM16), Geum et al. (4) (GK16) and Janjua et al. (7) (JM16) by using the nonlinear functions given in Table 1. Table 1 also includes the exact roots α and initial approximations x_0 for different nonlinear functios which are calculated using Maple 16. The error $|x_n - \alpha|$ and the computational order of convergence (coc) for first three iterations of various methods is displayed in the Tables 2-7, which supports the theoretical order of convergence. The formula to compute the computational order of convergence (coc) is given by [14]:

$$coc \approx \frac{\log|(x_{n+1}-\alpha)/(x_n-\alpha)|}{\log|(x_n-\alpha)/(x_{n-1}-\alpha)|}. \tag{25}$$

Tables 2–7 show that the proposed iterative method (TM16) is comparable and competitive to the methods (CM16), (GK16) and (JM16).

5. Polynomiography

In this section, we compare the iterative methods under consideration by drawing their polynomiographs. Polynomiographs contain basins of attraction that allow us to see how wide is the set of initial guesses that leads us to the required roots. We consider a rectangle $D = [-2, 2] \times [-2, 2] \in \mathbb{C}$ with a mesh of 1000×1000 initial approximations. If the sequence produced by an iterative method for a given initial guess z_0, reaches to a root of the polynomial

TABLE 1. Test Functions

Example	Test Functions	Exact root α	x_0
1	$f_1(x) = (2+x^3)\cos(\frac{\pi x}{2}) + \log(x^2+2x+2)$	-1	-0.93
2	$f_2(x) = x^2e^x + x\cos\frac{1}{x^3} + 1$	$-1.5650602...$	-1.25
3	$f_3(x) = xe^x + \log(1+x+x^4)$	0	-0.5
4	$f_4(x) = (x-1)(x+1+\log(2+x+x^2))$	1	1.05
5	$f_5(x) = -20x^5 - \frac{x}{2} + \frac{1}{2}$	0.42767729...	0.38
6	$f_6(x) = e^{\sin(8x)} - 4x$	0.34985721...	7

TABLE 2. Numerical Results of Example 1

$f_1(x) = (2+x^3)\cos(\frac{\pi x}{2}) + \log(x^2+2x+2), x_0 = -0.93$				
	GK16	CM16	JM16	TM16
$\lvert x_1-\alpha\rvert$	$1.44(-10)$	$6.42(-10)$	$1.83(-10)$	$1.63(-10)$
$\lvert x_2-\alpha\rvert$	$8.08(-147)$	$9.99(-136)$	$2.58(-145)$	$1.50(-150)$
$\lvert x_3-\alpha\rvert$	$7.29(-2327)$	$1.18(-2148)$	$6.18(-2303)$	$2.10(-2350)$
coc	16.00	16.00	16.00	16.00

TABLE 3. Numerical Results of Example 2

$f_2(x) = x^2e^x + x\cos\frac{1}{x^3} + 1, x_0 = -1.25$				
	GK16	CM16	JM16	TM16
$\lvert x_1-\alpha\rvert$	$5.84(-13)$	$4.10(-6)$	$1.46(-11)$	$1.34(-13)$
$\lvert x_2-\alpha\rvert$	$2.58(-201)$	$2.16(-89)$	$4.07(-180)$	$2.10(-210)$
$\lvert x_3-\alpha\rvert$	$5.49(-3215)$	$7.88(-1422)$	$5.01(-2877)$	$4.5(-3342)$
coc	16.00	16.00	16.00	16.00

TABLE 4. Numerical Results of Example 3

$f_3(x) = xe^x + \log(1+x+x^4), x_0 = -0.5$				
	GK16	CM16	JM16	TM16
$\lvert x_1-\alpha\rvert$	*Div.*	*Div.*	$6.69(-10)$	$4.59(-11)$
$\lvert x_2-\alpha\rvert$	*Div.*	*Div.*	$2.43(-152)$	$1.23(-165)$
$\lvert x_3-\alpha\rvert$	*Div.*	*Div.*	$2.35(-2431)$	$2.35(-2535)$
coc	*Div.*	*Div.*	16.00	16.00

$p(z) = 0$ with the tolerance 10^{-5} and in a maximum of 30 iterations, we paint that initial guess in a color already chosen for the corresponding root. If the iterative method for an initial guess does not converge to any of the roots in 30 iterations, that initial guess is painted with blue color. We have considered five different polynomials to draw and comapre the polynomiographs

Table 5. Numerical Results of Example 4

$f_4(x) = (x-1)(x+1+\log(2+x+x^2)), x_0 = 1.05$				
	GK16	CM16	JM16	TM16
$\lvert x_1-\alpha\rvert$	$1.02(-23)$	$3.40(-21)$	$2.69(-21)$	$1.45(-22)$
$\lvert x_2-\alpha\rvert$	$2.44(-370)$	$1.83(-327)$	$7.83(-330)$	$2.53(-345)$
$\lvert x_3-\alpha\rvert$	0	0	0	0
coc	16.00	16.00	16.00	16.00

Table 6. Numerical Results of Example 5

$f_5(x) = -20x^5 - \frac{x}{2} + \frac{1}{2}, x_0 = 0.38$				
	GK16	CM16	JM16	TM16
$\lvert x_1-\alpha\rvert$	$1.65(-5)$	$1.28(-3)$	$2.78(-11)$	$1.58(-11)$
$\lvert x_2-\alpha\rvert$	$2.04(-66)$	$2.68(-34)$	$5.53(-160)$	$2.13(-158)$
$\lvert x_3-\alpha\rvert$	$5.69(-1041)$	$5.39(-525)$	$3.22(-2539)$	$2.32(-2540)$
coc	16.00	16.00	16.00	16.00

Table 7. Numerical Results of Example 6

$f_6(x) = e^{\sin(8x)} - 4x, x_0 = 7$				
	GK16	CM16	JM16	TM16
$\lvert x_1-\alpha\rvert$	Div.	$1.98(-2)$	$1.50(-2)$	$1.45(-2)$
$\lvert x_2-\alpha\rvert$	Div.	$3.89(-12)$	$3.31(-17)$	$3.71(-18)$
$\lvert x_3-\alpha\rvert$	Div.	$1.20(-168)$	$9.46(-225)$	$1.26(-235)$
coc	Div.	16.12	16.20	16.30

of the proposed method (TM16) with (CM16) and (GK16). In all Figures, the polynomiographs of (TM16) shows wider basins of attraction than polynomiographs of (CM16) and (GK16).

Example 1. *We consider the following polynomial of degree 3:*

$$p_1(z) = z^3 + 1.$$

The roots of the above cubic equation are at $0.5000 \pm 0.8660i$ *and* $-1.000 + 0.0000i$. *In Figures 1 and 2, polynomiographs of TM16, CM16 and GK16 are shown for* p_1.

Example 2. *We consider the following polynomial of degree 4:*

$$p_2(z) = z^4 + 1.$$

The roots of the above quartic equation are at $0.7071 \pm 0.7071i$ *and* $-0.7071 \pm 0.7071i$. *In Figures 3 and 4, polynomiographs of TM16, CM16 and GK16 are shown for* p_2.

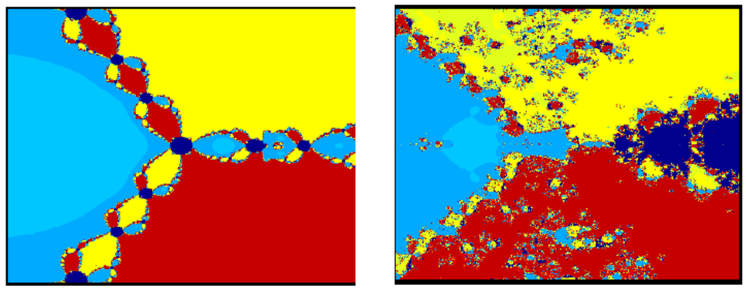


FIGURE 1. Polynomiographs of TM16 for p_1

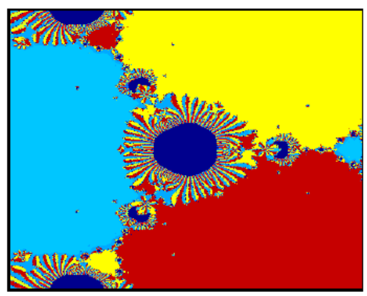


FIGURE 2. Polynomiograph of GK16 for p_1

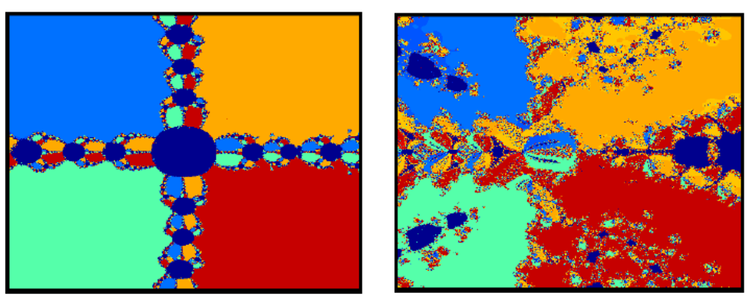


FIGURE 3. Polynomiographs of TM16 (Left) and CM16 (right) for p_2

Example 3. *Another polynomial of degree 4 is chosen as follows:*

$$p_3(z) = z^4 - z^3 + z^2 - z + 1.$$

The roots of the above quartic equation are at $0.8090 \pm 0.5878i$ *and* $-0.3090 \pm 0.9511i$.*In Figures 5 and 6, polynomiographs of TM16, CM16 and GK16 are plotted for* p_3.

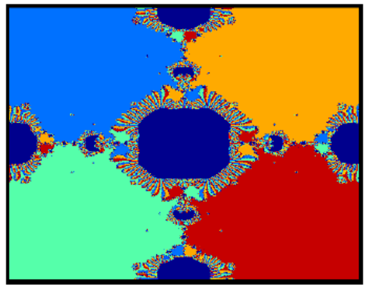


Figure 4. Polynomiograph of GK16 for p_2

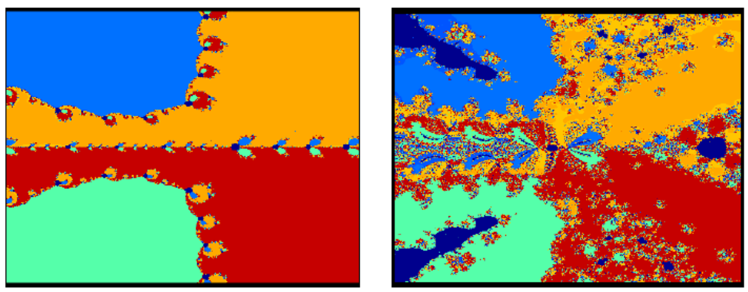


Figure 5. Polynomiographs of TM16 (Left) and CM16 (right) for p_3

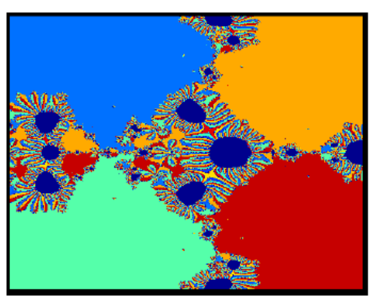
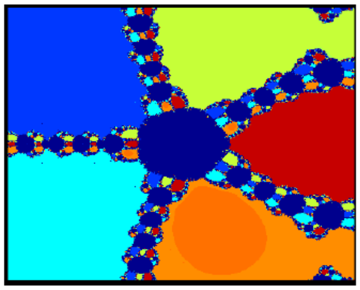


Figure 6. Polynomiograph of GK16 for p_3

Example 4. *The polynomial of degree 5 is considered as follows:*

$$p_4(z) = z^5 - 1.$$

The roots of the above equation of degree five are at $0.8090 \pm 0.9511i$, $-0.8090 \pm 0.5878i$ *and* $1.0000 + 0.0000i$. *In Figures 7 and 8, polynomiographs of TM16, CM16 and GK16 are shown for* p_4.



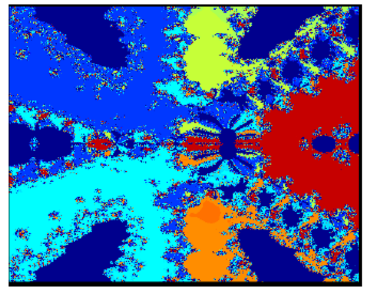


FIGURE 7. Polynomiographs of TM16 (Left) and CM16 (right) for p_4

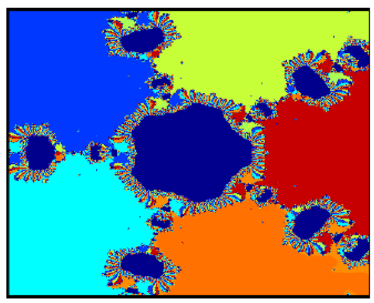


FIGURE 8. Polynomiograph of GK16 for p_4

Example 5. *We consider the following polynomial of degree 7:*

$$p_5(z) = z^7 - 1.$$

The roots of the above equation of degree seven are at $0.6325 \pm 0.7818i$, $-0.9010 \pm 0.4339i$, $-0.2225 \pm 0.9749i$ *and* $1.0000 \pm 0.0000i$. *In Figures* **??** *and 10, polynomiographs of TM16, CM16 and GK16 are shown for* p_5.

6. Conclusions

In this paper, we have presented a general family of optimal derivative free iterative methods of arbitrary high order for solving nonlinear equations by using Lagrange interpolation. The special cases of the family with optimal order of convergence two, four, eight and sixteen are obtained. These methods do not need the Newton's or Steffensen's iterations in the first step of their iterative schemes. Convergence analysis is also studied for the new iterative methods. The advantage of the new schemes is that they are also extendable to the iterative methods with-memory. Finally, numerical comparison and polynomiographs of different iterative methods are presented which support the theoretical results and illustrate that the new optimal derivative free

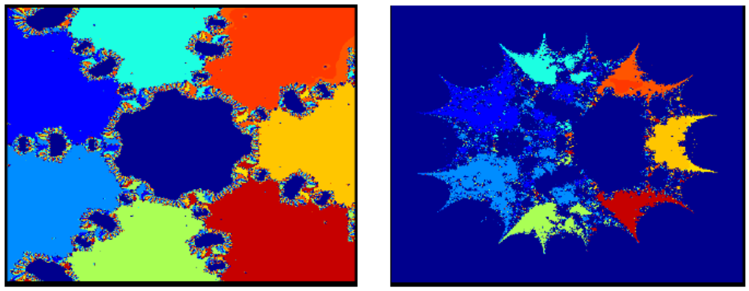


FIGURE 9. Polynomiographs of TM16 (Left) and CM16 (right) for p_5

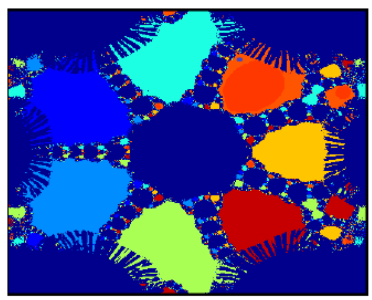


FIGURE 10. Polynomiograph of GK16 for p_5

methods are comparable and perform better than the existing methods of the same domain.

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