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Determinant Spectrum of Diagonal Block Matrix

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Abstract

It is known that in mathematical literature one of important questions of spectral theory of operators is to describe spectrum of diagonal block matrices in the direct sum of Banach spaces with the spectrums of their coordinate operators. This problem has been investigated in works [1] and [2]. Also for the singular numbers similar investigation has been made in [3]. In this paper the analogous question is researched. Namely, the relationships between ϵ-determinat spectrums of the diagonal block matrices and their block matrices are investigated. Later on, some applications are given.

Keywords: Eigenvalues, Spectrum, Determinant, Diagonal Block Matrix.
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1. Introduction

It is known that the determine of place of spectrum set of any linear densely defined closed operator in any linear normed space is one of central questions of spectral theory of operators. There are several generalizations of spectrum concept in mathematical literature. Knowing of these generalizations are pseudospectrum [4], condition spectrum [5], Ransford spectrum [6] and determinant spectrum [7]. Note that the determination of spectrum or resolvent sets have special and important place in Operator Theory. Unfortunately, in most cases the presence of these sets imposes great technical and tactical imperatives. In such cases the aim is at least the approximate location of spectrum sets in complex plane is of great importance.

The same difficulties arise the calculation of eigenvalues of large size square matrices. In this reason with the help of the determinant spectrum set it is possible to have an information.

In addition, the article of Krishna [7] has been a great motivation in the creation of the subject of this article.

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2. Determinant Spectrum

Definition 2.1. [7] Let $A \in \mathbb{C}^{n\times n}$ and $\epsilon > 0$. The ϵ-determinant spectrum of A is defined as

$$d_\epsilon(A) = \{z \in \mathbb{C} : |det(zI - A)| \leq \epsilon\}$$

The set $d_\epsilon(A)$ contains the all eigenvalues, i.e., $\sigma(A) \subseteq d_\epsilon(A)$ and $\sigma(A) = d_0(A)$ for any $n \times n$ matrix. For example, ϵ-determinant spectrum of matrix

$$A = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \in \mathbb{C}^{2\times 2}, \alpha, \beta \in \mathbb{C}$$

is in form

$$d_\epsilon(A) = \{z \in \mathbb{C} : |z^2 - \alpha\beta| \leq \epsilon\}.$$

At the same time the eigenvalues of A are in form

$$\lambda_\pm = \pm\sqrt{\alpha\beta}.$$

Now let m_j be a natural number for any $j = 1, 2, ..., n$ and $A_j \in \mathbb{C}^{m_j\times m_j}$. And also $A = \overset{n}{\underset{j=1}{\oplus}} A_j$, $\mathfrak{X} = \overset{n}{\underset{j=1}{\oplus}} \mathbb{C}^{m_j}$, $A : \mathfrak{X} \to \mathfrak{X}$. Here A is $\left(\sum_{j=1}^n m_j\right) \times \left(\sum_{j=1}^n m_j\right)$ square block matrix.

Theorem 2.2. *For any* $\epsilon_1 > 0, ..., \epsilon_n > 0$ *it is clear that*

$$\bigcap_{j=1}^n d_{\epsilon_j}(A_j) \subset d_\epsilon(A) \subset \bigcup_{j=1}^n d_{\epsilon_j}(A_j)$$

where $\epsilon = \epsilon_1 \cdot \epsilon_2 \cdots \epsilon_n$.

Proof. In this case when $z \in d_{\epsilon_j}(A_j)$, $j = 1, 2, ..., n$, then for each $\epsilon_j > 0, j = 1, 2, ..., n$ we have

$$|det(zI - A)| = |det\,(zI_1 - A_1)\,| \cdots |det\,(zI_n - A_n)\,| \leq \epsilon_1 \cdot \epsilon_2 \cdots \epsilon_n.$$

From this it is obtained that

$$z \in d_\epsilon(A)\ , \quad \epsilon = \epsilon_1 \cdot \epsilon_2 \cdots \epsilon_n.$$

Now assume that $z \in \bigcap_{j=1}^n d^c_{\epsilon_j}(A_j)$ for each $\epsilon_j > 0$, $j = 1, 2, ..., n$. Then

$$|det(zI - A)| = |det\,(zI_1 - A_1)\,| \cdots |det\,(zI_n - A_n)\,| > \epsilon_1 \cdot \epsilon_2 \cdots \epsilon_n.$$

That is $z \in d^c_\epsilon(A)$, $\epsilon = \epsilon_1 \cdot \epsilon_2 \cdots \epsilon_n$. Consequently

$$\bigcap_{j=1}^n d^c_{\epsilon_j}(A_j) \subset d^c_\epsilon(A).$$

So

$$d_\epsilon(A) \subset \bigcup_{j=1}^n d_{\epsilon_j}(A_j).$$

□

Theorem 2.3. *Under assumptions for each $\epsilon > 0$ it is true that*

$$d_\epsilon(A) = \bigcup_{\substack{\epsilon_1>0,\dots,\epsilon_n>0 \\ \epsilon = \epsilon_1\cdot\epsilon_2\cdots\epsilon_n}} \left(\bigcap_{j=1}^{n} d_{\epsilon_j}(A_j) \right)$$

Proof. In this case when $z \in d_{\epsilon_j}(A_j),\ j = 1, 2, ..., n$ for any $\epsilon_1 > 0, ..., \epsilon_n > 0$, then it is easy to see that

$$|det(zI - A)| = \prod_{j=1}^{n} |det\,(zI_j - A_j)\,| \le \epsilon_1\cdot \epsilon_2 \cdots \epsilon_n.$$

So $z \in d_\epsilon(A)$, $\epsilon = \epsilon_1\cdot \epsilon_2 \cdots \epsilon_n$. In this situation it implies that

$$\bigcup_{\substack{\epsilon_1>0,\dots,\epsilon_n>0 \\ \epsilon = \epsilon_1\cdot\epsilon_2\cdots\epsilon_n}} \left(\bigcap_{j=1}^{n} d_{\epsilon_j}(A_j) \right) \subset d_\epsilon(A).$$

Let prove the contrary part of last inclusion in case when $n = 2$. That is assumed that

$$A_j \in \mathbb{C}^{m_j\times m_j}, j = 1, 2,\ \mathfrak{X} = \mathbb{C}^{m_1\times m_1} \oplus \mathbb{C}^{m_2\times m_2},\ A = A_1 \oplus A_2,\ A : \mathfrak{X} \to \mathfrak{X}.$$

Let us $z_0 \in d_\epsilon(A)$ for any $\epsilon > 0$. Then

$$|det\,(z_0I - A)\,| \le \epsilon.$$

This means that

$$|det\,(z_0I_1 - A_1)\,|\cdot |det\,(z_0I_2 - A_2)\,| \le \epsilon.$$

If it is denoted by $\epsilon_1 > 0$ number satisfying inequality

$$|det\,(z_0I_1 - A_1)\,| \le \epsilon_1,$$

then from last relation it is obtained that

$$|det\,(z_0I_2 - A_2)\,| \le \epsilon/\epsilon_1.$$

Consequently, if $z_0 \in d_{\epsilon_1}(A_1)$, then $z_0 \in d_{\epsilon_2}(A_2), \epsilon_2 = \epsilon/\epsilon_1$. This means that with condition $\epsilon = \epsilon_1\cdot \epsilon_2$

$$d_\epsilon(A) \subset (\ d_{\epsilon_1}(A_1) \cap d_{\epsilon_2}(A_2)\ ).$$

Therefore in case when $n = 2$

$$d_\epsilon(A) = \bigcup_{\substack{\epsilon_1>0,\epsilon_2>0 \\ \epsilon = \epsilon_1\cdot\epsilon_2}} (\ d_{\epsilon_1}(A_1) \cap d_{\epsilon_2}(A_2)\ ).$$

The general case could be proved similarly. □

The following proposition is true from the definition of determinant spectrum.

Theorem 2.4. *Let $f : \{1, ..., n\} \to \{1, ..., n\}$ is a one-to-one onto function. Then for each $\epsilon > 0$*

$$d_\epsilon \left(A_{f(1)} \oplus \cdots \oplus A_{f(n)} \right) = d_\epsilon(A).$$

Theorem 2.5. *Let us $f_j : \mathbb{C}^{m_j} \to \mathbb{C}^{m_j}, j = 1, 2, ..., n$ be a linear, unital and determinant preserving function and $f = \overset{n}{\underset{j=1}{\oplus}} f_j : \mathfrak{X} \to \mathfrak{X}$. Then f preserves ϵ-determinant for each $\epsilon > 0$.*

Proof. In this case by Theorem 3.3 [7] for any $1 \le j \le n$, f_j preserves ϵ-determinant spectrum for all $\epsilon > 0$. Then for any $A : \mathfrak{X} \to \mathfrak{X}$ we have

$$|det f(zI - A)| = \prod_{j=1}^{n} |det\,(zf(I_j) - f(A))\,| = \prod_{j=1}^{n} |det\,(zI_j - f(A))\,|.$$

From this it is obtained that for each $\epsilon > 0$

$$d_\epsilon(f(A)) = d_\epsilon(A).$$

□

3. Applications

Example 3.1. Let us

$$A_1, A_2 \in \mathbb{C}^{2\times 2}, A_1 = \begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 2 & \alpha_2 \\ 0 & 2 \end{pmatrix}, \alpha_1, \alpha_2 \in \mathbb{C}$$

Then for each $\epsilon > 0$

$$\begin{aligned} d_\epsilon(A_1) &= \{\ z \in \mathbb{C} : |det\,(zI_1 - A_1)\,| \le \epsilon\ \} = \{\ z \in \mathbb{C} : |z - 1| \le \sqrt{\epsilon}\ \}, \\ d_\epsilon(A_2) &= \{\ z \in \mathbb{C} : |det\,(zI_2 - A_2)\,| \le \epsilon\ \} = \{\ z \in \mathbb{C} : |z - 2| \le \sqrt{\epsilon}\ \}. \end{aligned}$$

In this case by Theorem 2.5, we have for each $\epsilon > 0$

$$\begin{aligned} d_\epsilon\,(A_1 \oplus A_2) &= \bigcup_{\substack{\epsilon_1 > 0, \epsilon_2 > 0 \\ \epsilon\, =\, \epsilon_1 \cdot \epsilon_2}} (\ d_{\epsilon_1}(A_1) \cap d_{\epsilon_2}(A_2)\ ) \\ &= \bigcup_{\substack{\epsilon_1 > 0, \epsilon_2 > 0 \\ \epsilon\, =\, \epsilon_1 \cdot \epsilon_2}} \{\ z \in \mathbb{C} : |z - 1| \le \sqrt{\epsilon_1} \quad \text{and} \quad |z - 2| \le \sqrt{\epsilon_2}\ \} \end{aligned}$$

Example 3.2. Let us

$$A_1, A_2 \in \mathbb{C}^{3\times 3}, A_1 = \begin{pmatrix} \alpha & 0 & 1 \\ 0 & \alpha & 0 \\ 1 & 0 & \alpha \end{pmatrix}, A_2 = \begin{pmatrix} \beta & 0 & 1 \\ 0 & \beta & 0 \\ 1 & 0 & \beta \end{pmatrix}, \alpha, \beta \in \mathbb{C},\ A = A_1 \oplus A_2,\ A \in \mathbb{C}^{6\times 6}$$

In this case for each $\epsilon > 0$

$$d_\epsilon(A_1) = \{\ z \in \mathbb{C} : |det\,(zI_1 - A_1)\,| \le \epsilon\ \} = \{\ z \in \mathbb{C} : |(z - \alpha)^3 - (z - \alpha)| \le \epsilon\ \}$$

$$d_\epsilon(A_2) = \{\ z \in \mathbb{C} : |det\,(zI_2 - A_2)\,| \le \epsilon\ \} = \{\ z \in \mathbb{C} : |(z - \beta)^3 - (z - \beta)| \le \epsilon\ \}$$

Therefore by Theorem 2.5, we have for each $\epsilon > 0$

$$\begin{aligned} d_\epsilon\,(A_1 \oplus A_2) &= \bigcup_{\substack{\epsilon_1 > 0, \epsilon_2 > 0 \\ \epsilon\, =\, \epsilon_1 \cdot \epsilon_2}} (\ d_{\epsilon_1}(A_1) \cap d_{\epsilon_2}(A_2)\ ) \\ &= \bigcup_{\substack{\epsilon_1 > 0, \epsilon_2 > 0 \\ \epsilon\, =\, \epsilon_1 \cdot \epsilon_2}} \{\ z \in \mathbb{C} : |(z - \alpha)^3 - (z - \alpha)| \le \epsilon_1 \quad \text{and} \quad |(z - \beta)^3 - (z - \beta)| \le \epsilon_2\ \}. \end{aligned}$$

4. Conclusion

In this paper, connections between ϵ-determinant spectrums of the diagonal block matrices and their block matrices have been determined. Then, obtained results have been supported by some applications. On the other hand, these results will provide of the localization of the spectrum in cases where it is difficult to find eigenvalues of the large sizes block diagonal matrices. Also, it is predicted that these results will be used in linear algebra and matrices theory.

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