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FRACTIONAL OPTIMAL CONTROL FOR A CORRUPTION MODEL

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Abstract. In this work, a fractional optimal control of corruption model is investigated. The variable controls are included in the model to optimize the best strategy in reducing the corruption in the society. The fraction derivative employed in the study is in Atangana–Beleanu–Caputo (ABC) sense based on generalized Mittag–Leffler. The uniqueness and existence of solution of the corruption model is established. The necessary and sufficient condition for establishing fractional optimal control in ABC sense is determined. A numerical algorithm for obtaining fractional optimal control solution is presented. The numerical solution results show that the best strategy in controlling corruption in the society is to optimize all the thee controls simultaneously.

Key words: Corruption, fractional optimal control, Mittag–Leffler, Atangana–Beleanu–Caputo, existence and uniqueness.
MSC: Primary 14H50, 14H20, 32S15.

1. Introduction

Let Corruption has been one major social disease that has ruined many nations. Unfortunately, the corruption disease cuts across the entire globe. The most affected part of the world is the sub-Saharan Africa [1, 2]. Some African and Asian countries are still under developed because of corruption. Many public and private sectors of these economies do not have strong regulatory bodies to deal with corruption related issues [1, 3]. Jain [4] observes that, three pre–conditions necessitates the existence of corruption as economic related issues, discretionary power due to procedures and inadequate punishment. Mishra [5] observed that, corruption can go up in any community

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through an evolutionary game. Corruption related issues have been studied by many researchers.

Corruption is an unavoidable part of human social interaction, prevalent in every society at any time since the very beginning of human history till today. Corruption is a difficult social disease which has no race or colour and it has been with humans since creation [6]. The social and economic wars across the world and unstable societies are highly attributed to corruption related issues. The worst continent is Africa where weak democratic institutions are found [7]. Politicians are able to use the offices they hold to get access to resources for their personal gains.

Mathematical model on corruption was first formulated and analyzed in 1975 and since then this subject has attracted many mathematical modelers due to its effect on society [3, 8–10]. There has been several epidemiological corruption models presenting several strategies to understand the dynamics of corruption in the society [6, 7, 11–16]. Eguda et al., [17], constructed a mathematical model on corruption to explore the dynamics of the long-term negative impact on public procurement. They further observed that government should raise the minimum wage to discourage corruption practices. Lemecha, [18] constructed a corruption mathematical model taking into consideration the awareness provided by anti-corruption institutions and counselling services rendered to people in jail because of corruption. In their study, they observed that corruption is higher is Ethiopia than New Zealand.

However, very few works have been explored in the light of mathematical modelling in providing detailed account of corruption dynamics in society. Mathematical modelling has been noticed to be as one of the most powerful tools for obtaining qualitative information in the absence of real data for the purpose of decision making. There has been several integer mathematical models on social issues such as corruption. The concept of fractional calculus gained attention in the middle of the 21st century[19–22] because of its ability to predict accurately, complex real issues.

The common fractional derivative in early stages was hinged on power law that was not able to capture complex related issues (Caputo, 1971) [23–26]. It is upon this, that Caputo-Fabrizio (CF) introduced a new operator based exponential law and this was also not able to capture more complex systems because the kernel local and also non-singular [21, 27]. In recent times, a new operator based on generalized Mittage–Leffler function was developed by Atangana-Baleanu in Caputo sense which is non-local and non-singular (Atangana) [24, 28, 29] has become very popular because of its ability to capture more complex phenomena in real world. This operator has the property of crossover that enhances accurate predictions in complex systems.

The aim of this paper is to use fractional optional time control in ABC to obtain the best strategy in minimizing corruption in our society in the light of corruption model.

2. Mathematical Preliminaries

This section presents the following fractional order derivative definitions that are relevant in this work

Definition 1. *The Liouville-Caputo FO derivative is defined as in [2, 30]:*

$$ {}_a^cD_t^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-p)^{-\alpha} g(p)dp, 0 < \alpha \leq 1 \tag{1} $$

Definition 2. *The Atangana-Baleanu fractional derivative in the Liouville-Caputo sense is defined as in [19, 30].*

$$ {}_a^{ABC}D_t^\alpha g(t) = \frac{B(\alpha)}{(1-\alpha)} \int_0^t (E_\alpha(-a\frac{(t-q)^\alpha}{(1-\alpha)}))g(p)dp \tag{2} $$

where $B(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$ is the normalization function.

Definition 3. *The corresponding fractional integral concerning the Atangana-Baleanu-Caputo derivative is defined as [19, 30]*

$$ {}_a^{ABC}I_t^\alpha g(t) = \frac{(1-\alpha)}{B(\alpha)} g(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-p)^{\alpha-1} g(p)dp \tag{3} $$

They established that when $\aleph$ is zero, they recovered the initial function, and if $\aleph$ is 1, they get the ordinary integral. Further, they computed the Laplace transform of both derivatives and arrived at the following:

$$ \$ \left\{ {}_a^{ABC}D_t^\alpha g(t) \right\} = \frac{B\,(a)\,G\,(p)\,p^\alpha - p^{\alpha-1} g\,(0)}{(1-\alpha)\left(p^\alpha + \frac{\alpha}{(1-\alpha)}\right)} \tag{4} $$

Theorem 1. *For a function $g \in C[a,b]$, the following result holds [20, 30]:*

$$ \|{}_a^{ABC}\, D_t^\alpha g(t) \| < \frac{B(\alpha)}{(1-\alpha)} \| g(t) \|, where \| g(t) \| = \max_{a \leq t \leq b} |g(t)| \tag{5} $$

Further, the Atangana-Baleanu-Caputo derivatives fulfill the Lipschitz condition [20, 30]:

$$ \|{}_a^{ABC}\, D_t^\alpha g_1(t) - {}_a^{ABC}\, D_t^\alpha g_2(t) \| < \varphi \| g_1(t) - g_2(t) \| \tag{6} $$

3. Mathematical Model Formulation

This section reformulates model proposed by Bijal [31], in which the model sub-divides the total human population at time t, donated by $N(t)$, into the following sub-populations susceptible individuals who are not corrupt $S(t)$, those that are endemic in corruption and can corrupt others $I(t)$, those corrupt individuals punished $P(t)$, those corrupt individuals that get away with punishment $F(t)$ and those individuals who repented from corruption $R(t)$. The human mortality rate is μ and force of infection of corruption is given by $\beta_1 S(c_1 E + c_2 I + c_3 P + c_4 F)$. The modification parameters for exposed, infected, punished and unpunished individuals are c_1, c_2, c_3, c_4 respectively. The rate of infection of corruption is β. The rate individuals move from exposed to endemic corruption is given by ϕ. q denotes proportion of individuals who get punished for being corrupt at a rate η while $(1-q)$ is the proportion of individuals who get unpunished for being corrupted at a rate γ. Fraction of unpunished individuals n move to endemic class at a rate σ while fraction $(1-n)$ move to repent class at a rate ψ_2 The recruitment rate into human population is denoted by Λ. The corruption wanning rate is donated by m. Punished corrupt individuals of a fraction k move to susceptible at a α while a fraction $(1-k)$ get repented after being punished at a rate ψ_1. The following nonlinear fractional differential equations in ABC sense are arrived at, based on the interrelationship between the compartments.

$$\begin{cases} {}^{\mathrm{ABC}}_{0}D_t^{\alpha} S = \Lambda + \alpha K P + mR - \beta_1 S(c_1 I + c_2 P + c_3 F) - \mu S \\ {}^{\mathrm{ABC}}_{0}D_t^{\alpha} E = \beta_1 S(c_1 I + c_2 P + c_3 F) - (\phi + \mu)E \\ {}^{\mathrm{ABC}}_{0}D_t^{\alpha} I = \phi E + n\sigma F - q\eta I - (1-q)\gamma I - \mu I \\ {}^{\mathrm{ABC}}_{0}D_t^{\alpha} P = q\eta I - \alpha k P - (1-k)\psi_1 P - \mu P \\ {}^{\mathrm{ABC}}_{0}D_t^{\alpha} F = (1-q)\gamma I - (1-n)\psi_2 F - n\sigma F - \mu F \\ {}^{\mathrm{ABC}}_{0}D_t^{\alpha} R = (1-k)\psi_1 P + (1-n)\psi_2 F - (m+\mu)R \end{cases} \tag{7}$$

4. Stability analysis

Lemma 2. *The closed set*

$$\Psi = \left\{ (S, E, I, P, F, R) \in \mathbb{R}_+^6 : S + E + I + P + F + R = \frac{\Lambda}{\mu} \right\} \tag{8}$$

is positively invariant with regard to the system given by Eq. (7).

Proof. For system (7), let $N(t) = S + E + I + P + F + R$ represent the total population, hence the FD of the total population is expressed as

$$ {}^{\mathrm{ABC}}_{0}D^{\alpha}_{t} N = \Lambda - \mu N(t). \tag{9} $$

Making uses of the Laplace transform, from Eq. (7) we derive

$$ N(t) = \left(\frac{B(\alpha)}{B(\alpha) + (1-\alpha)\mu} N(0) + \frac{(1-\alpha)\Lambda}{B(\alpha) + (1-\alpha)\mu} \right) E_{\alpha,1}(-\beta t^{\alpha}) + \frac{\alpha\Lambda}{B(\alpha) + (1-\alpha)\mu} E_{\alpha,\alpha+1}(-\beta t^{\alpha}), \tag{10} $$

where $\beta = \frac{\alpha\mu}{B(\alpha)+(1-\alpha)\mu}$ and $E_{\alpha,\beta}$ is the two parameter Mittag-Leffler function defined by

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \tag{11} $$

Taking into account the asymptotic characteristic of the Mittag-Leffler function (Beleanu et al.[5]), we observe that $N(t) = \Lambda/\mu$ as $t \to \infty$. Thus, the solution of the model system (7) for initial conditions in Ψ stays in Ψ for every $t > 0$.Therefore, Ψ is a positively invariant region with regard to system (7). □

The system given by Eq. (9) has two equilibrium points E_0 and E_1. $E_0 = (\frac{\Lambda}{\mu}, 0, 0, 0, 0, 0)$ is the corruption-free equilibrium and the Jacobian matrix J_{E_0} of (9) evaluated at the corruption-free equilibrium E_0 is given by
$J_{E_0} =$

$$ \begin{pmatrix} -\mu & 0 & -\frac{\beta_1 c_1 \Lambda}{\mu} & k\alpha - \frac{\beta_1 c_2 \Lambda}{\mu} & -\frac{\beta_1 c_3 \Lambda}{\mu} & m \\ 0 & -\mu - \phi & \frac{\beta_1 c_1 \Lambda}{\mu} & \frac{\beta_1 c_2 \Lambda}{\mu} & \frac{\beta_1 c_3 \Lambda}{\mu} & 0 \\ 0 & \phi & -(1-q)\gamma - q\eta - \mu & 0 & n\sigma & 0 \\ 0 & 0 & q\eta & -k\alpha - \mu - (1-k)\psi_1 & 0 & 0 \\ 0 & 0 & (1-q)\gamma & 0 & -mu - n\sigma - (1-n)\psi_2 & 0 \\ 0 & 0 & 0 & (1-k)\psi_1 & (1-n)\psi_2 & -m - \mu \end{pmatrix} \tag{12} $$

The transmission matrix F and transition matrix V are obtained as

$$ F = \begin{pmatrix} 0 & \frac{\beta_1 c_1 \Lambda}{\mu} & \frac{\beta_1 c_2 \Lambda}{\mu} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, $$

$$ V = \begin{pmatrix} \mu + \phi & 0 & 0 \\ -\phi & (1-q)\gamma + q\eta + \mu & 0 \\ 0 & -q\eta & k\alpha + \mu + (1-k)\psi_1 \end{pmatrix} \tag{13} $$

According to the next generation matrix method, the next generation matrix is defined by FV^{-1} with the basic reproduction number $\mathcal{R}_0$ of system 7 given by the spectral radius of FV^{-1} as

$$\mathcal{R}_0 = \frac{\Lambda\beta_1\phi\left(c_1\left(k\alpha+\mu+\psi_1-k\psi_1\right)+q\eta c_2\right)}{\mu(\mu+\phi)(\gamma-q\gamma+q\eta+\mu)\left(k\alpha+\mu+\psi_1-k\psi_1\right)} \tag{14}$$

5. Existence and Uniqueness of the solution

In this section, the existence and uniqueness of the solution of the fractional-order corruption model 7 is throughly examined . In order to do this we initially reorganize the fractional corruption system (7) in the form.

$$\begin{cases} {}_0^{ABC}D_t^{\alpha} y\left(t\right) = g\left(y\left(t\right)\right), 0 \leq t < T < \infty, \\ y\left(0\right) = y_0. \end{cases} \tag{15}$$

where y is the state vector presented as (S, E, I, P, F, R), a is a real-valued continues vector fraction expressed as

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} \Lambda + \alpha KP + mR - \beta_1 S\left(c_1 I + c_2 P + c_3 F\right) - \mu S \\ \beta_1 S\left(c_1 I + c_2 P + c_3 F\right) - \left(\phi + u\right)E \\ \phi E + n\sigma F - qnI - \left(1-q\right)\gamma I - uI \\ qnI - \alpha KP - \left(1-K\right)\psi_1 P - \mu P \\ \left(1-q\right)\gamma I - \left(1-n\right)\psi_2 F - n\sigma F - uF \\ \left(1-K\right)\psi_1 P + \left(1-n\right)\psi_2 F - \left(m+u\right)R \end{bmatrix} \tag{16}$$

and y_0 denotes the initial state vector. Since a is a quadratic vector function, it satisfied the Lipschitz condition, i.e there exists a constant W such that

$$\parallel a\left(y\left(t\right)\right) - a\left(x\left(t\right)\right) \parallel \leq W \parallel y\left(t\right) - x\left(t\right) \parallel \tag{17}$$

It can be noted that the existence and uniqueness of the solution of the classical fractional differential equation (7) in the Caputo sense has been developed and analysed in [30]. In the following theorem, we shall investigate this matter theoretically for the ABC fractional operator with ML nonsingular kernel in details.

Theorem 3. *(Existence and uniqueness) the fractional corruption system (7) has a unique solution if the following condition exists*

$$\frac{(1-\alpha)}{B\left(\alpha\right)}W + \frac{\alpha}{B\left(\alpha\right)T\left(\alpha\right)}WT^{\alpha} < 1 \tag{18}$$

Making use of the ABC fractional integral operator (3), we have

$$y(t) = y_0 + \frac{1-\alpha}{B(\alpha)} a\left(x(t)\right) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int\limits_0^t (t-\varepsilon)^{\alpha-1} a\left(y(\varepsilon)\right) d\varepsilon \tag{19}$$

We let $J = (0, T)$ and the operator is defined $A : C(J, R)^6 \to C\left(J, R^6\right)$ as

$$A\left[y\left(t\right)\right] = y_0 + \frac{1-\alpha}{B\left(\alpha\right)} a\left(y\left(t\right)\right) + \frac{\alpha}{B\left(\alpha\right) T\left(\alpha\right)} \int_0^t \left(t-\varepsilon\right)^{\alpha-1} a\left(y\left(\varepsilon\right)\right) d\varepsilon \tag{20}$$

The Eq (19) is rearranged as follows .

$$y\left(t\right) = A\left[y\left(t\right)\right] \tag{21}$$

Let $\parallel . \parallel_J$ represent the supremum norm on J, Thus, $\parallel y\left(t\right) \parallel_J = \sup_{t\in J} \parallel y\left(t\right) \parallel$ $y\left(t\right) \in C\left(J, R^6\right)$ Then $C\left(J, R^6\right)$ with $\parallel . \parallel_J$ is a Banach spaced. Furthermore, it is simply shown that

$$\parallel \int_0^t K\left(t,\varepsilon\right) y\left(\varepsilon\right) d\varepsilon \parallel_J \leq T \parallel K\left(t,\varepsilon\right) \parallel_J \parallel y\left(\varepsilon\right) \parallel_J \tag{22}$$

where $y\left(t\right) \in C\left(J, R^6\right)$, $K\left(t,\varepsilon\right) \in C\left(J^2, R\right)$ and

$$\parallel K\left(t,\varepsilon\right) \parallel_J = \sup_{t\in J} \parallel K\left(t,\varepsilon\right) \parallel K\left(t,\varepsilon\right) \in C\left(J^2, R\right) \tag{23}$$

Employing the definition of operator G in the equation (20) together with equations (17) and (22) we obtain

$$\begin{aligned} &\parallel A\left[y\left(t\right)\right] - A\left[x\left(t\right)\right] \parallel_J \leq \tfrac{1-\alpha}{B(\alpha)} \parallel a\left(y\left(t\right)\right) - a\left(x\left(t\right)\right) \parallel_J \\ &+ \tfrac{\alpha}{B(\alpha)T(\alpha)} T^\alpha \parallel a\left(y\left(\varepsilon\right)\right) - a\left(x\left(\varepsilon\right)\right) \parallel_J \\ &\leq \left(\tfrac{1-\alpha}{B(\alpha)} W + \tfrac{\alpha}{B(\alpha)T(\alpha)} W T^\alpha\right) \parallel y\left(t\right) - x\left(t\right) \parallel_J \end{aligned} \tag{24}$$

Hence, we have

$$\parallel A\left(y\left(t\right)\right) - A\left(x\left(t\right)\right) \parallel_J \leq L \parallel y\left(t\right) - x\left(t\right) \parallel_J \tag{25}$$

where $L = \frac{1-\alpha}{B(\alpha)} W + \frac{\alpha}{B(\alpha)T(\alpha)} W T^\alpha$. If condition stated in equation (18) is field, the operator A will have a condition on $C\left(J, R^6\right)$. It can be therefore be concluded that because of Banach fixed theorem, the system (15) possesses a unique solution.

6. Fractional optimal control

In this section we include three controls, u_1 seeks to prevent individuals who have been exposed, corrupt, punished and made away with corruption in the society. The control u_2 seeks to present measures which serve as a social medication that encourages individuals punished before to repent from corruption for good.While control u_3 also provides some social medication to encourage individuals who have ever involved in corruption without being punished willingly repent for good. The main objective is to reduce the number of individual who are corrupt while reducing the costs associated with the strategies. In this study we employ will ABC fractional optimal control for the analysis since the operator is efficient and effective based on generalised Mittage- Leffler function.

$$\begin{cases} {}^{\text{ABC}}_{0}D_t^{\alpha} S = \Lambda + \alpha K P + mR - (1-u_1)\beta_1 S(c_1 I + c_2 P + c_3 F) - \mu S \\ {}^{\text{ABC}}_{0}D_t^{\alpha} E = (1-u_1)\beta_1 S(c_1 I + c_2 P + c_3 F) - (\phi + \mu) E \\ {}^{\text{ABC}}_{0}D_t^{\alpha} I = \phi E + n\sigma F - q\eta I - (1-q)\gamma I - \mu I \\ {}^{\text{ABC}}_{0}D_t^{\alpha} P = q\eta I - \alpha k P - (1-k) u_2 \psi_1 P - \mu P \\ {}^{\text{ABC}}_{0}D_t^{\alpha} F = (1-q)\gamma I - (1-n) u_3 \psi_2 F - n\sigma F - \mu F \\ {}^{\text{ABC}}_{0}D_t^{\alpha} R = (1-k) u_2 \psi_1 P + (1-n) u_3 \psi_2 F - (m+\mu) R \end{cases} \tag{26}$$

7. Control problem formulation

The state system Equation (7) is considered together with the admissible control functions. $\Omega = \{(u_1(.), u_2(.), u_3(.))\}\, u_i$ is Lebsegue measurable on [0,1], $0 \leq (u_1(.), u_2(.), u_3(.)) \leq 1\ t \in [0, T_{ff}], i = \{1,2,3\}$. where T_f is the terminal time and $u_1(.)$,$u_2(.)$ and $u_3(.)$ represent the controls functions. The objective function is therefore defined as follows:

$$J(u_1, u_2, u_3) = \int_0^{T_f} \sigma \left(I + P + F + \frac{D_1}{2} u_1^2(t) + \frac{D_2}{2} u_2^2(t) + \frac{D_3}{2} u_3^2(t) \right) dt \tag{27}$$

where D_1, D_2 and D_3 stand for the relative cost connected with the controls u_1, u_2 and u_3 . We determine the optional controls u_1, u_2 and u_3 that seeks to minimize the cost function

$$J(u_1, u_2, u_3) = \int_0^{T_f} \sigma(S, E, I, P, F, R, u_1, u_2, u_3, t)\, dt \tag{28}$$

Subject to the constraint

$$
{}_{0}^{ABC}D_{t}^{\alpha}S=\lambda_1, {}_{0}^{ABC}D_{t}^{\alpha}E=\lambda_2, {}_{0}^{ABC}D_{t}^{\alpha}I=\lambda_3, {}_{0}^{ABC}D_{t}^{\alpha}P=\lambda_4,
$$

$$
{}_{0}^{ABC}D_{t}^{\alpha}F=\lambda_5, {}_{0}^{ABC}D_{t}^{\alpha}R=\lambda_6 \tag{29}
$$

where $\lambda_i = \lambda\left(S,E,I,P,F,R,u_1,u_2,u_3\right)$ $i=1,......,6$, and also the following initial conditions are taking into consideration $S(0)=S_0, E(0)=E_0, I(0)=I_0, F(0)=F_0, R(0)=R_0$. In order to present a definition of the fractional optional control problem the following cost function [31] are reformulated.

$$
J=\int_0^{T_f}\left[H_b\left(S,E,I,P,F,R,u_j,t\right)-\sum\nolimits_{i=1}^{6}\lambda i\varepsilon\left(S,E,I,P,F,R,u_i,t\right)\right]dt \tag{30}
$$

where $j=1,2,3$ and also $i=1,2,3$

The Hamiltonian for the FOCP is stated as follows:

$$
H_b\left(S,E,I,P,F,R,u_j,t\right)=\sigma\left(S,E,I,P,F,R,u_j,t\right)+\sum\nolimits_{i=1}^{6}\lambda i\varepsilon\left(S,E,I,P,F,R,u_j,t\right) \tag{31}
$$

where $j=1,2,3$ and also $i=1,2,3$. Following equations (30) and (31), the necessary and sufficient conditions required for establishing the FOCP are as follows:

$$
{}_{t}^{ABC}D_{t_f}^{\alpha}\lambda_1=\frac{\partial H_b}{\partial S}, {}_{t}^{ABC}D_{t_f}^{\alpha}\lambda_2=\frac{\partial H_b}{\partial E}, {}_{t}^{ABC}D_{t_f}^{\alpha}\lambda_3=\frac{\partial H_b}{\partial I},
$$

$$
{}_{t}^{ABC}D_{t_f}^{\alpha}\lambda_4=\frac{\partial H_b}{\partial P}, {}_{t}^{ABC}D_{t_f}^{\alpha}\lambda_5=\frac{\partial H_b}{\partial F}, {}_{t}^{ABC}D_{t_f}^{\alpha}\lambda_6=\frac{\partial H_b}{\partial R} \tag{32}
$$

$$
0=\frac{\partial H}{\partial u_k} \tag{33}
$$

$$
{}_{0}^{ABC}D_{t}^{\alpha}S=\frac{\partial H_b}{\partial\lambda_1}, {}_{0}^{ABC}D_{t}^{\alpha}E=\frac{\partial H_b}{\partial\lambda_2}, {}_{0}^{ABC}D_{t}^{\alpha}I=\frac{\partial H_b}{\partial\lambda_3},
$$

,

$$
{}_{0}^{ABC}D_{t}^{\alpha}P=\frac{\partial H_b}{\partial\lambda_4}, {}_{0}^{ABC}D_{t}^{\alpha}F=\frac{\partial H_b}{\partial\lambda_5}, {}_{0}^{ABC}D_{t}^{\alpha}R=\frac{\partial H_b}{\partial\lambda_6} \tag{34}
$$

Furthermore,

$$
\lambda_1\left(T_f\right)=0, \lambda_1=j=1,2,3.....,6 \tag{35}
$$

constitutes the Lagrange multipliers equations. Equation (7) and (33) present the necessary conditions for establishing FOCP in terms of Hamiltonian. The following theorem is given:

Theorem 4. *Let $S^*, E^*, I^*, P^*, F^*, R^*$ be the solutions of the state system and u_i, $i=1,2,3$ be the given optional controls. Then, therefore there exists adjoint-state variables $\lambda_1=j=1,2,3.....,6$ satisfy the following:*

$$
\begin{aligned}
{}_{t}^{ABC}D_{t_f}^{\alpha}\lambda_1^* &= u\lambda_1^* + (1-u_1)\,\beta_1\,(c_1I + c_2P + c_3F)\,(\lambda_1^* - \lambda_2^*)\\
{}_{t}^{ABC}D_{t_f}^{\alpha}\lambda_2^* &= (\phi+\mu)\,\lambda_2^* - \phi\lambda_3^*\\
{}_{t}^{ABC}D_{t_f}^{\alpha}\lambda_3^* &= (1-u_1)\,\beta_1 Sc_1\,(\lambda_1^* - \lambda_2^*) + q\eta\lambda_3^* + (1-q)\,\gamma\lambda_3^*\\
&\quad +u\lambda_3^* - q\eta\lambda_4^* - (1-q)\,\gamma\lambda_5^*\\
{}_{t}^{ABC}D_{t_f}^{\alpha}\lambda_4^* &= u\lambda_4^* + (1-k)\,u_2\psi_1\lambda_4^* + \alpha k\lambda_4^* - \alpha k\lambda_1^*\\
&\quad -(1-k)\,u_2\psi_1\lambda_5^* + \beta_1 S\,(1-u_1)\,(\lambda_1^* - \lambda_2^*)\\
{}_{t}^{ABC}D_{t_f}^{\alpha}\lambda_5^* &= u\lambda_5^* + \eta\sigma\lambda_5^* + (1-\eta)\,u_3\psi_3\lambda_5^* - (1-\eta)\,\psi_3\psi_3\lambda_5^*\\
&\quad +(1-u_1)\,\beta_1 Sc_3\,(\lambda_1^* - \lambda_2^*)\\
{}_{t}^{ABC}D_{t_f}^{\alpha}\lambda_6^* &= (m+u)\,\lambda_6^* - m\lambda_1^*
\end{aligned}
\tag{36}
$$

Transversality conditions:

$$\lambda_j^*(T_f) = 0, j = 1, 2, ..., 6 \tag{37}$$

i. We state the optimality condition for FOCP:

$$
\begin{aligned}
&H_b\,(S^*, E^*, I^*, P^*, F^*, R^*, u_1^*, u_2^*, u_3^*, \lambda_j)\\
&= \min_{0\le u_1^*, u_2^*, u_3^* \le 1} H\,(S^*, E^*, I^*, P^*, F^*, R^*, u_1^*, u_2^*, u_3^*, \lambda_j)
\end{aligned}
\tag{38}
$$

$$u_1^* = \min\left\{1, \max\left\{\frac{\beta_1 S\,(C_1I + C_2P + C_3F)\,(\lambda_2 - \lambda_1)}{D_1}\right\}\right\} \tag{39}$$

$$u_2^* = \min\left\{1, \max\left\{\frac{(1-K)\,\psi_1 P\,(\lambda_6 - \lambda_4)}{D_2}\right\}\right\} \tag{40}$$

$$u_3^* = \min\left\{1, \max\left\{\frac{(1-n)\,\psi_2 F\,(\lambda_6 - \lambda_5)}{D_3}\right\}\right\} \tag{41}$$

Proof. The costate is obtained from equation (36) and (7) where

$$
\begin{aligned}
H_b^* &= I^* + P^* + F^* + \tfrac{D_1}{2}u_1^2(t) + \tfrac{D_2}{2}u_2^2(t) + \tfrac{D_3}{2}u_3^2(t)\\
&+\lambda_{1_b}^*\,{}_{b}^{ABC}D_t^{\alpha}S^* + \lambda_{2_b}^*\,{}_{b}^{ABC}D_t^{\alpha}E^* + \lambda_{3_b}^*\,{}_{b}^{ABC}D_t^{\alpha}I^*\\
&+\lambda_{4_b}^*\,{}_{b}^{ABC}D_t^{\alpha}P^* + \lambda_{5_b}^*\,{}_{b}^{ABC}D_t^{\alpha}F^* + \lambda_{6_b}^*\,{}_{b}^{ABC}D_t^{\alpha}R^*
\end{aligned}
\tag{42}
$$

represents the Hamiltonian. Also, the condition observed in equation (35) and optional control characterization in equation (39) - (41) can be obtained from equation (33). Making a substitution for $u_i = 1, 2, 3$ in equation (26) the following state system can be arrived at:

$$\begin{aligned}
&{}_{b}^{ABC}D_t^{\alpha} S^* = \Lambda + \alpha k P^* + m R^* - (1 - u_1^*)(c_1 I^* + c_2 P^* + c_3 F^*)\beta_1 S^* - u S^*, \\
&{}_{b}^{ABC}D_t^{\alpha} E^* = (1 - u_1^*)\beta_1 S^* (c_1 I^* + c_2 P^* + c_3 F^*) - (\phi + \mu) E^*, \\
&{}_{b}^{ABC}D_t^{\alpha} I^* = \phi E^* + \eta\sigma F^* - q\eta I^* - (1 - q)\gamma I^* - u I^*, \\
&{}_{b}^{ABC}D_t^{\alpha} P^* = q\eta I^* - \alpha k P^* - (1 - k) u_2^* \psi_1 P^* - u P^*, \\
&{}_{b}^{ABC}D_t^{\alpha} F^* = (1 - q)\gamma I^* - (1 - \eta) u_3^* \psi_2 F^* - \eta\sigma F^* - u F^*, \\
&{}_{b}^{ABC}D_t^{\alpha} R^* = (1 - k) u_2^* \psi_1 P^* + (1 - \eta) u_3^* \psi_2 F^* - (m + u) R^*.
\end{aligned} \tag{43}$$

□

8. Numerical approach for the FOCP model

In this regard, a general initial value problem is taken into consideration:

$$ {}_{a}^{ABC}D^{\alpha} x(t) = v(t, x(t)), x(0) = x_0 \tag{44}$$

Applying fundamental theorem of fractional calculus on equation (44), then we have

$$x(t) - x(0) = \frac{1-\alpha}{B(\alpha)} v(t, x(t)) + \frac{\alpha}{T(\alpha) B(\alpha)} \int_0^t v\left(\theta, x(\theta)(t-\theta)^{\alpha-1}\right) d\theta \tag{45}$$

where $B(\alpha) = 1 - \alpha + \frac{\alpha}{T(\alpha)}$ is a normalization function, and t_{n+1} we obtain

$$x_{n+1} - x_0 = \frac{T(\alpha)(1-\alpha)}{T(\alpha)(1-\alpha)+\alpha} v(t_n, x(t_n))$$

$$+ \frac{\alpha}{T(\alpha) + \alpha(1 - \tau(\alpha))} \sum_{m=0}^{n} \int_{tm}^{t_{m+1}} v.(t_{m+1} - \theta)^{\alpha-1} d\theta \tag{46}$$

At this instance $v(\theta, x(\theta))$ will be approximated in the interval of $[t_k, t_{k+1}]$ by employing a two-step language interpolation technique. The two-step language interpolation for the model is expressed as [30, 32]

$$Q = \frac{v(t_m, x_m)}{h} (\theta - t_{m-1}) - \frac{\lambda(t_{m-1}, x_{m-1})}{h} (\theta - t_m) \tag{47}$$

Replacing equation (47) by (8) and undertaking some steps as in [30, 33] we get:

$$\begin{aligned} &x_{n+1} - x_0 = \tfrac{T(\alpha)(1-\alpha)}{T(\alpha)(1-\alpha)+\alpha} v\left(t_n, x\left(t_n\right)\right) \\ &+ \tfrac{1}{(\alpha+1)(1-\alpha)T(\alpha)+\alpha} \textstyle\sum_{m=0}^{n} h^{\alpha} v\left(t_m, x\left(t_m\right)\right)(n+1-m)^{\alpha} \\ &(n-m+2+\alpha) - (n-m)^{\alpha}(n-m+2+2\alpha) \\ &-h^{\alpha} v\left(t_{m-1}\right), x\left(t_{m-1}\right)(n+1-m)^{\alpha+1} \\ &(n-m+2+\alpha) - (n-m)^{\alpha}(n-m+1+\alpha) \end{aligned} \tag{48}$$

In order to consider the stability of the numerical method, a simple reorganization of equation (48) is made. This new modification substitutes the step size h with $\phi(h)$ in a manner that

$$\phi(h) = h + 0\left(h^2\right), 0 \leq \phi(h) \leq 1 \tag{49}$$

Readers can consult authors [34]. This new scheme established is referred to as the nonstandard and two-step language interpolation technique (NS2LIM) which is expressed as follows:

$$\begin{aligned} &{}_{n+1} - x_0 = \tfrac{T(\alpha)(1-\alpha)}{T(\alpha)(1-\alpha)+\alpha} v\left(t_n, y\left(t_n\right)\right) \\ &+ \tfrac{1}{(\alpha+1)(1-\alpha)T(\alpha)+\alpha} \textstyle\sum_{m=0}^{n} \phi(h)^{\alpha} v\left(t_m, y\left(t_{m-1}\right)\right) \\ &(n+1-m)^{\alpha}(n-m+2+\alpha) - (n-m)^{\alpha}(n-m+2+2\alpha) \\ &-\phi(h)^{\alpha} v\left(t_{m-1}, y\left(t_{m-1}\right)\right) \\ &(n+1-m)^{\alpha+1}(n-m+2+\alpha) - (n-m)^{\alpha}(n-m+1+\alpha) \end{aligned} \tag{50}$$

The new scheme derived is then used to obtain solution to the equation (43) and implicit finite difference approach is also used to get the co-state system equations (36) solution with the transversality condition in equation (37)

9. Numerical simulations

In this section, we present the new scheme in Eq. (50) to numerically simulate the fractional- order optimal system in Eq. (43) and Eq. (36) with the transversality condition in Eq. (37) using the parameters given in system model (7) and $\varphi(h) = A(1-e^{-h})$ where A constitutes a certain positive number less than or equal to 0.001. The initial condition for the simulation is given as

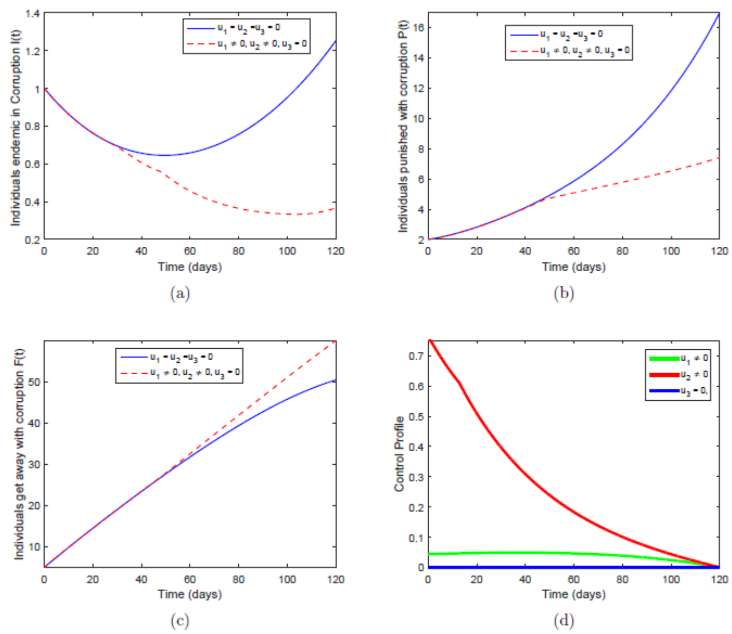

FIGURE 1. Simulations showing the effect of prevention and punished repented corruption on transmission

$S(0) = 2000, E(0) = 100, I(0) = 20, P(0) = 40, F(0) = 30, R(0) = 10$. Figures (1)–(4) show the numerical simulation results of the newly developed scheme $NS2LIM$ as in [30].

9.1. **Prevention (u_1) and Treatment (u_2) only.** The prevention control u_1 which seeks to prevent individuals from being corrupted and also social medication control u_2 which deals with individuals repented after being punished from corruption are employed to optimize the objective function J which control u_3 is made zero. Thus, control u_3 has no effect in this strategy. It is clearly shown in Figures 2(a) and 2(b) that there is a significant difference in the number of individuals endemic corruption I and those punished with corruption controlled cases and without controls respectively. In Figure 2(c) the control strategy is not effective since the control strategy has no effect on individuals who get corrupted and manage to get away with it in the society. Figure 2(d) shows the control profile employed for this strategy in ABC sense and control u_1 is 5% activated and gradually reduced till the end of the intervention. The control u_2 is initially, fully activated and immediately reduced till the end of the intervention while control u_3 is set to zero.

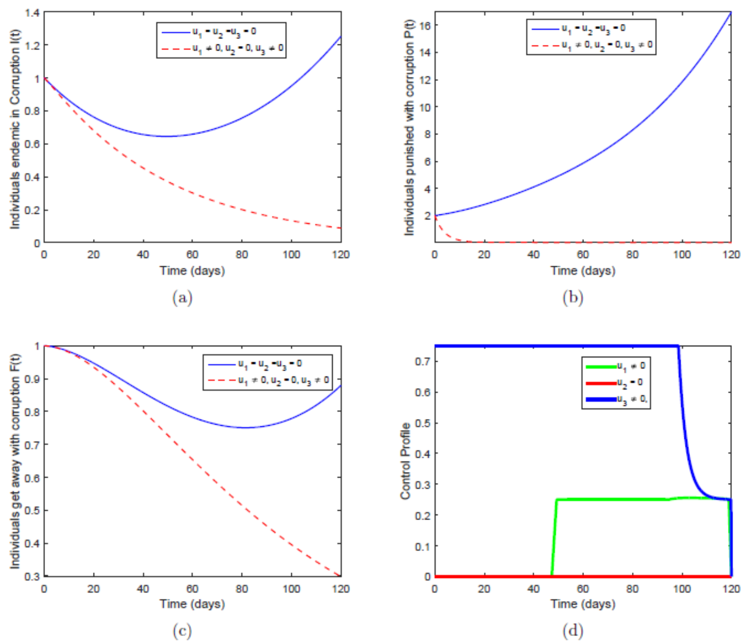

FIGURE 2. Simulations showing the effect of prevention and willingly repented corruption on transmission

9.2. **Prevention (u_1) and Treatment (u_3) only.** The prevention control u_1 which seeks to prevent individuals from being corrupted and control u_3 is a social medication designed to encourage individuals willingly repented from corruption to remain uncorrupted are activated to optimize the objective function J while control u_2 is put to zero. There is a vast difference between application for control strategy and uncontrolled cases in Figures 2(a)–2(c) in ABC sense. The Atangana– Beneanu operator has a crossover property which is able to unearth hidden outcomes. This control strategy is very effective and efficient since in all the three infected compartments, the strategy has been effective in controlling corruption in the society. The control profile is depicted in Figure 2(d) in which control u_1 is activated at 48th day and rose up to 24% which was kept constant while control u_2 was set zero. The control u_3 in this strategy is fully kept at 100% for 98 days and gradually reduced till the end of the intervention.

9.3. **Treatment (u_2) and Treatment (u_3) only.** The social treatment control u_2 which is designed to encourage individuals who have been punished for being corrupt and repented, treatment control u_3 which is social medication designed to punish individuals who willingly repent from being corrupt is used to optimize the objective while control u_1 is set to zero. In Figures 2(a)–2(c)

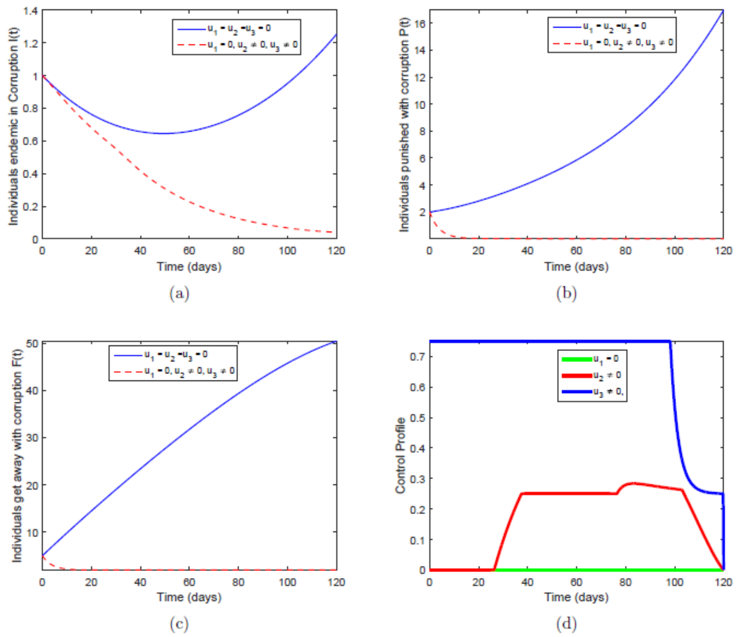

FIGURE 3. Simulations showing the effect of treatment punished corruption and treatment of willingly repented corruption on transmission

there are substantial difference in controlled cases and without controlled cases which show that the control strategy is effective in minimizing corruption in the society. This implies that ABC operator is very effective in predicting accurately. The control profile is depicted in Figure 2(d) in which control u_2 is activated at 28th day and rise up to 24% constantly till 100th day and finally reduces to zero during the rest of the intervention. The control u_1 is set to zero during this intervention. The control u_3 in this strategy is fully kept at 100% for 98 days and gradually reduced till the end of the intervention.

9.4. **Prevention, Treatment and Treatment (u_1, u_2, u_3).** In this strategy, all the three controls are activated to optimize the objective function. Figures 2(a)–2(c) show tremendous positive effect as there is much significant difference between the use of control strategy, and without the application of the control strategy. The crossover effect in ABC operator has enhanced the accurate prediction of the model outcomes. The control strategy is therefore very effective in controlling the spread of corruption in the society. Figures 2(d) is the control profile, control u_1 is constantly kept at 2% and eventually reduced to zero during the rest of the intervention. The control u_2 initially

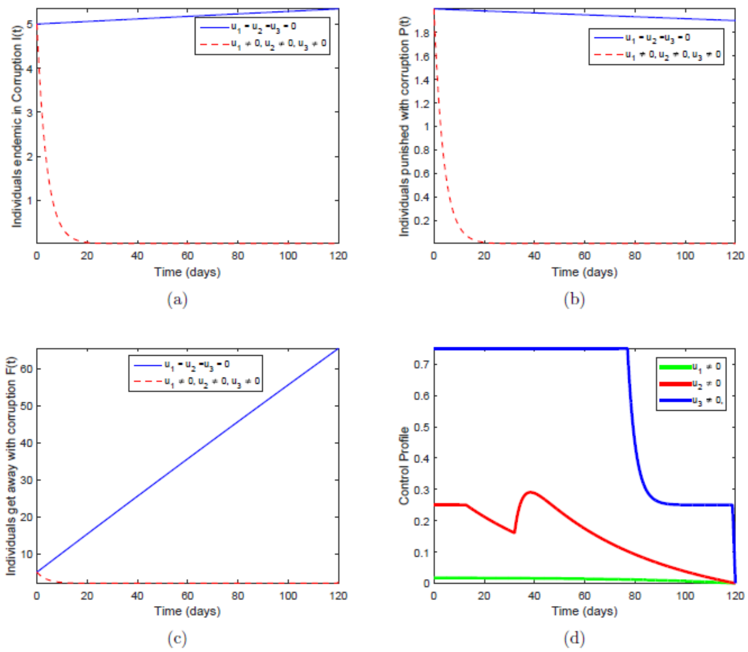

FIGURE 4. Simulations showing the effect of prevention and treatment with social medications on transmission

set at 26% for 18 days, then reduced till 38th day.This rose again to 30% on 40th day and eventually reduced to zero during the rest of the intervention.

10. Conclusions

In this study, a fractional optimal control formulated in ABC sense was employed in investigating corruption model. The model was explored making use of non-local and non-singular kernel. This study utilized three controls u_1, u_2 and u_3 that were designed to make corruption unattractive in the society. It can be inferred that this fractional –order in ABC sense has the ability to describe the complexity in real life problems due to the crossover characteristics than in the case of integer order. The numerical schemes employed for this work was NS2LIM which produced accurate predictions. Some figures are presented to depict the effectiveness of the numerical scheme and of the optimal.The utilization of the all the controls at the same time, is the most effective strategy in controlling corruption in the society.

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