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On a third-order fuzzy difference equation

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Abstract

In this paper, we investigate the qualitative behavior of the fuzzy difference equation

$$z_{n+1} = \frac{z_{n-2}}{C + z_{n-2}z_{n-1}z_n}$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, (z_n) is a sequence of positive fuzzy numbers, C and initial conditions z_{-2}, z_{-1}, z_0 are positive fuzzy numbers.

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1. Introduction

Difference equations appear naturally as discrete analogs and as numerical solutions of differential equations and delay differential equations having many applications in population statistics and analysis, economics, biology, computer sciences, engineering, etc ([1, 6, 9, 13] and the references therein). Fuzzy difference equation is a difference equation that initial conditions and parameters are fuzzy numbers and its solutions are sequence of fuzzy numbers. Recently, there has been a lot of work concerning the fuzzy difference equations because many real life problems are modeled by these equations naturally. For example, in [5], Deeba and Korvin studied the second order fuzzy difference equation

$$x_{n+1} = x_n - abx_{n-1} + c,\ n \in \mathbb{N}_0 \tag{1.1}$$

where (x_n) is a sequence of fuzzy numbers and a, b, c, x_{-1}, x_0 are fuzzy numbers. This equation is a linearized model of a nonlinear model which determines the carbondioxide (CO_2) level in the blood.

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In [11], Papaschinopoulos and Papadopoulos studied the existence, the boundedness and the asymptotic behavior of the positive solutions of the fuzzy difference equation

$$x_{n+1} = A + \frac{B}{x_n}, \ n \in \mathbb{N}_0 \tag{1.2}$$

where (x_n) is a sequence of fuzzy numbers and A, B, x_0 are fuzzy numbers.

In [2], Bajo and Liz investigated the global behavior of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a + bx_{n-1}x_n}, \ n \in \mathbb{N}_0 \tag{1.3}$$

where the parameters a, b and initial conditions x_{-1}, x_0 are real numbers.

Moreover, in [14], Rahman et al. investigated the qualitative behavior of the second-order rational fuzzy difference equation

$$x_{n+1} = \frac{x_{n-1}}{A + Bx_{n-1}x_n}, \ n \in \mathbb{N}_0 \tag{1.4}$$

where A, B and initial conditions x_{-1}, x_0 are positive fuzzy numbers.

For more works on fuzzy difference equations, see [4, 7, 16] and the references cited therein.

In [15], Shojaei et al. investigated the stability and periodic character of the rational third-order difference equation

$$x_{n+1} = \frac{\alpha x_{n-2}}{\beta + \gamma x_{n-2}x_{n-1}x_n}, \ n \in \mathbb{N}_0 \tag{1.5}$$

where the parameters α, β, γ and initial conditions x_{-2}, x_{-1}, x_0 are real numbers.

Moreover, in [18], Zhang et al. investigated the dynamical behavior of positive solutions of the system of difference equations

$$x_{n+1} = \frac{x_{n-2}}{B + y_{n-2}y_{n-1}y_n}, \ y_{n+1} = \frac{y_{n-2}}{A + x_{n-2}x_{n-1}x_n}, \ n \in \mathbb{N}_0 \tag{1.6}$$

where A, B and initial conditions $x_{-2}, x_{-1}, x_0, \ y_{-2}, y_{-1}, y_0$ are positive real numbers. If $A < 1, \ B < 1$, then system (1.6) has equilibrium $(0, 0)$ and $(\sqrt[3]{1-A}, \ \sqrt[3]{1-B})$. In addition, if $A < 1, \ B = 1$, then system (1.6) has an equilibrium $(\sqrt[3]{1-A}, 0)$ and if $A = 1, \ B < 1$, then system (1.6) has an equilibrium $(0, \sqrt[3]{1-B})$. Finally, if $A > 1, \ B > 1$, then system (1.6) has an equilibrium $(0, 0)$.

In this paper, we investigate the qualitative behavior of the fuzzy difference equation

$$z_{n+1} = \frac{z_{n-2}}{C + z_{n-2}z_{n-1}z_n}, \ n \in \mathbb{N}_0 \tag{1.7}$$

where (z_n) is a sequence of positive fuzzy numbers, C and initial conditions z_{-2}, z_{-1}, z_0 are positive fuzzy numbers.

2. Preliminaries

In this section, we give some definitions which will be used in this paper [12]. For more details see [3, 8, 10, 17].

Definition 2.1. Let A be any set.

(a) The set A is said to be *fuzzy* if A is a function from $\mathbb{R}^+$ into the interval $[0, 1]$.

(b) The set A is said to be *convex* if for every $t \in [0, 1]$ and $x_1, x_2 \in \mathbb{R}^+$ we have $A(tx_1 + (1-t)x_2) \geq \min\{A(x_1), A(x_2)\}$.

(c) The set A is said to be *normal* if there exists $x \in \mathbb{R}^+$ such that $A(x) = 1$.

(d) An α-cut for fuzzy set A, $\alpha \in (0, 1]$, is the set $[A]_\alpha = \{x \in \mathbb{R}^+, A(x) \geq \alpha\}$.

(e) For a set H we denote by $\overline{H}$ the closure of H. We say that A is a fuzzy number if the following conditions hold:

  (i) A is a normal set,

  (ii) A is a convex fuzzy set,

  (iii) A is upper semicontinuous,

  (iv) A is compactly supported i.e., $\overline{\{x \in \mathbb{R} : A(x) > 0\}}$ is compact.

(f) We say that a fuzzy number A is a positive if $supp\, A \subset (0, \infty)$.

Definition 2.2. (a) Let A, B be any fuzzy numbers with $[A]_\alpha = [A_{l,\alpha}, A_{r,\alpha}]$ and $[B]_\alpha = [B_{l,\alpha}, B_{r,\alpha}]$ for $\alpha \in (0, 1]$. We define a norm on the fuzzy numbers space as follow;

$$||A|| = \sup \max \{|A_{l,\alpha}|, |A_{r,\alpha}|\}$$

where sup is taken for all $a \in (0, 1]$. Then from the above norm we take the following metric

$$D(A, B) = \sup \{\max \{|A_{l,\alpha} - B_{l,\alpha}|, |A_{r,\alpha} - B_{r,\alpha}|\}\}$$

where sup is taken for all $a \in (0, 1]$.

(b) Let (x_n) be a sequence of positive fuzzy numbers and x is a fuzzy number. Then we say that

$$\lim_{n\to\infty} x_n = x \text{ if } \lim_{n\to\infty} D(x_n, x) = 0.$$

The fuzzy analog of concept of boundedness and persistence are given in [11] as follows:

Definition 2.3. (a) We say that a sequence of positive fuzzy numbers (x_n) is bounded and persists if there exist $n_0 \in \mathbb{N}$ and positive fuzzy numbers C, D such that $\min \{x_n, C\} = C$ and $\min \{x_n, D\} = D$ for $n \geq n_0$.

(b) We say that (x_n) for $n \in \mathbb{N}_0$ is an unbounded sequence if the norm $||x_n||$ for $n \in \mathbb{N}_0$ is an unbounded sequence.

3. Main Results

In this section, we prove our main results. Firstly, we will study the existence of the positive solutions of Eq.(1.7). We say (z_n) is a positive solution of Eq.(1.7) if (z_n) is a sequence of positive fuzzy numbers which satisfies Eq.(1.7).

We need the following lemma which is a generalization of Lemma 2.1 of [12].

Lemma 3.1. *Let $f : \mathbb{R}^+ \times \mathbb{R}^+ \times ... \times \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function and $A_1, A_2, ..., A_k$ be fuzzy numbers. Then,*

$$[f (A_1, A_2, ..., A_k)]_\alpha = f ([A_1]_\alpha, [A_2]_\alpha, ..., [A_k]_\alpha)$$

for all $\alpha \in (0, 1]$.

Proof. It is sufficient to prove that for every $w \in \mathbb{R}^+$ the

$$\sup_{x\in f^{-1}(w)} \{\min\{A_1(x_1), A_2(x_2), ..., A_k(x_k)\}\},$$

where $x = (x_1, x_2, ..., x_k)$, is attained. We define the function $h : \mathbb{R}^+ \times \mathbb{R}^+ \times ... \times \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$h(x_1, x_2, ..., x_k) = \min\{A_1(x_1), A_2(x_2), ..., A_k(x_k)\}.$$

It is clear that if $(x_1, x_2, ..., x_k) \neq S_{A_1} \times S_{A_2} \times ... \times S_{A_k}$ where S_{A_1} (resp. S_{A_i} for $i = 2, 3, \ldots, k$) is the support of A_1 (resp. A_i for $i = 2, 3, \ldots, k$), then $h(x_1, x_2, ..., x_k) = 0$. Therefore,

$$\sup_{x\in f^{-1}(w)} h(x_1, x_2, ..., x_k) = \sup_{x\in f^{-1}(w)\cap(S_{A_1}\times S_{A_2}\times...\times S_{A_k})} h(x_1, x_2, ..., x_k).$$

From the condition (iv) of the definition of fuzzy numbers we have that $S_{A_1} \times S_{A_2} \times ... \times S_{A_k}$ is compact. Moreover since f is a continuous function we have that $f^{-1}(w)$ is a closed set. Therefore the set $f^{-1}(w) \cap (S_{A_1} \times S_{A_2} \times ... \times S_{A_k})$ is compact.

In addition since from the condition (iii) of the definition of fuzzy numbers, the fuzzy numbers $A_1, A_2, ..., A_k$ are upper semicontinuous it is clear that the function h is also upper semicontinuous. Therefore, there exist $\overline{x}_1, \overline{x}_2, ..., \overline{x}_k, (\overline{x}_1, \overline{x}_2, ..., \overline{x}_k) \in f^{-1}(w) \cap (S_{A_1} \times S_{A_2} \times ... \times S_{A_k})$ such that

$$\begin{aligned}\sup_{x\in f^{-1}(w)} h(x_1, x_2, ..., x_k) &= \sup_{x\in f^{-1}(w)\cap(S_{A_1}\times S_{A_2}\times...\times S_{A_k})} h(x_1, x_2, ..., x_k)\\ &= h(\overline{x}_1, \overline{x}_2, ..., \overline{x}_k).\end{aligned}$$

This completes the proof. □

Theorem 3.2. *Consider Eq.(1.7) where C is a positive fuzzy number. Then for any positive fuzzy numbers z_{-2}, z_{-1}, z_0 there exists a unique positive solution (z_n) of Eq.(1.7) with the initial conditions z_{-2}, z_{-1}, z_0.*

Proof. Suppose that there exists a sequence of positive fuzzy numbers (z_n) satisfying Eq.(1.7) with the initial conditions z_{-2}, z_{-1}, z_0. Consider the α-cuts

$$\begin{cases}[z_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}],\ n = -2, -1, ...,\\ [C]_\alpha = [C_{l,\alpha}, C_{r,\alpha}]\end{cases} \tag{3.1}$$

for $\alpha \in (0, 1]$. Then from (1.7), (3.1) and Lemma 3.1 it follows that

$$\begin{aligned}[z_{n+1}]_\alpha &= \left[\frac{z_{n-2}}{C + z_{n-2}z_{n-1}z_n}\right]_\alpha\\ &= \frac{[z_{n-2}]_\alpha}{[C]_\alpha + [z_{n-2}]_\alpha[z_{n-1}]_\alpha[z_n]_\alpha}\\ &= \frac{[L_{n-2,\alpha}, R_{n-2,\alpha}]}{[C_{l,\alpha}, C_{r,\alpha}] + [L_{n-2,\alpha}, R_{n-2,\alpha}][L_{n-1,\alpha}, R_{n-1,\alpha}][L_{n,\alpha}, R_{n,\alpha}]}\\ &= \left[\frac{L_{n-2,\alpha}}{C_{r,\alpha} + R_{n-2,\alpha}R_{n-1,\alpha}R_{n,\alpha}}, \frac{R_{n-2,\alpha}}{C_{l,\alpha} + L_{n-2,\alpha}L_{n-1,\alpha}L_{n,\alpha}}\right]\end{aligned}$$

from which we get

$$L_{n+1,\alpha} = \frac{L_{n-2,\alpha}}{C_{r,\alpha} + R_{n-2,\alpha}R_{n-1,\alpha}R_{n,\alpha}},\ R_{n+1,\alpha} = \frac{R_{n-2,\alpha}}{C_{l,\alpha} + L_{n-2,\alpha}L_{n-1,\alpha}L_{n,\alpha}} \tag{3.2}$$

for $\alpha \in (0, 1]$ and $n \in \mathbb{N}_0$. Then it is clear that for any $(L_{j,\alpha}, R_{j,\alpha})$, $j = -2, -1, 0$ there exists a unique solution $(L_{n,\alpha}, R_{n,\alpha})$ with the initial conditions $(L_{j,\alpha}, R_{j,\alpha})$, $j = -2, -1, 0$ for $\alpha \in (0, 1]$.

Now, we prove that $[L_{n,\alpha}, R_{n,\alpha}]$ for $\alpha \in (0,1]$ where $(L_{n,\alpha}, R_{n,\alpha})$ is the solution of the system (3.2) with the initial conditions $(L_{j,\alpha}, R_{j,\alpha})$, $j = -2,-1,0$ determines the solution (z_n) of Eq.(1.7) with the initial conditions z_{-2}, z_{-1}, z_0 such that

$$[z_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}],\ \alpha \in (0,1],\ n = -2,-1,.... \tag{3.3}$$

Since C and z_{-2}, z_{-1}, z_0 are positive fuzzy numbers for any $\alpha_1, \alpha_2 \in (0,1]$ and $\alpha_1 \le \alpha_2$, we get

$$\begin{cases} 0 < C_{l,\alpha_1} \le C_{l,\alpha_2} \le C_{r,\alpha_2} \le C_{r,\alpha_1} \\ 0 < L_{j,\alpha_1} \le L_{j,\alpha_2} \le R_{j,\alpha_2} \le R_{j,\alpha_1} \end{cases} \tag{3.4}$$

for $j = -2,-1,0$. We prove by the induction that

$$L_{n,\alpha_1} \le L_{n,\alpha_2} \le R_{n,\alpha_2} \le R_{n,\alpha_1} \tag{3.5}$$

for $n \in \mathbb{N}_0$. From (3.4) we have that (3.5) hold for $n = -2,-1,0$. Suppose that (3.5) are true for $n \le k$, $k \in \{1,2,...\}$. Then from (3.2), (3.4) and (3.5) for $n \le k$ it follows that

$$\begin{aligned} L_{k+1,\alpha_1} &= \frac{L_{k-2,\alpha_1}}{C_{r,\alpha_1} + R_{k-2,\alpha_1}R_{k-1,\alpha_1}R_{k,\alpha_1}} \le \frac{L_{k-2,\alpha_2}}{C_{r,\alpha_2} + R_{k-2,\alpha_2}R_{k-1,\alpha_2}R_{k,\alpha_2}} = L_{k+1,\alpha_2}, \\ L_{k+1,\alpha_2} &= \frac{L_{k-2,\alpha_2}}{C_{r,\alpha_2} + R_{k-2,\alpha_2}R_{k-1,\alpha_2}R_{k,\alpha_2}} \le \frac{R_{k-2,\alpha_2}}{C_{l,\alpha_2} + L_{k-2,\alpha_2}L_{k-1,\alpha_2}L_{k,\alpha_2}} = R_{k+1,\alpha_2} \\ R_{k+1,\alpha_2} &= \frac{R_{k-2,\alpha_2}}{C_{l,\alpha_2} + L_{k-2,\alpha_2}L_{k-1,\alpha_2}L_{k,\alpha_2}} \le \frac{R_{k-2,\alpha_1}}{C_{l,\alpha_1} + L_{k-2,\alpha_1}L_{k-1,\alpha_1}L_{k,\alpha_1}} = R_{k+1,\alpha_1}. \end{aligned}$$

Therefore (3.5) are satisfied. Moreover from (3.2) we get

$$L_{1,\alpha} = \frac{L_{-2,\alpha}}{C_{r,\alpha} + R_{-2,\alpha}R_{-1,\alpha}R_{0,\alpha}},\ R_{1,\alpha} = \frac{R_{-2,\alpha}}{C_{l,\alpha} + L_{-2,\alpha}L_{-1,\alpha}L_{0,\alpha}} \tag{3.6}$$

for $\alpha \in (0,1]$. Then, since C and z_{-2}, z_{-1}, z_0 are positive fuzzy numbers, we have that $C_{l,\alpha}$, $C_{r,\alpha}$, $L_{-2,\alpha}$, $R_{-2,\alpha}$, $L_{-1,\alpha}$, $R_{-1,\alpha}$, $L_{0,\alpha}$ and $R_{0,\alpha}$ are left continuous. So, from (3.6) we have that $L_{1,\alpha}$ and $R_{1,\alpha}$ are also left continuous. Working inductively we can easily prove that $L_{n,\alpha}$ and $R_{n,\alpha}$ are left continuous for $n \in \mathbb{N}$.

Now, we prove that $\overline{\cup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]}$ is compact. It is sufficient to prove that $\cup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]$ is bounded. Let $n = 1$, since C and z_{-2}, z_{-1}, z_0 are positive fuzzy numbers there exist constants $M_C, N_C, M_j, N_j > 0$ for $j = -2,-1,0$ such that

$$[C_{l,\alpha}, C_{r,\alpha}] \subset [M_C, N_C],\ [L_{j,\alpha}, R_{j,\alpha}] \subset [M_j, N_j],\ j = -2,-1,0. \tag{3.7}$$

Therefore, from (3.6) and (3.7) we can easily obtain that

$$[L_{1,\alpha}, R_{1,\alpha}] \subset \left[\frac{M_{-2}}{N_C + N_{-2}N_{-1}N_0}, \frac{N_{-2}}{M_C + M_{-2}M_{-1}M_0}\right]$$

from which it is clear that

$$\cup_{\alpha\in(0,1]} [L_{1,\alpha}, R_{1,\alpha}] \subset \left[\frac{M_{-2}}{N_C + N_{-2}N_{-1}N_0}, \frac{N_{-2}}{M_C + M_{-2}M_{-1}M_0}\right] \tag{3.8}$$

for $\alpha \in (0,1]$. (3.8) implies that $\overline{\cup_{\alpha\in(0,1]}[L_{1,\alpha}, R_{1,\alpha}]}$ is compact and

$$\overline{\cup_{\alpha\in(0,1]}[L_{1,\alpha}, R_{1,\alpha}]} \subset (0,\infty).$$

Working inductively we can easily obtain that

$$\overline{\cup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]} \text{ is compact, } \overline{\cup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]} \subset (0,\infty) \tag{3.9}$$

for $n \in \mathbb{N}$. Therefore, (3.5), (3.9) and since $L_{n,\alpha}$, $R_{n,\alpha}$ are left continuous we get that $[L_{n,\alpha}, R_{n,\alpha}]$ determines a sequence of positive fuzzy numbers (z_n) such that (3.3) holds.

We prove now that (z_n) is the solution of Eq.(1.7) with the initial conditions z_{-2}, z_{-1}, z_0. Since

$$\begin{aligned}[z_{n+1}]_\alpha &= [L_{n+1,\alpha}, R_{n+1,\alpha}] \\ &= \left[\frac{L_{n-2,\alpha}}{C_{r,\alpha} + R_{n-2,\alpha}R_{n-1,\alpha}R_{n,\alpha}}, \frac{R_{n-2,\alpha}}{C_{l,\alpha} + L_{n-2,\alpha}L_{n-1,\alpha}L_{n,\alpha}}\right] \\ &= \left[\frac{z_{n-2}}{C + z_{n-2}z_{n-1}z_n}\right]_\alpha\end{aligned}$$

for all $\alpha \in (0,1]$, we have that (z_n) is the solution of Eq.(1.7) with the initial conditions z_{-2}, z_{-1}, z_0.

Suppose that there exists another solution $(\widetilde{z}_n)$ of Eq.(1.7) with the initial conditions z_{-2}, z_{-1}, z_0. Then arguing as above we can easily prove that

$$[\widetilde{z}_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}] \tag{3.10}$$

for $\alpha \in (0,1]$ and $n \in \mathbb{N}_0$. Then from (3.3) and (3.10) we have that $[z_n]_\alpha = [\widetilde{z}_n]_\alpha$ for $\alpha \in (0,1]$ and $n = -2, -1, 0, ...$ from which it holds $z_n = \widetilde{z}_n$ for $n = -2, -1, 0, ...$ Thus, the proof is completed. □

To study the dynamics of the fuzzy difference Eq.(1.7), we need the following theorem concerning the behavior of the solutions of the system (1.6) which has been proved in [18]:

Theorem 3.3. *Let (x_n, y_n) be any positive solution of system (1.6), then the following statements are true:*

(1) For all $k \geq 0$, the following results hold:

$$0 \leq x_n \leq \begin{cases} \left(\frac{1}{B}\right)^{k+1} x_{-2}, & n = 3k+1, \\ \left(\frac{1}{B}\right)^{k+1} x_{-1}, & n = 3k+2, \\ \left(\frac{1}{B}\right)^{k+1} x_0, & n = 3k+3, \end{cases} \tag{3.11}$$

and

$$0 \leq y_n \leq \begin{cases} \left(\frac{1}{A}\right)^{k+1} y_{-2}, & n = 3k+1, \\ \left(\frac{1}{A}\right)^{k+1} y_{-1}, & n = 3k+2, \\ \left(\frac{1}{A}\right)^{k+1} y_0, & n = 3k+3. \end{cases} \tag{3.12}$$

(2) If $A > 1$ and $B > 1$, then every solution of system (1.6) is bounded.

(3) For the equilibriums of system (1.6), the following statements are true:

(i) If $A > 1$ and $B > 1$, then (x_n, y_n) converges exponentially to the equilibrium $(0,0)$ and the equilibrium $(0,0)$ is locally asymptotically stable.

(ii) If $A < 1$ and $B < 1$, then the equilibriums $(0,0)$ and $(\sqrt[3]{1-A}, \sqrt[3]{1-B})$ are locally unstable.

(4) If $A < 1$ and $B < 1$, then the following statements are true for $j = -2, -1, 0$:

(i) If $(x_j, y_j) \in (0, \sqrt[3]{1-A}) \times (\sqrt[3]{1-B}, \infty)$, then $(x_n, y_n) \in (0, \sqrt[3]{1-A}) \times (\sqrt[3]{1-B}, \infty)$.

(ii) If $(x_j, y_j) \in (\sqrt[3]{1-A}, \infty) \times (0, \sqrt[3]{1-B})$, then $(x_n, y_n) \in (\sqrt[3]{1-A}, \infty) \times (0, \sqrt[3]{1-B})$.

The following corollary is obtained from (4) of Theorem 3.3:

Corollary 3.4. *Assume that $A < 1$, $B < 1$ and (x_n, y_n) is a positive solution of system (1.6). Then, the following statements are true:*

(i) If $x_{-2}, x_{-1}, x_0 < \sqrt[3]{1-A}$ and $y_{-2}, y_{-1}, y_0 > \sqrt[3]{1-B}$, then $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = \infty$.

(ii) If $x_{-2}, x_{-1}, x_0 > \sqrt[3]{1-A}$ and $y_{-2}, y_{-1}, y_0 < \sqrt[3]{1-B}$, then $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty} y_n = 0$.

Theorem 3.5. *Consider the Eq.(1.7). Then the following statements are true:*

(i) If $C_{l,\alpha} > 1$ for all $\alpha \in (0,1]$, then every positive solution (z_n) of Eq.(1.7) is bounded and persists.
(ii) If there exists an $\overline{\alpha} \in (0,1]$ such that $C_{r,\overline{\alpha}} < 1$, then the Eq.(1.7) has unbounded solutions.

Proof. (i) Consider the system (1.6) where

$$[C]_\alpha = [C_{l,\alpha}, C_{r,\alpha}] \subset \overline{\cup_{\alpha\in(0,1]}[C_{l,\alpha}, C_{r,\alpha}]} \subset [A, B].$$

Let (x_n, y_n) be a solution of system (1.6) with the initial conditions $x_j = A_j$, $y_j = B_j$ where A_j, B_j are given

$$[L_{j,\alpha}, R_{j,\alpha}] \subset \overline{\cup_{\alpha\in(0,1]}[L_{j,\alpha}, R_{j,\alpha}]} \subset [A_j, B_j], j = -2, -1, 0 \tag{3.13}$$

then, it folows that

$$x_1 = \frac{x_{-2}}{B + y_{-2}y_{-1}y_0} = \frac{A_{-2}}{B + B_{-2}B_{-1}B_0} \le \frac{L_{-2,\alpha}}{C_{r,\alpha} + R_{-2,\alpha}R_{-1,\alpha}R_{0,\alpha}} = L_{1,\alpha} \tag{3.14}$$

and

$$y_1 = \frac{y_{-2}}{A + x_{-2}x_{-1}x_0} = \frac{B_{-2}}{A + A_{-2}A_{-1}A_0} \ge \frac{R_{-2,\alpha}}{C_{l,\alpha} + L_{-2,\alpha}L_{-1,\alpha}L_{0,\alpha}} = R_{1,\alpha}. \tag{3.15}$$

Hence by induction one can obtain $x_n \le L_{n,\alpha}$ and $R_{n,\alpha} \le y_n$ for $n \in \mathbb{N}$. Assume that $C_{l,\alpha} > 1$ for all $\alpha \in (0,1]$, then it follows that $A > 1$ and $B > 1$. From (2) of Theorem 3.3, the solution (x_n, y_n) of system (1.6) is bounded and persists, which is the solution (z_n) of Eq.(1.7). This completes the proof of (i).

(ii) Suppose that there exists an $\overline{\alpha} \in (0,1]$ such that $C_{r,\overline{\alpha}} < 1$. If $C_{l,\overline{\alpha}} = A$, $C_{r,\overline{\alpha}} = B$, $L_{n,\overline{\alpha}} = x_n$ and $R_{n,\overline{\alpha}} = y_n$ for $n = -2, -1, ...$, then we can apply (i) of Corollary 3.4 to system (3.2). If there exists an $\overline{\alpha} \in (0,1]$ such that $A \le B = C_{r,\overline{\alpha}} < 1$ and $x_{-2},\ x_{-1}, x_0 < \sqrt[3]{1-A}$ and $y_{-2},\ y_{-1}, y_0 > \sqrt[3]{1-B}$, then there exist solutions (x_n, y_n) of system (3.2) where $\overline{\alpha} = \alpha$ with initial conditions (x_{-j}, y_{-j}) for $j = 0, 1, 2$ such that

$$\lim_{n\to\infty} x_n = 0 \text{ and } \lim_{n\to\infty} y_n = \infty. \tag{3.16}$$

Moreover, if $x_{-j} < y_{-j}$ ($j = 0, 1, 2$), we can find positive fuzzy numbers z_{-j} ($j = 0, 1, 2$) such that

$$[z_j]_\alpha = [L_{j,\alpha}, R_{j,\alpha}] \tag{3.17}$$

for $\alpha \in (0,1]$ and

$$[z_j]_{\overline{\alpha}} = [L_{j,\overline{\alpha}}, R_{j,\overline{\alpha}}] = [x_j, y_j],\ j = -2, -1, 0. \tag{3.18}$$

Let (z_n) be a positive solution of Eq.(1.7) with the initial conditions z_{-j} ($j = 0, 1, 2$) and $[z_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}]$ for $\alpha \in (0,1]$. Since (3.17) and (3.18) hold and $(L_{n,\alpha}, R_{n,\alpha})$ satisfies system (3.2) we have

$$[z_n]_{\overline{\alpha}} = [L_{n,\overline{\alpha}}, R_{n,\overline{\alpha}}] = [x_n, y_n]. \tag{3.19}$$

Therefore, from (3.16), (3.19) and since

$$||z_n|| = \sup_{\alpha\in(0,1]} \max\{|L_{n,\alpha}|, |R_{n,\alpha}|\} \ge \max\{|L_{n,\overline{\alpha}}|, |R_{n,\overline{\alpha}}|\} = R_{n,\overline{\alpha}}$$

where sup is taken for all $\alpha \in (0,1]$, it is clear that solution (z_n) is unbounded. This completes the proof of (ii). Similarly, one can prove by applying (ii) of Corollary 3.4.

□

Theorem 3.6. *If $C_{l,\alpha} > 1$ for all $\alpha \in (0,1]$, then every positive solution (z_n) of Eq.(1.7) converges to 0 as $n \to \infty$.*

Proof. Let (z_n) be a positive solution of Eq.(1.7) such that (3.1) holds with $C_{l,\alpha} > 1$ for all $\alpha \in (0,1]$. Then, we can apply (i) of 3 of Theorem 3.3 to system (3.2). So, we get

$$\lim_{n\to\infty} L_{n,\alpha} = \lim_{n\to\infty} R_{n,\alpha} = 0. \tag{3.20}$$

Therefore, from (3.18) we get

$$\lim_{n\to\infty} D(z_n, 0) = \lim_{n\to\infty} \left(\sup_{\alpha\in(0,1]} \{\max\{|L_{n,\alpha} - 0|, |R_{n,\alpha} - 0|\}\} \right) = 0.$$

This completes the proof. □

4. Numerical Examples

In this section, to verify our theatrical results, we give some numerical examples for the solutions of Eq.(1.7) regard to the different values of C with the initial conditions z_{-2}, z_{-1}, z_0 are satisfied

$$\begin{cases} z_{-2}(x) = \begin{cases} \frac{5x-0.50}{2}, & 0.10 \le x \le 0.50, \\ \frac{4.50-5x}{2}, & 0.50 \le x \le 0.90, \end{cases} \\ z_{-1}(x) = \begin{cases} 20x - 10, & 0.50 \le x \le 0.55, \\ 12 - 20x, & 0.55 \le x \le 0.60, \end{cases} \\ z_0(x) = \begin{cases} 20x - 4, & 0.20 \le x \le 0.25, \\ 6 - 20x, & 0.25 \le x \le 0.30. \end{cases} \end{cases} \tag{4.1}$$

From (4.1), we get

$$\begin{cases} [z_{-2}]_\alpha = \left[\frac{2\alpha+0.50}{5}, \frac{4.50-2\alpha}{5}\right], \\ [z_{-1}]_\alpha = \left[\frac{\alpha+10}{20}, \frac{12-\alpha}{20}\right], \\ [z_0]_\alpha = \left[\frac{\alpha+4}{20}, \frac{6-\alpha}{20}\right] \end{cases} \tag{4.2}$$

for $\alpha \in (0,1]$. Therefore, it follows that

$$\begin{cases} \overline{\cup_{\alpha\in(0,1]}[z_{-2}]_\alpha} = [0.10, 0.90], \\ \overline{\cup_{\alpha\in(0,1]}[z_{-1}]_\alpha} = [0.50, 0.60], \\ \overline{\cup_{\alpha\in(0,1]}[z_0]_\alpha} = [0.20, 0.30]. \end{cases} \tag{4.3}$$

Example 4.1. Consider Eq.(1.7) where (z_n) is a sequence of positive fuzzy numbers, the initial conditions z_{-2}, z_{-1}, z_0 are satisfied (4.1) and C is satisfied

$$C = \begin{cases} 4x - 1, & 0.25 \le x \le 0.50, \\ 3 - 4x, & 0.50 \le x \le 0.75. \end{cases} \tag{4.4}$$

From (4.4), we get $[C]_\alpha = \left[\frac{\alpha+1}{4}, \frac{3-\alpha}{4}\right]$. Therefore, it follows that $\overline{\cup_{\alpha\in(0,1]}[C]_\alpha} = [0.25, 0.75]$.

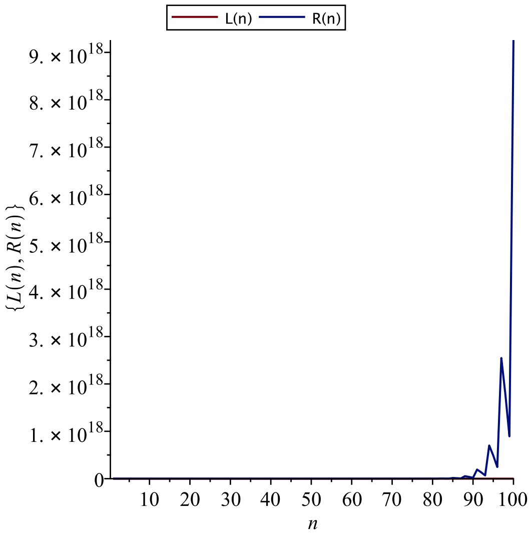

Figure 1: $\alpha = 0.1$.

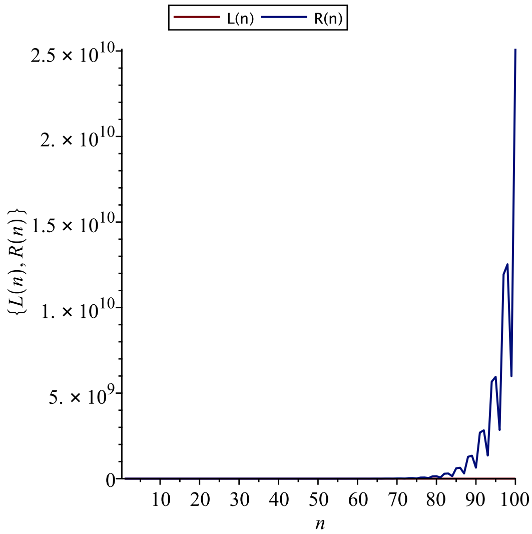

Figure 2: $\alpha = 0.9$.

Example 4.2. Consider Eq.(1.7) where (z_n) is a sequence of positive fuzzy numbers, the initial conditions z_{-2}, z_{-1}, z_0 are satisfied (4.1) and C is satisfied

$$C = \begin{cases} x-1, & 1 \le x \le 2, \\ 3-x, & 2 \le x \le 3. \end{cases} \tag{4.5}$$

From (4.5), we get $[C]_\alpha = [\alpha+1, 3-\alpha]$. Therefore, it follows that $\overline{\cup_{\alpha\in(0,1]}[C]_\alpha} = [1,3]$.

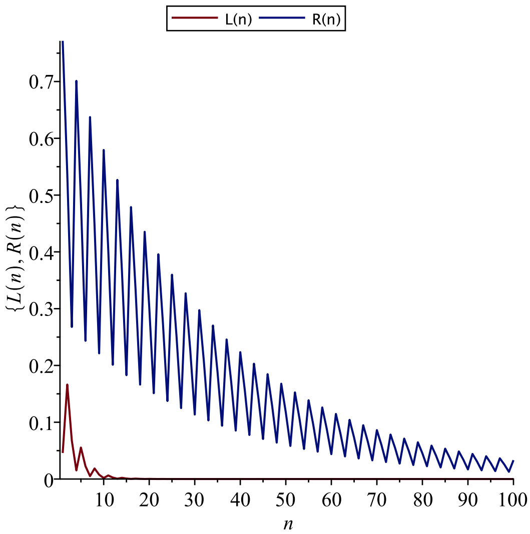

Figure 3: The solution of (1.7) in $\alpha = 0.1$.

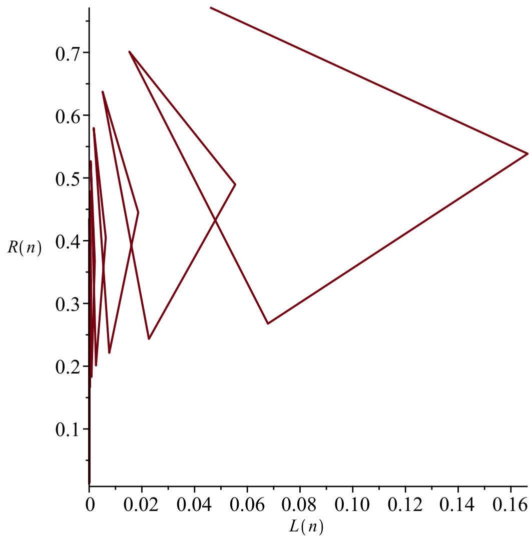

Figure 4: Dynamics of (1.7) in $\alpha = 0.1$.

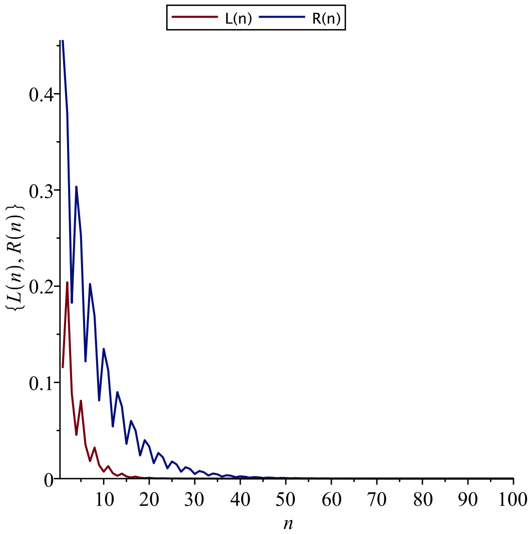

Figure 5: The solution of (1.7) in $\alpha = 0.5$.

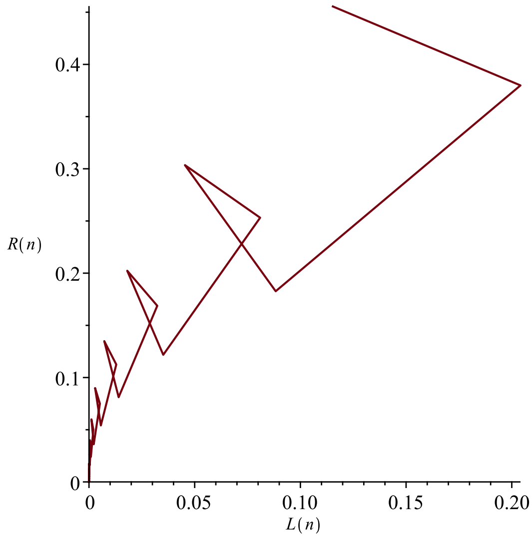

Figure 6: Dynamics of (1.7) in $\alpha = 0.5$.

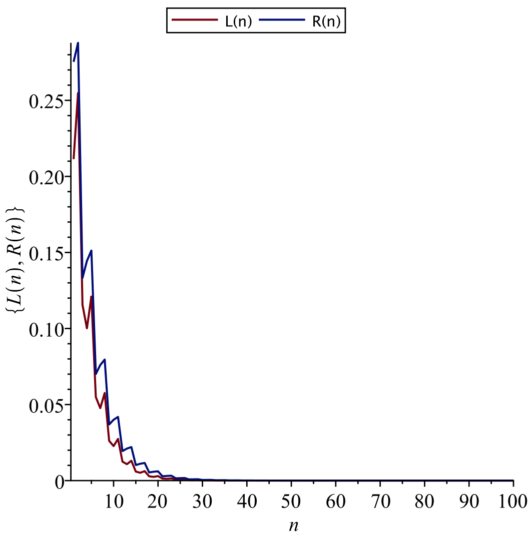

Figure 7: The solution of (1.7) in $\alpha = 0.9$.

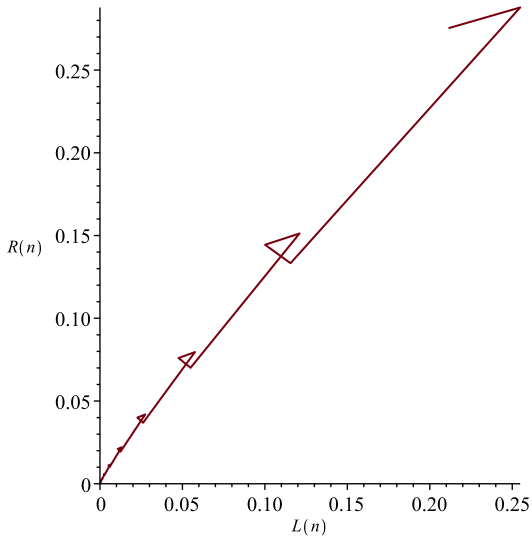

Figure 8: Dynamics of (1.7) in $\alpha = 0.9$.

5. Conclusion

In this paper we analyzed the positive solutions of the fuzzy difference equation given in the abstract in the fuzzy environment. We have shown that the positive solutions of the fuzzy difference equation converge under certain conditions to the only positive equilibrium point of the equation. We have also evaluated the case where the solutions are unbounded. Finally, we have supported our theoretical results via some numerical examples and their drawings.

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