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On algebraic aspects of SSC associated to the subdivided prism graph

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Abstract

In this article, some important combinatorial and algebraic properties of spanning simplicial complex associated to the subdivided prism graph $P(n,m)$ are presented. The $f-$vector of the spanning simplicial complex $\Delta_s(P(n,m))$ and the Hilbert series for the face ring $K\left[\Delta_s(P(n,m))\right]$ are computed. Further, the associated primes of the facet ideal $I_{\mathcal{F}}(\Delta_s(P(n,m)))$ are determined. Finally, the Cohen-Macaulay characterization of the SR-ring of $\Delta_s(P(n,m))$ is discussed.

Keywords: Simplicial Complexes, $f-$vector, Spanning Trees, Face Ring, Hilbert Series, Cohen Macaulay.
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1. Introduction

Let $G(V(G),\mathcal{E}(G))$ be a simple, finite and connected graph. Here, $V(G)$ will be called the set of vertices, $\mathcal{E}(G)=\{uv \mid u,v\in V(G)\}$ will be called the edge set and uv the edges of the graph $G(V(G),\mathcal{E}(G))$. A connected graph having all of its vertices and edges on a single straight line is a path graph and a path with same initial and final vertices is called a cycle. A tree is connected graph without any cycle or closed path. A subgraph of a graph $G(V(G),\mathcal{E}(G))$ is a graph whose set of vertices and and edge set are subsets of those of G. A subgraph of a graph G having no cycle and containing all of its vertices is called a spanning tree. The set having all edge sets of the spanning trees of $G(V(G),\mathcal{E}(G))$ is represented by $s(G)$ i.e. $s(G)=\{\mathcal{E}(T)\subset\mathcal{E}(G) \mid \text{T is a spanning tree of G}\}$. An edge uv of a graph $G(V(G),\mathcal{E}(G))$ is said to be subdivided if it is replaced by a path uwv, where w is a new vertex. A subdivision of a graph $G(V(G),\mathcal{E}(G))$ is a graph obtained by subdividing at least one edge of the graph G. For further details of the graph theory and notation we refer to [1].

The prism graph $P(n,1)$ is obtained by two disjoint cycles each of n vertices, namely C_n^1 and C_n^2, where $V(C_n^1)=\{x_1,x_2,\dots,x_n\}$, $V(C_n^2)=\{w_1,w_2,\dots,w_n\}$, $\mathcal{E}(C_n^1)=\{x_ix_{i+1},x_1x_n|i\in\{1,2,\dots,n-1\}\}$,

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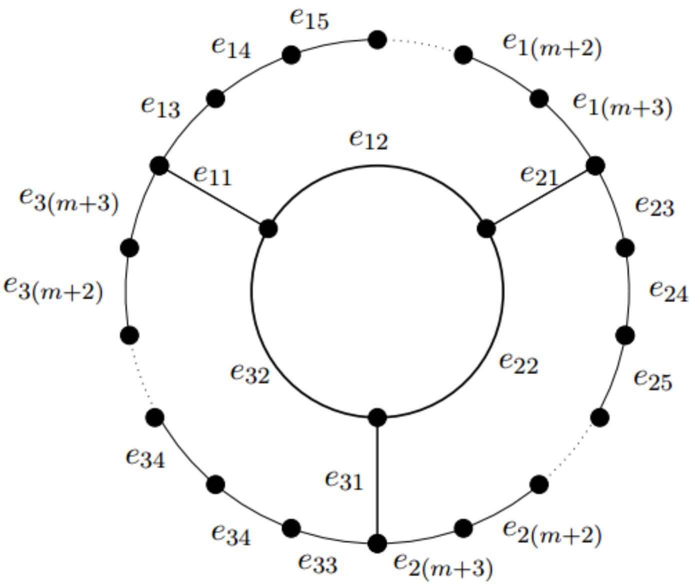

Figure 1: The subdivided prism graph $P(3, m)$

$\mathcal{E}(C_n^2) = \{w_iw_{i+1}, w_1w_n | i \in \{1, 2, \ldots, n-1\}\}$ together with additional edges w_ix_i, where $i \in \{1, 2, \ldots, n\}$. The subdivision of the prism graph obtained by inserting m vertices between each edge of the cycle C_n^2 will be called subdivided prism graph denoted by $P(n, m)$. The subdivided prism graph $P(3, m)$ is shown in the Figure 1. The edge set of the subdivided prism graph $P(n, m)$ is given by

$$\mathcal{E}(P(n,m)) = \{e_{11}, e_{12}, \ldots, e_{1(m+2)}, e_{1(m+3)}, e_{21}, e_{22}, \ldots, e_{2(m+2)}, e_{2(m+3)}, \\ e_{31}, e_{32}, \ldots, e_{3(m+2)}, e_{3(m+3)} \ldots, e_{n1}, e_{n2}, \ldots, e_{n(m+2)}, e_{n(m+3)}\}. \quad (1.1)$$

The edge sets of the apparent cycles $C_0, C_1, C_2, \ldots, C_n$, and C_{n+1} of the subdivided prism graph $P(n, m)$ are $\mathcal{E}(C_0) = \{e_{12}, e_{22}, e_{32}, \ldots e_{n2}\}$, $\mathcal{E}(C_n) = \{e_{n1}, e_{n2}, e_{n3}, e_{11}\}$, $\mathcal{E}(C_t) = \{e_{t1}, e_{t2}, e_{t3}, e_{(t+1)1}\}$ for $t \in \{1, 2, \ldots, n-1\}$ and $\mathcal{E}(C_{n+1}) = \{e_{13}, e_{14}, \ldots, e_{1(m+3)}, e_{23}, e_{24}, \ldots, e_{2(m+3)}, \ldots, e_{n3}, e_{n4}, \ldots, e_{n(m+3)}\}$. Let $[n] = \{1, 2, 3, \ldots, n\}$ be a finite set of positive integers. Then a collection Δ of subsets of $[n]$ is called a simplicial complex (SC) on $[n]$, if **(a)** $\{j\} \in \Delta$ for all $j \in [n]$ and **(b)** $F \in \Delta$ implies $F' \in \Delta$, for each $F' \subseteq F$. A member F of a SC Δ is called its face and its dimension is defined as $\dim(F) = |F| - 1$. The maximal faces of the SC Δ with respect to inclusion are called its facets. The dimension of a SC Δ is defined as $\dim \Delta = \max\{\dim F | F \in \Delta\}$. A SC Δ with facets $\{F_0, F_1, \ldots, F_p\}$ is represented as $\Delta = \langle F_0, F_1, \ldots, F_p \rangle$. The $f-$vector of the SC Δ of dimension $\mathcal{D}$ is defined as a $\mathcal{D}+1$-tuple $f(\Delta) = (f_0, f_1, \ldots, f_{\mathcal{D}})$, where f_j denotes the number of $j-$dimensional faces of the SC Δ, where $0 \le j \le \mathcal{D} - 1$. Let Δ be a SC defined on the set of vertices $[y_1, y_2, \ldots, y_n]$ and $S = K[y_1, y_2, \ldots, y_n]$ be the respective polynomial ring, where K is a field. Then the Stanley-Reisner ideal of SC Δ is a monomial ideal $I_{\mathcal{N}}(\Delta)$ generated by the square free monomials in polynomial ring $S = K[y_1, y_2, \ldots, y_n]$ associated to the non faces of the SC Δ by assigning one variable to each vertex of the SC Δ. The Stanley-Reisner ring of the SC Δ is a standard graded algebra denoted by $K[\Delta] = S/I_{\mathcal{N}}(\Delta)$. The details related to Stanley-Reisner ring and simplicial complex can be seen in [2] and [3]. In [4], Anwar et al. introduced the notion of the spanning simplicial complex (SSC) on a simple, finite and connected graph as follows:

Definition 1. [4] *Let $s(G) = \{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_t\}$ be the collection of the edge sets of all the spanning trees of the graph $G(V(G), \mathcal{E}(G))$. A simplicial complex $\Delta_s(G)$ defined on $\mathcal{E}(G)$ such that its facets are the members of $s(G)$ is called* spanning simplicial complex *(SSC) of $G(V(G), \mathcal{E}(G))$. Mathematically, it can be expressed as follows:*

$$\Delta_s(G) = \langle \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_t \rangle.$$

Further in [4], the authors characterized all spanning trees of the unicyclic graph $U_{n,m}$ and computed $h-$vector and Hilbert series of the SR-ring $K\big[\Delta_s(U_{n,m})\big]$. They showed that the SSC $\Delta_s(U_{n,m})$ is shifted and shellable. The algebraic properties of SSC of some other classes of graph were discussed in [5, 6, 7, 8, 9]. These classes include $r-$cyclic graph $G_{n,r}$, cyclic graph having n edges and r cycles with exactly one common edge between every two consecutive cycles $G^1_{n,r}$, $n-$cyclic graphs $G_{t_1,t_2,\ldots,t_n}$ with a common edge, Jahangir's graph $\mathcal{J}_{m,n}$ and wheel graph W_n. In [8], the authors stated that computing the SSC for an arbitrary graph G is an NP-hard problem. This encourages us to discuss algebraic and combinatorial properties of SSC associated to some more general classes of simple, finite and connected graph.

In this article, some combinatorial and algebraic properties of the subdivided prism graph $P(n,m)$ are explored. The rest of the article is organized as follows: The Section 2 enlights the combinatorial properties of the spanning trees of the subdivided prism graph $P(n,m)$. In Section 3, $f-$vectors associated to the SSC $\Delta_s(P(n,m))$ and Hilbert series of the SR-ring $K\big[\Delta_s(P(n,m))\big]$ are presented. Further, in Section 4, all associated primes of the facet ideal $I_{\mathcal{F}}(\Delta_s(P(n,m)))$ of the SSC $\Delta_s(P(n,m))$ are discussed. Finally, the Section 5 debates the Cohen-Macaulayness of the SR-ring associated to the SSC $\Delta_s(P(n,m))$.

2. Combinatorial characteristics of the subdivided prism graph graph $P(n,m)$

In the following section, some important combinatorial characterizations of the subdivided prism graph $P(n,m)$ are discussed. The following lemma gives the count of the total edges in a spanning tree of the subdivided prism graph $P(n,m)$.

Lemma 2.1. The number of edges in a spanning tree of the subdivided prism graph $P(n,m)$ is $|\mathcal{E}(P(n,m))|-(n+1)$.

Proof. Since a spanning tree of a graph is a connected subgraph with no cycles. Therefore, to get the spanning tree of the subdivided prism graph $P(n,m)$, n edges are deleted from the n successive cycles $C_1, C_2, C_3, \ldots, C_n$ such that one edge is removed from each cycle and one from the cycle C_0, keeping in mind that when a shared edge is deleted from two or more than two cycles then one edge should be deleted from unshared edges of the resulting big cycle. If more than one edges are deleted from the n successive cycles in $P(n,m)$, then a disconnection is obtained which is not a spanning tree. Therefore, spanning tree has exactly $|\mathcal{E}(P(n,m))|-(n+1)$ edges. This completes the proof. □

It is evident from the Lemma 2.1, the spanning trees of the subdivided prism graph $P(n,m)$ are obtained by deleting exactly $(n+1)$ edges from the graph $P(n,m)$ following the cutting down method as described below:

1. Exactly one edge is to be removed from the unshared edges of the cycles and one from the cycle C_0.
2. If a shared edge between two or more than two successive cycles is removed, then exactly one edge must be removed from the resulting big cycle.
3. All shared edges can not be removed at a time.
4. If all edges of the cycle C_0 are deleted, then no shared edge is to be deleted to keep the graph connected and exactly one edge will be deleted from the outer big cycle.

The set of all spanning trees of the subdivided prism graph $P(3,1)$ obtained by the cutting down method is given as follows:
$s(P(3,1)) = \{$ $\{e_{11}, e_{21}, e_{31}, e_{13}, e_{22}, e_{32}, e_{23}, e_{33}\}$, $\{e_{11}, e_{21}, e_{31}, e_{13}, e_{22}, e_{32}, e_{23}, e_{34}\}$, $\{e_{11}, e_{21}, e_{31}, e_{13}, e_{22}, e_{32}, e_{24}, e_{33}\}$, $\{e_{11}, e_{21}, e_{31}, e_{13}, e_{22}, e_{32}, e_{24}, e_{34}\}$, $\{e_{11}, e_{21}, e_{31}, e_{14}, e_{22}, e_{32}, e_{23}, e_{33}\}$, $\{e_{11}, e_{21}, e_{31}, e_{14}, e_{22}, e_{32}, e_{23}, e_{34}\}$, $\{e_{11}, e_{21}, e_{31}, e_{14}, e_{22}, e_{32}, e_{24}, e_{33}\}$, $\{e_{11}, e_{21}, e_{31}, e_{14}, e_{22}, e_{32}, e_{24}, e_{34}\}$, $\{e_{11}, e_{21}, e_{31}, e_{13}, e_{12}, e_{32}, e_{23}, e_{33}\}$, $\{e_{11}, e_{21}, e_{31}, e_{13}, e_{12}, e_{32}, e_{23}, e_{34}\}$, $\{e_{11}, e_{21}, e_{31}, e_{13}, e_{12}, e_{32}, e_{24}, e_{33}\}$, $\{e_{11}, e_{21}, e_{31}, e_{13}, e_{12}, e_{32}, e_{24}, e_{34}\}$, $\{e_{11}, e_{21}, e_{31}, e_{14}, e_{12}, e_{32}, e_{23}, e_{33}\}$, $\{e_{11}, e_{21}, e_{31}, e_{14}, e_{12}, e_{32}, e_{23}, e_{34}\}$, $\{e_{11}, e_{21}, e_{31}, e_{14}, e_{12}, e_{32}, e_{24}, e_{33}\}$, $\{e_{11}, e_{21}, e_{31}, e_{14}, e_{12}, e_{32}, e_{24}, e_{34}\}$, $\{e_{11}, e_{21}, e_{31}, e_{13}, e_{12}, e_{22}, e_{23}, e_{33}\}$,

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$\{e_{11}, e_{21}, e_{23}, e_{24}, e_{22}, e_{32}, e_{33}, e_{13}\}$, $\{e_{11}, e_{21}, e_{23}, e_{24}, e_{22}, e_{32}, e_{34}, e_{13}\}$, $\{e_{11}, e_{21}, e_{23}, e_{24}, e_{22}, e_{32}, e_{33}, e_{14}\}$, $\{e_{11}, e_{21}, e_{23}, e_{24}, e_{22}, e_{32}, e_{34}, e_{14}\}$, $\{e_{11}, e_{21}, e_{23}, e_{24}, e_{12}, e_{32}, e_{33}, e_{13}\}$, $\{e_{11}, e_{21}, e_{23}, e_{24}, e_{12}, e_{32}, e_{34}, e_{13}\}$, $\{e_{11}, e_{21}, e_{23}, e_{24}, e_{12}, e_{32}, e_{33}, e_{14}\}$, $\{e_{11}, e_{21}, e_{23}, e_{24}, e_{12}, e_{32}, e_{34}, e_{14}\}$, $\{e_{11}, e_{21}, e_{22}, e_{23}, e_{24}, e_{33}, e_{34}, e_{13}\}$, $\{e_{11}, e_{21}, e_{22}, e_{23}, e_{24}, e_{33}, e_{34}, e_{14}\}$, $\{e_{11}, e_{21}, e_{22}, e_{13}, e_{14}, e_{33}, e_{34}, e_{23}\}$, $\{e_{11}, e_{21}, e_{22}, e_{13}, e_{14}, e_{33}, e_{34}, e_{24}\}$, $\{e_{11}, e_{21}, e_{22}, e_{23}, e_{24}, e_{13}, e_{14}, e_{33}\}$, $\{e_{11}, e_{21}, e_{22}, e_{23}, e_{24}, e_{13}, e_{14}, e_{34}\}$, $\{e_{11}, e_{21}, e_{32}, e_{23}, e_{24}, e_{33}, e_{34}, e_{13}\}$, $\{e_{11}, e_{21}, e_{32}, e_{23}, e_{24}, e_{33}, e_{34}, e_{14}\}$, $\{e_{11}, e_{21}, e_{32}, e_{13}, e_{14}, e_{33}, e_{34}, e_{23}\}$, $\{e_{11}, e_{21}, e_{32}, e_{13}, e_{14}, e_{33}, e_{34}, e_{24}\}$, $\{e_{11}, e_{21}, e_{32}, e_{23}, e_{24}, e_{13}, e_{14}, e_{33}\}$, $\{e_{11}, e_{21}, e_{32}, e_{23}, e_{24}, e_{13}, e_{14}, e_{34}\}$, $\{e_{11}, e_{22}, e_{32}, e_{23}, e_{24}, e_{33}, e_{34}, e_{13}\}$, $\{e_{11}, e_{22}, e_{32}, e_{23}, e_{24}, e_{33}, e_{34}, e_{14}\}$, $\{e_{11}, e_{22}, e_{32}, e_{13}, e_{14}, e_{33}, e_{34}, e_{23}\}$, $\{e_{11}, e_{22}, e_{32}, e_{13}, e_{14}, e_{33}, e_{34}, e_{24}\}$, $\{e_{11}, e_{22}, e_{32}, e_{13}, e_{14}, e_{23}, e_{24}, e_{33}\}$, $\{e_{11}, e_{22}, e_{32}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}$, $\{e_{11}, e_{12}, e_{32}, e_{23}, e_{24}, e_{33}, e_{34}, e_{13}\}$, $\{e_{11}, e_{12}, e_{32}, e_{23}, e_{24}, e_{33}, e_{34}, e_{14}\}$, $\{e_{11}, e_{12}, e_{32}, e_{13}, e_{14}, e_{33}, e_{34}, e_{23}\}$, $\{e_{11}, e_{12}, e_{32}, e_{13}, e_{14}, e_{33}, e_{34}, e_{24}\}$, $\{e_{11}, e_{12}, e_{32}, e_{13}, e_{14}, e_{23}, e_{24}, e_{33}\}$, $\{e_{11}, e_{12}, e_{32}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}$, $\{e_{11}, e_{12}, e_{22}, e_{23}, e_{24}, e_{33}, e_{34}, e_{13}\}$, $\{e_{11}, e_{12}, e_{22}, e_{23}, e_{24}, e_{33}, e_{34}, e_{14}\}$, $\{e_{11}, e_{12}, e_{22}, e_{13}, e_{14}, e_{33}, e_{34}, e_{23}\}$, $\{e_{11}, e_{12}, e_{22}, e_{13}, e_{14}, e_{33}, e_{34}, e_{24}\}$, $\{e_{11}, e_{12}, e_{22}, e_{13}, e_{14}, e_{23}, e_{24}, e_{33}\}$, $\{e_{11}, e_{12}, e_{22}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}$, $\{e_{21}, e_{22}, e_{32}, e_{23}, e_{24}, e_{33}, e_{34}, e_{13}\}$, $\{e_{21}, e_{22}, e_{32}, e_{23}, e_{24}, e_{33}, e_{34}, e_{14}\}$, $\{e_{21}, e_{22}, e_{32}, e_{13}, e_{14}, e_{33}, e_{34}, e_{23}\}$, $\{e_{21}, e_{22}, e_{32}, e_{13}, e_{14}, e_{33}, e_{34}, e_{24}\}$, $\{e_{21}, e_{22}, e_{32}, e_{13}, e_{14}, e_{23}, e_{24}, e_{33}\}$, $\{e_{21}, e_{22}, e_{32}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}$, $\{e_{21}, e_{12}, e_{32}, e_{23}, e_{24}, e_{33}, e_{34}, e_{13}\}$, $\{e_{21}, e_{12}, e_{32}, e_{23}, e_{24}, e_{33}, e_{34}, e_{14}\}$, $\{e_{21}, e_{12}, e_{32}, e_{13}, e_{14}, e_{33}, e_{34}, e_{23}\}$, $\{e_{21}, e_{12}, e_{32}, e_{13}, e_{14}, e_{33}, e_{34}, e_{24}\}$, $\{e_{21}, e_{12}, e_{32}, e_{13}, e_{14}, e_{23}, e_{24}, e_{33}\}$, $\{e_{21}, e_{12}, e_{32}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}$, $\{e_{21}, e_{12}, e_{22}, e_{23}, e_{24}, e_{33}, e_{34}, e_{13}\}$, $\{e_{21}, e_{12}, e_{22}, e_{23}, e_{24}, e_{33}, e_{34}, e_{14}\}$, $\{e_{21}, e_{12}, e_{22}, e_{13}, e_{14}, e_{33}, e_{34}, e_{23}\}$, $\{e_{21}, e_{12}, e_{22}, e_{13}, e_{14}, e_{33}, e_{34}, e_{24}\}$, $\{e_{21}, e_{12}, e_{22}, e_{13}, e_{14}, e_{23}, e_{24}, e_{33}\}$, $\{e_{21}, e_{12}, e_{22}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}$, $\{e_{21}, e_{22}, e_{32}, e_{23}, e_{24}, e_{33}, e_{34}, e_{13}\}$, $\{e_{21}, e_{22}, e_{32}, e_{23}, e_{24}, e_{33}, e_{34}, e_{14}\}$, $\{e_{31}, e_{22}, e_{32}, e_{13}, e_{14}, e_{33}, e_{34}, e_{23}\}$, $\{e_{31}, e_{22}, e_{32}, e_{13}, e_{14}, e_{33}, e_{34}, e_{24}\}$, $\{e_{31}, e_{22}, e_{32}, e_{13}, e_{14}, e_{23}, e_{24}, e_{33}\}$, $\{e_{31}, e_{22}, e_{32}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}$, $\{e_{31}, e_{12}, e_{32}, e_{23}, e_{24}, e_{33}, e_{34}, e_{13}\}$, $\{e_{31}, e_{12}, e_{32}, e_{23}, e_{24}, e_{33}, e_{34}, e_{14}\}$, $\{e_{31}, e_{12}, e_{32}, e_{13}, e_{14}, e_{33}, e_{34}, e_{23}\}$, $\{e_{31}, e_{12}, e_{32}, e_{13}, e_{14}, e_{33}, e_{34}, e_{24}\}$, $\{e_{31}, e_{12}, e_{32}, e_{13}, e_{14}, e_{23}, e_{24}, e_{33}\}$, $\{e_{31}, e_{12}, e_{32}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}$, $\{e_{31}, e_{12}, e_{22}, e_{23}, e_{24}, e_{33}, e_{34}, e_{13}\}$, $\{e_{31}, e_{12}, e_{22}, e_{23}, e_{24}, e_{33}, e_{34}, e_{14}\}$, $\{e_{31}, e_{12}, e_{22}, e_{13}, e_{14}, e_{33}, e_{34}, e_{23}\}$, $\{e_{31}, e_{12}, e_{22}, e_{13}, e_{14}, e_{33}, e_{34}, e_{24}\}$, $\{e_{31}, e_{12}, e_{22}, e_{13}, e_{14}, e_{23}, e_{24}, e_{33}\}$, $\{e_{31}, e_{12}, e_{22}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}$ $\}$.

The subdivided prism graph $P(n,m)$ has more cycles other than the cycles $C_0, C_1, C_2, \ldots, C_n, C_{n+1}$. These cycles are formed by removing the shared edges from the successive cycles. If $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$ where $i_t \in \{0, 1, \ldots, n\}$ are the successive cycles of the graph $P(n,m)$, then the new cycle formed by deleting the shared edges is represented by $C_{i_1,i_2,\ldots,i_k}$. The number of edges in the new cycles is denoted by $\rho_{i_1,i_2,\ldots,i_k} = \left|C_{i_1,i_2,\ldots,i_k}\right|$. The cycles in the subdivided prism graph $P(n,m)$ can be represented as

$$C_{i_1,i_2,\ldots,i_k}, \text{ where } i_j \in \{1, 2, \ldots, n\} \text{ and } 0 \leq k \leq n,$$

with $i_{j+1} = i_j + 1$ when $i_j \neq n$, $i_{j+1} = 1$ when $i_j = n$ and $i_{j+1} = l$ when $i_j = 0$, where $l \in \{1, 2, \ldots, n\}$. The total number of cycles in the subdivided prism graph $P(n,m)$ and the number of the edges in these cycles is determined in the following lemma.

Lemma 2.2. The total number of the cycles in the subdivided prism graph $P(n,m)$ having cycles C_0, $C_1, C_2, \ldots, C_n$,C_{n+1} are

$$\gamma = n^2 + n + 2$$

such that the cardinality of edge set of the cycle $C_{i_1,i_2,\ldots,i_k}$ with m subdivisions is

$$\rho_{i_1,i_2,\ldots,i_k} = \begin{cases} k(m+2)+2, & 1 \leq k \leq n-1 \\ k(m+2)+1, & k = n \end{cases}$$

and the length of the cycle $C_{0,k}$ with m subdivisions is

$$\rho_{o,k} = m + n + 2, \text{ where } k \in \{1, 2, \ldots, n\}.$$

Proof. The subdivided prism graph $P(n,m)$ consists of cycles C_0,$C_1,C_2,C_3,\ldots,C_n$,C_{n+1}. The rest of the cycles $C_{i_1,i_2,\ldots,i_k}$ are obtained by deleting shared edges from the neighboring cycles $C_{i_1},C_{i_2},\ldots,C_{i_k}$. Therefore, the resulting cycles are $C_{n,1},C_{n-1,n},C_{n-2,n-1}\ldots,C_{2,3},C_{1,2}$ when one shared edge is removed, $C_{n,1,2},C_{n-1,n,1},C_{n-2,n-1,n},\ldots,C_{1,2,3}$ when two shared edges are removed and in the similar manner we have $C_{n,1,2,\ldots,n-1},C_{n-1,n,1\ldots,n-2},C_{n-2,n-1,n,\ldots,2,1},\ldots,C_{1,2,3\ldots,n}$, when $n-1$ shared edges are removed. In addition to these cycles there are more cycles obtained by deleting the shared edge from the cycle C_0 and the successive cycles $C_1,C_2,C_3,\ldots,C_n$. These cycles are n in number and include $C_{0,1},C_{0,2},C_{0,3},\ldots,C_{0,n}$ cycles. Thus, the total new cycles formed by the deletion of the shared edges are:
$C_{0,n},C_{0,n-1},C_{0,n-2},\ldots,C_{0,1},C_{n,1},C_{n-1,n},C_{n-2,n-1}\ldots,C_{2,3},C_{1,2},C_{n,1,2},C_{n-1,n,1},C_{n-2,n-1,n},\ldots,C_{1,2,3},\ldots,$ $C_{n,1,2,\ldots,n-1},C_{n-1,n,1\ldots,n-2},C_{n-2,n-1,n,\ldots,2,1},\ldots,C_{1,2,3\ldots,n}$.

Adding these cycles with the $n+2$ cycles we get total cycles of the subdivided prism graph graph $P(n,m)$ equals to γ. Since the cycle $C_{i_1,i_2,\ldots,i_k}$ is formed by removing shared edges from the successive cycles $C_{i_1},C_{i_2},\ldots,C_{i_k}$ which are $(k-1)$ in number. Therefore, the total edges in the subdivided cycle $C_{p_1,p_2,\ldots,p_k}$ are calculated by adding the orders of all $C_{i_1},C_{i_2},\ldots,C_{i_k}$ that is $k(m+4)$ as the order of each subdivided cycle of $P(n,m)$ is $(m+4)$ and subtracting $2(k-1)$ from it since the shared edges are considered two times in the sum. This shows that for $1\le k\le n-1$

$$\rho_{i_1,i_2,\ldots,i_k}=\left|C_{i_1,i_2,\ldots,i_k}\right|=\sum_{l=1}^{t}\left|C_{i_l}\right|-2(k-1)=k(m+4)-2(k-1)=k(m+2)+2.$$

Similarly, for $k=n$ when only shared edge is left:

$$\rho_{i_1,i_2,\ldots,i_k}=\left|C_{i_1,i_2,\ldots,i_k}\right|=\sum_{l=1}^{t}\left|C_{i_l}\right|-2(k-1)-1=k(m+4)-2(k-1)-1=k(m+2)+1.$$

and the length of the cycle $C_{0,n}$ with m subdivisions is as follows:

$$\rho_{o,k}=\left|C_k\right|+\left|C_0\right|-2=(m+4)+n-2=m+n+2.$$

□

In the following propositions, we take any two cycles $C_{p_1,p_2,\ldots,p_r}$ and $C_{q_1,q_2,\ldots,q_t}$ where $r,t\in\{1,2,\ldots,n\}$ of the subdivided prism graph $P(n,m)$. We use a new notation $y\to z$ which shows that y immediate proceeds z.

Proposition 2.3. Let $\{p_1,p_2,\ldots,p_r\}\subseteq\{q_1,q_2,\ldots,q_t\}$. Then

$$\left|C_{p_1,p_2,\ldots,p_r}\cap C_{q_1,q_2,\ldots,q_t}\right|=\begin{cases}\rho_{p_1,p_2,\ldots,p_r}-2, & \{p_1,p_r\}\nsubseteq\ \{q_1,q_t\}\\ \rho_{p_1,p_2,\ldots,p_r}-1, & p_1\in\{q_1,q_t\}\ \&\ p_r\notin\{q_1,q_t\}\\ \rho_{p_1,p_2,\ldots,p_r}-1, & p_p\in\{q_1,q_t\}\ \&\ p_1\notin\{q_1,q_t\}\\ \rho_{p_1,p_2,\ldots,p_r}, & p_1=q_1,p_r=q_t\ or\\ & p_1=q_t,p_r=q_t.\end{cases}$$

Proof. Since the cycles $C_{p_1,p_2,\ldots,p_r}$ and $C_{q_1,q_2,\ldots,q_t}$ are formed by removing the shared edges from the cycles $C_{p_1},C_{p_2},\ldots,C_{p_r}$ and $C_{q_1},C_{q_2},\ldots,C_{q_t}$ respectively, then $\{p_1,p_r\}\nsubseteq\ \{q_1,q_t\}$ shows that $\{p_1,p_2,\ldots,p_r\}\subset\{q_1,q_2,\ldots,q_t\}$. Hence, the intersection of the cycles $C_{p_1,p_2,\ldots,p_r}$ and $C_{q_1,q_2,\ldots,q_t}$ contains only the unshared edges of the cycle $C_{p_1,p_2,\ldots,p_r}$ eliminating the two edges on its extreme ends. This implies that the order of the intersection is $\rho_{p_1,p_2,\ldots,p_r}-2$. Remaining cases can be proved in the same way.

□

Proposition 2.4. If $\{\overline{p}_1,\overline{p}_2,\ldots,\overline{p}_\varrho\}\subseteq\{q_1,q_2,\ldots,q_t\}$ and $\overline{p}_s\in\{p_1,p_2,\ldots,p_r\}$ and$\overline{p}_{s-1}\to\overline{p}_s$ such that with $s\le\varrho<r$. Then

$$\left|C_{p_1,p_2,\ldots,p_r} \cap C_{q_1,q_2,\ldots,q_t}\right| = \begin{cases} \rho_{\overline{p}_1,\overline{p}_2,\ldots,\overline{p}_\varrho} - 1, & \overline{p}_1 = q_1 \ \& \ q_t \to p_1 \\ \rho_{\overline{p}_1,\overline{p}_2,\ldots,\overline{p}_\varrho} - 2, & \overline{p}_1 = q_1 \ \& \ q_t \not\to p_1 \\ \rho_{\overline{p}_1,\overline{p}_2,\ldots,\overline{p}_\varrho} - 1, & \overline{p}_\varrho = q_t \ \& \ p_r \to q_1 \\ \rho_{\overline{p}_1,\overline{p}_2,\ldots,\overline{p}_\varrho} - 2, & \overline{p}_\varrho = q_t \ \& \ p_r \not\to q_1. \end{cases}$$

Proof. The cycles $C_{\overline{p}_1}, C_{\overline{p}_2}, \ldots, C_{\overline{p}_\varrho}$ are the ϱ successive neighboring cycles of the cycle $C_{p_1,p_2,\ldots,p_r}$ overlapped with the ϱ successive neighboring cycles from the cycle $C_{q_1,q_2,\ldots,q_t}$ as well. When the neighboring cycle $C_{\overline{p}_1}$ from the cycle $C_{p_1,p_2,\ldots,p_r}$ is overlapping with the initial neighboring cycle C_{q_1} from the cycle $C_{q_1,q_2,\ldots,q_t}$ and cycle C_{q_t} immediately proceeds the cycle C_{p_1}, then by Proposition 2.3 the order of the intersection is $\rho_{\overline{p}_1,\overline{p}_2,\ldots,\overline{p}_\varrho} - 1$. Similarly, when C_{q_t} and C_{p_1} are not successive cycles, then there is no shared edge in intersection, then by above Proposition $\left|C_{p_1,p_2,\ldots,p_r} \cap C_{q_1,q_2,\ldots,q_t}\right| = \rho_{\overline{p}_1,\overline{p}_2,\ldots,\overline{p}_\varrho} - 2$. The remaining cases can be shown in the similar way. □

Remark 1. In Proposition 2.4 when the cycles $C_{\overline{p}_1}, C_{\overline{p}_2}, \ldots, C_{\overline{p}_{s_0-1}}, C_{\overline{p}_{s_0}}, \ldots, C_{\overline{p}_\varrho}$ are not ϱ successive neighboring cycles of the cycle $C_{p_1,p_2,\ldots,p_r}$ such that there is a $s_0 < \varrho < r$ and $C_{\overline{p}_{s_0-1}}$ and $C_{\overline{p}_{s_0}}$ are not successive cycles. Then the Proposition 2.4 is applied on the parts which are overlapped to compute the the order of the intersection $C_{p_1,p_2,\ldots,p_r} \cap C_{q_1,q_2,\ldots,q_t}$.

Proposition 2.5. Let $\{p_1, p_2, \ldots, p_r\} \cap \{q_1, q_2, \ldots, q_t\} = \phi$ and $r \leq t$. Then

$$\left|C_{p_1,p_2,\ldots,p_r} \cap C_{q_1,q_2,\ldots,q_t}\right| = \begin{cases} 1, & p_r \to q_1 \ \& \ q_t \not\to p_1 \\ 1, & p_r \not\to q_1 \ \& \ q_t \to p_1 \\ 2, & p_r \to q_1 \ \& \ q_t \to p_1 \\ 0, & otherwise. \end{cases}$$

Proof. Since the intersection of $\{p_1, p_2, \ldots, p_r\}$ and $\{q_1, q_2, \ldots, q_t\}$ is empty then the cycles $C_{p_1,p_2,\ldots,p_r}$ and $C_{q_1,q_2,\ldots,q_t}$ are non-overlapping cycles. If $C_{q_t} \not\to C_{p_1}$ and $C_{p_r} \to C_{q_1}$, then the last cycle C_{p_r} from the neighboring cycle $C_{p_1,p_2,\ldots,p_r}$ shares one edge with the initial cycle C_{q_1} from the neighboring cycle $C_{q_1,q_2,\ldots,q_t}$. Hence, the intersection of $C_{p_1,p_2,\ldots,p_r}$ and $C_{q_1,q_2,\ldots,q_t}$ has only one edge. Remaining can be proved in the same way. □

Proposition 2.6. For the subdivided prism graph $P(n, m)$

$$\left|C_{n+1} \cap C_{i_1,i_2,\ldots,i_k}\right| = \begin{cases} \rho_{i_1,i_2,\ldots,i_k} - (k+2), & i_k \leq n \\ \rho_{i_1,i_2,\ldots,i_k} - (k+1), & i_k = n. \end{cases}$$

Proof. Since the cycle $C_{i_1,i_2,\ldots,i_k}$ is obtained by removing the shared edges from the cycles $C_{i_1}, C_{i_2}, C_{i_3}, \ldots, C_{i_k}$. If $i_k \leq n$, then the intersection will contain only unshared edges of the cycle $C_{i_1,i_2,\ldots,i_k}$ excluding the two shared edges from its extreme ends and the k edges shared with the inner small cycle C_0 giving order of the of intersection equals to $\rho_{i_1,i_2,\ldots,i_k} - (k+2)$. Similarly, when $i_k = n$ the intersection will contain only unshared edge of $C_{i_1,i_2,\ldots,i_k}$ excluding one shared edge on one extreme end and k shared edges of the inner small cycle giving the required order of intersection. This completes the proof. □

In the following three propositions $\mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)})$ is taken as a subset of $\mathcal{E}(P(n, m))$ as defined in Eq. (1.1) with $\omega_\eta \in \{1, 2, \ldots, n\}$ and $\lambda_\eta \in \{1, 2, 3, \ldots, (m+3)\}$ and the conditions required for $\mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)})$ to be the spanning tree of the subdivided prism graph $P(n, m)$ are described.

Proposition 2.7. $\mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)}) \in s(P(n, m))$ when $\omega_\eta\lambda_\eta \neq \omega_\eta 1, \forall \eta$ if and only if $\mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)}) = \mathcal{E}(P(n, m)) \setminus \{e_{1\lambda_1}, e_{2\lambda_2}, \ldots, e_{n\lambda_n}\}$.

Proof. To obtain a spanning tree of the subdivided prism graph $P(n, m)$ when none of shared edges $e_{11}, e_{21}, \ldots, e_{n1}$ are deleted, we need to remove exactly one edge from the unshared edges from each cycle and one from the cycle C_0 keeping the graph connected and acyclic by cutting down method. This completes the proof. □

Proposition 2.8. $\mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)}) \in s(P(n,m))$ when $\omega_\eta\lambda_\eta = \omega_\eta 1$, $\forall\eta$ if and only if

$$\mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)}) = \mathcal{E}(P(n,m)) \setminus \{e_{\omega_1\lambda_1}, e_{\omega_2\lambda_2}, \ldots, e_{\omega_n\lambda_n}\}$$

where $\{e_{\omega_1\lambda_1}, e_{\omega_2\lambda_2}, \ldots, e_{\omega_n\lambda_n}\}$ will carry exactly one edge from $C_{(\omega_\eta-1)(\omega_\eta)} \setminus \{e_{(\omega_\eta-1)1}, e_{(\omega_\eta+1)1}\}$ except $e_{\omega_\eta 1}$.

Proof. To get a spanning tree of the subdivided prism graph $P(n,m)$ using cutting down method when exactly one shared edge $e_{\omega_\eta 1}$ is deleted, we have to delete exactly n edges from the remaining edges. However, we have to delete only one edge from unshared edges of the cycle $C_{(\omega_\eta-1)(\omega_\eta)}$ except $e_{\omega_\eta 1}$ to keep the graph connected. This completes the proof. □

Proposition 2.9. $\mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)}) \in s(P(m,n,1))$, where $\omega_\eta\lambda_\eta = \omega_\eta 1$ and $\eta \in \{\vartheta_1, \vartheta_2, \ldots, \vartheta_r\} \subset \{1, 2, \ldots, n\}$, if and only if the following hold:

1. If the shared edges $e_{\omega_{\vartheta_1}1}, e_{\omega_{\vartheta_2}1}, \ldots, e_{\omega_{\vartheta_r}1}$ are from successive cycles, then

$$\mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)}) = \mathcal{E}(P(n,m)) \setminus \{e_{\omega_1\lambda_1}, e_{\omega_2\lambda_2}, \ldots, e_{\omega_n\lambda_n}\}$$

   such that $\{e_{\omega_1\lambda_1}, e_{\omega_2\lambda_2}, \ldots, e_{\omega_n\lambda_n}\}$ will have only one edge from the cycle $C_{\omega_{\vartheta_0}\omega_{\vartheta_1}\cdots\omega_{\vartheta_r}}$ except $e_{\omega_{\vartheta_1}1}, e_{\omega_{\vartheta_2}1}, \ldots, e_{\omega_{\vartheta_r}1}$, where ω_{ϑ_0} immediately proceeds ω_{ϑ_1}.
2. If none of the shared edges $e_{\omega_{\vartheta_1}1}, e_{\omega_{\vartheta_2}1}, \ldots, e_{\omega_{\vartheta_r}1}$ are from successive cycles, then

$$\mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)}) = \mathcal{E}(P(n,m)) \setminus \{e_{\omega_1\lambda_1}, e_{\omega_2\lambda_2}, \ldots, e_{\omega_n\lambda_n}\}$$

   such that for each edge $e_{\omega_{\vartheta_r}1}$, the Proposition 2.5is satisfied.
3. If some of the shared edges $e_{\omega_{\vartheta_1}1}, e_{\omega_{\vartheta_2}1}, \ldots, e_{\omega_{\vartheta_r}1}$ are from successive cycles, then

$$\mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)}) = \mathcal{E}(P(n,m)) \setminus \{e_{\omega_1\lambda_1}, e_{\omega_2\lambda_2}, \ldots, e_{\omega_n\lambda_n}\}$$

   such that for the shared edges of successive cycles and for the remaining shared edges Proposition 2.9.1 and 2.9.2 hold respectively.

Proof. For the first case, when $(\vartheta_r - \vartheta_1)$ shared edges are deleted from the r successive cycles $C_{\omega_{\vartheta_1}}, C_{\omega_{\vartheta_2}}, \ldots, C_{\omega_{\vartheta_r}}$, then the remaining edges to be removed to get the spanning tree are $(n+1) - (\vartheta_r - \vartheta_1)$. Therefore, to get the spanning tree of the graph $P(n,m)$ exactly one edge must be deleted from the unshared edges of the cycle $C_{\omega_{\vartheta_0},\omega_{\vartheta_1},\ldots,\omega_{\vartheta_r}}$ and the rest of $n - (\vartheta_r - \vartheta_1)$ cycles in the graph $P(n,m)$. This proves the first case of the proposition. Using Propositions 2.7 and 2.8 the remaining cases of the proposition can be proved. This concludes the proof of the proposition. □

Proposition 2.10. $\mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)}) \in s(P(n,m))$ when $\omega_\eta\lambda_\eta = \omega_\eta 2$, $\forall\eta$ if and only if

$$\mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)}) = \mathcal{E}(P(n,m)) \setminus \{e_{\omega_1\lambda_1}, e_{\omega_2\lambda_2}, \ldots, e_{\omega_n\lambda_n}\}$$

where $\{e_{\omega_1\lambda_1}, e_{\omega_2\lambda_2}, \ldots, e_{\omega_n\lambda_n}\}$ will carry exactly one edge from the unshared edges of the cycle C_{n+1} except the shared edges $e_{\omega_\eta 1}$.

Proof. To get a spanning tree of the subdivided prism graph $P(n,m)$ using cutting down method when all edges $e_{\omega_\eta 2}$ from the cycle C_0 are deleted, we have to delete exactly one edge from the unshared edges of the cycle C_{n+1} except the shared edges to keep the graph connected. This completes the proof. □

Remark 2. Let the different classes of the subsets of the edge set $\mathcal{E}(P(n,m))$ of the subdivided prism graph $P(n,m)$ discussed in the Propositions 2.7 to 2.10 be denoted by $\Omega_{P1}, \Omega_{P2}, \Omega_{P2}, \Omega_{P3a}, \Omega_{P3b}, \Omega_{P3c}$ and Ω_{P4} respectively. Then the spanning set $s(P(n,m))$ of the subdivided prism graph $P(n,m)$ can be represented as:

$$s(P(n,m)) = \Omega_{P1} \cup \Omega_{P2} \cup \Omega_{P3a} \cup \Omega_{P3b} \cup \Omega_{P3c} \cup \Omega_{P4}.$$

3. The Hilbert series of SR-ring $K\big[\Delta_s(P(n,m))\big]$

In this section, the formulation of the $f-$vector associated to the SSC $\Delta_s(P(n,m))$ of the subdivided prism graph $P(n,m)$ is presented which is further used to compute the Hilbert series of the SR-ring $K\big[\Delta_s(P(n,m))\big]$.

Theorem 3.1. *Let $\Delta_s(P(n,m))$ be a SSC of the subdivided prism graph $P(n,m)$ with m subdivisions. Then the dimension of SSC $\Delta_s(P(n,m))$ is*

$$\mathcal{D} = dim(\Delta_s(\mathcal{P}(n,1)) = n(m+2)-1.$$

The $f-$vector $f(\Delta_s(P(n,m)) = (f_0, f_1, \ldots, f_{\mathcal{D}})$ of the SSC $\Delta_s(P(n,m))$ can be determined as follows:

$$f_j = \begin{pmatrix} n(m+3) \\ j+1 \end{pmatrix} + \sum_{k=1}^{\gamma}(-1)^k \left[\sum_{\{j_1,j_2,\ldots,j_k\}\in C_J^k} \begin{pmatrix} n(m+3) - \sum\limits_{w=1}^{k}\rho_{j_w} + \sum\limits_{\{j_p,j_q\}\subseteq\{j_r\}_{r=1}^k} \left|C_{j_p}\bigcap C_{j_q}\right| \\ j+1-\sum\limits_{w=1}^{k}\rho_{j_w} + \sum\limits_{\{j_p,j_q\}\subseteq\{j_r\}_{r=1}^k} \left|C_{j_p}\bigcap C_{j_q}\right| \end{pmatrix} \right]$$

where $0 \le j \le \mathcal{D}$. $J = \{j_1 j_2 \ldots j_n\}$, where $j_i \in \{1,2,\ldots,n\}$ with $j_{i+1} = j_i + 1$ when $j_i \neq n$, $j_{i+1} = 1$ when $j_i = n$ and $j_{i+1} = l$ when $j_i = 0$, where $l \in \{1,2,\ldots,n\}$ and C_J^k are the subsets of J having order k.

Proof. Let the edge set of the subdivided prism graph $P(n,m)$ be $\mathcal{E}(P(n,m))$ as defined in Eq. (1.1). The different classes of the spanning trees according to the Propositions 2.7 to 2.10 and the Remark 2 are $\Omega_{P1}, \Omega_{P2}, \Omega_{P3a}, \Omega_{P3b}, \Omega_{P3c}$ and Ω_{P4}. Therefore, the SSC $\Delta_s(P(n,m))$ of the subdivided prism graph $P(n,m)$ by the Definition 1 is

$$\Delta_s(P(n,m)) = \Big\langle \Omega_{P1} \cup \Omega_{P2} \cup \Omega_{P3a} \cup \Omega_{P3b} \cup \Omega_{P3c} \cup \Omega_{P4} \Big\rangle.$$

Since the Propositions 2.7 to 2.10 explain that the facets $\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)} = \mathcal{E}(T_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)})$ are formed by the deletion of the $n+1$ edges from the edge set $\mathcal{E}(P(n,m))$ of the subdivided prism graph $P(n,m)$. Therefore, the cardinality of all the facets is same and equals to $n(m+2)$ which shows that all facets have same dimension equal to $n(m+2)-1$. Hence,

$$\dim(\Delta_s(P(n,m)) = n(m+2)-1$$

The definition of the SSC $\Delta_s(P(n,m))$ shows that it has only those subsets of the edge set $\mathcal{E}(P(n,m))$ which do not carry any cycles in them. The Lemma 2.2 gives the total number of cycles in $P(n,m)$ which is equal to γ. Let $\mathcal{F}$ be a subset of the edge set $\mathcal{E}(P(n,m))$ such that it has no cycle in it and its cardinatlity $j+1$. Infact the total count of these subsets is f_j, where $0 \le j \le n(m+2)-1$. This number can be found by the inclusion exclusion principle. Hence, $f_j =$ Total count of subsets of $\mathcal{E}(P(n,m))$ having cardinality $j+1$ not carrying any of the cycles $C_{i_1,i_2,\ldots,i_k}$ where $i_j \in \{1,2,\ldots,n\}$ and $0 \le k \le n$, with $i_{j+1} = i_j + 1$ when $i_j \neq n$, $i_{j+1} = 1$ when $i_j = n$ and $i_{j+1} = l$ when $i_j = 0$, where $l \in \{1,2,\ldots,n\}$.

By Inclusion Exclusion Principle and above notations we get $f_j = \Big($ Total count of the subsets of $\mathcal{E}(P(n,m))$ having cardinaltiy $j+1\Big) - \sum\limits_{\{j_1\}\in C_J^1}\Big($ Total count of the subset of $\mathcal{E}(P(n,m))$ carrying C_{j_w} for $w = 1$ and cardinality $j+1\Big) + \sum\limits_{\{j_1,j_2\}\in C_J^2}\Big($ Total count of the subset of $\mathcal{E}(P(n,m))$ carrying both $C_{j_w}, \forall$ $1 \le w \le 2$ and cardinality $j+1\Big) - \cdots + (-1)^{\gamma} \sum\limits_{\{j_1,j_2,\ldots,j_\gamma\}\in C_J^\gamma}\Big($Total count of the subset of $\mathcal{E}(P(n,m))$ carrying each C_{j_w} together for all $1 \le w \le \gamma$ and cardinality $j+1\Big)$.

This implies that

$$f_j = \binom{n(m+3)}{j+1} - \Big[\sum_{\{j_1\}\in C_J^1} \binom{n(m+3)-\rho_{j_1}}{j+1-\rho_{j_1}} \Big]$$

$$+ \left[\sum_{\{j_1,j_2\}\in C_J^2} \begin{pmatrix} n(m+3) - \sum\limits_{w=1}^{2} \rho_{j_w} + \sum\limits_{\{j_p,j_q\}\subseteq\{j_r\}_{r=1}^{2}} |C_{j_p}\cap C_{j_q}| \\ j+1 - \sum\limits_{w=1}^{2} \rho_{j_w} + \sum\limits_{\{j_p,j_q\}\subseteq\{j_r\}_{r=1}^{2}} |C_{j_p}\cap C_{j_q}| \end{pmatrix} \right]$$

$$- \cdots + (-1)^{\gamma} \left[\sum_{\{j_1,j_2,\ldots,j_\gamma\}\in C_J^{\gamma}} \begin{pmatrix} n(m+3) - \sum\limits_{w=1}^{\gamma} \rho_{j_w} + \sum\limits_{\{j_p,j_q\}\subseteq\{j_r\}_{r=1}^{\gamma}} |C_{j_p}\cap C_{j_q}| \\ j+1 - \sum\limits_{w=1}^{\gamma} \rho_{j_w} + \sum\limits_{\{j_p,j_q\}\subseteq\{j_r\}_{r=1}^{\gamma}} |C_{j_p}\cap C_{j_q}| \end{pmatrix} \right].$$

This implies

$$f_j = \binom{n(m+3)}{j+1} + \sum_{k=1}^{\gamma} (-1)^k \left[\sum_{\{j_1,j_2,\ldots,j_k\}\in C_J^{k}} \begin{pmatrix} n(m+3) - \sum\limits_{w=1}^{k} \rho_{j_w} + \sum\limits_{\{j_p,j_q\}\subseteq\{j_r\}_{r=1}^{k}} |C_{j_p}\cap C_{j_q}| \\ j+1 - \sum\limits_{w=1}^{k} \rho_{j_w} + \sum\limits_{\{j_p,j_q\}\subseteq\{j_r\}_{r=1}^{k}} |C_{j_p}\cap C_{j_q}| \end{pmatrix} \right].$$

□

In the following example, the above theorem is applied on the subdivided prism graph $P(3,1)$.

Example 1. Let $\Delta_s(P(3,1))$ be a SSC of the subdivided prism graph with $m = 1$ subdivisions , then the $dim(\Delta_s(P(3,1,1)) = 8$ and $\gamma = 3^2 + 3 + 2 = 14$. Therefore, f−vector $f(\Delta_s(P(3,1,1)) = (f_0, f_1, \ldots, f_8)$ and

$$f_j = \binom{12}{j+1} - \Big[\sum_{\{j_1\}\in C_J^1} \binom{12-\rho_{j_1}}{j+1-\rho_{j_1}} \Big]$$

$$+ \left[\sum_{\{j_1,j_2\}\in C_J^2} \begin{pmatrix} 12 - \sum\limits_{w=1}^{2} \rho_{j_w} + \sum\limits_{\{j_p,j_q\}\subseteq\{j_r\}_{r=1}^{2}} |C_{j_p}\cap C_{j_q}| \\ j+1 - \sum\limits_{w=1}^{2} \rho_{j_w} + \sum\limits_{\{j_p,j_q\}\subseteq\{j_r\}_{r=1}^{2}} |C_{j_p}\cap C_{j_q}| \end{pmatrix} \right]$$

$$- \cdots + (-1)^{14} \left[\sum_{\{j_1,j_2,\ldots,j_{14}\}\in C_J^{14}} \begin{pmatrix} 12 - \sum\limits_{w=1}^{12} \rho_{j_w} + \sum\limits_{\{j_p,j_q\}\subseteq\{j_r\}_{r=1}^{14}} |C_{j_p}\cap C_{j_q}| \\ j+1 - \sum\limits_{w=1}^{12} \rho_{j_w} + \sum\limits_{\{j_p,j_q\}\subseteq\{j_r\}_{r=1}^{14}} |C_{j_p}\cap C_{j_q}| \end{pmatrix} \right]$$

where $0 \le j \le 8$.

In the following theorem, the Hilbert series of the SR-ring $K\big[\Delta_s(P(n,m))\big]$ associated to the SSC of the subdivided prism graph $P(n,m)$ is computed using Theorem 3.1 .

Theorem 3.2. *Let $\Delta_s(P(n,m))$ be the SSC of the subdivided prism graph $P(n,m)$. Then the Hilbert series $\hbar_\chi(A)$ of the SR-ring $K\big[\Delta_s(P(n,m))\big]$ is given as follows:*

$$\hbar(K[\Delta_s(P(n,m))],\chi) = 1 + \sum_{j=0}^{\mathcal{D}} \frac{\binom{n}{j+1}\chi^{j+1}}{(1-\chi)^{j+1}}$$

$$+ \sum_{j=0}^{\mathcal{D}} \sum_{t=1}^{\gamma} (-1)^t \left[\sum_{\{j_1,i_2,\ldots,j_k\}\in C_J^{k}} \begin{pmatrix} n(m+3) - \sum\limits_{w=1}^{t} \rho_{j_w} + \sum\limits_{\{j_p,j_q\}\subseteq\{j_r\}_{r=1}^{t}} |C_{j_p}\bigcap C_{j_q}| \\ j+1 - \sum\limits_{w=1}^{t} \rho_{j_w} + \sum\limits_{\{j_p,j_q\}\subseteq\{j_r\}_{r=1}^{t}} |C_{j_p}\bigcap C_{j_q}| \end{pmatrix} \right] \frac{\chi^{j+1}}{(1-\chi)^{j+1}}.$$

Proof. Let Δ be a SC with $f-$vector $f(\Delta) = (f_0, f_1, \ldots, f_{\mathcal{D}})$ and dimension $\mathcal{D}$. Then by [3], the Hilbert series of the SR-ring $K[\Delta]$ is given as follows:

$$\hbar(K[\Delta], \chi) = 1 + \sum_{j=0}^{\mathcal{D}} \frac{f_j \chi^{j+1}}{(1-\chi)^{j+1}}.$$

The required result is obtained by substituting the values of $f-$vector from Theorem 3.1 in above expression. □

4. The facet ideal $I_{\mathcal{F}}(\Delta_s(P(n,m)))$ and its associated primes

In this section, all the associated primes of the facet ideal $I_{\mathcal{F}}(\Delta_s(P(n,m)))$ associated to the SSC $\Delta_s(P(m,n,1))$ of the subdivided prism graph $P(n,m)$ are computed.

Lemma 4.1. If $\Delta_s(P(n,m))$ be the SSC of the subdivided prism graph $P(n,m)$, then

$$I_{\mathcal{F}}(\Delta_s(P(n,m))) = (\{x_{11}, x_{21}, x_{31}, \ldots, x_{n1}\}) \bigcap \left(\bigcap_{1\leq\lambda\leq n} (x_{\lambda 1} x_{\lambda 3} x_{(\lambda-1)(m+3)}) \right)$$

$$\bigcap \left(\bigcap_{\lambda\in\{0,1,2,\ldots,n-1\}} (x_{\lambda 1} x_{\lambda 2} x_{(\lambda-1)2}) \right) \bigcap \left(\bigcap_{1\leq\lambda\leq n, 3\leq i\leq(m+3)-j} (x_{\lambda i} x_{\lambda(i+j)}) \right)$$

$$\bigcap \left(\{x_{\lambda 1}\}_{\lambda=1}^{n} \setminus \{\{x_{i1}\} \bigcup \{x_{i3} x_{(i-1)(m+3)}\}\} \right).$$

Proof. Let us consider the SSC $\Delta_s(P(m,n,1))$ of subdivided prism graph $P(n,m)$ having n successive cycles. Let $I_{\mathcal{F}}(\Delta_s(P(n,m)))$ be the facet ideal associated to the SSC $\Delta_s(P(n,m))$. It is well known from [10] that the mvc (minimal vertex cover) of the SSC Δ_s and mp (minimal prime ideal) of Δ have $1-1$ correspondence. Hence the mvc of the $\Delta_s(P(n,m))$ will provide the primary decomposition of the facet ideal $I_{\mathcal{F}}(\Delta_s(P(n,m)))$. Using the definition of $\Delta_s(P(n,m))$ and Propositions 2.7 to 2.10 we get $\{e_{\lambda 1}\}$ for all $\lambda \in \{1,2,3,\ldots,n\}$ as mvc of $\Delta_s(P(n,m))$ with $\{e_{\lambda 1}\} \notin C_i$, $\forall i \in \{1,2,\ldots,n\}$ as $\{e_{\lambda 1}\} \in \hat{\mathcal{E}}_{(\omega_1\lambda_1, \omega_2\lambda_2, \ldots, \omega_n\lambda_n)}$ for any $\omega_\eta \in \{1,2,\ldots,n\}$ and $\lambda_\eta \in \{1,2,3,\ldots,(m+3)\}$. Also, $\{e_{\lambda 1}, e_{\lambda 3}, e_{(\lambda-1)(m+3)}\}$ and $\{e_{\lambda 1}, e_{\lambda 2}, e_{(\lambda-1)2}\}$ with $\lambda \in \{0,1,2,\ldots,n-1\}$ are mvc of $\Delta_s(P(n,m))$ as at least one of the member of the sets $\{e_{\lambda 1}, e_{\lambda 3}, e_{(\lambda-1)(m+3)}\}$ and $\{e_{\lambda 1}, e_{\lambda 2}, e_{(\lambda-1)2}\}$ belongs to $\hat{\mathcal{E}}_{(\omega_1\lambda_1, \omega_2\lambda_2, \ldots, \omega_n\lambda_n)}$. Moreover, $\{e_{\lambda i}, e_{\lambda(i+j)}\}$ where $1 \leq \lambda \leq n$ and $3 \leq i \leq (m+3)-j$ and $\{e_{\lambda 1}\}_{\lambda=1}^{n} \setminus \{e_{i1} \bigcup \{e_{i3}, e_{(i-1)(m+3)}\}$ are also minimal vertex covers of $\Delta_s(P(n,m))$ because they have non empty intersection with $\hat{\mathcal{E}}_{(\omega_1\lambda_1, \omega_2\lambda_2, \ldots, \omega_n\lambda_n)}$. This completes the proof. □

5. The Cohen-Macaulay Characterization of the SR-ring $K[\Delta_s(P(m,m,1))]$

The following section describes the Cohen-Macaulay characterization of the SR-ring $K[\Delta_s(P(n,m))]$. The definition in [11] will be helpful in sequel.

Definition 2. [11] Let I be a monomial ideal such that $G(I) = \{g_1, g_2, \ldots, g_t\}$ is an ordered system of generators. Then I is said to have *linear residuals* if $\mathrm{Res}(I_v) = \{w_1, w_2, \ldots, w_{v-1}\}$ such that it is minimally generated by linear monomials, $\forall\, 1 \leq v \leq t$ when

$$w_k = \frac{m_v}{\gcd(m_k, m_v)}.$$

The authors in [11] presented a criteria for a pure SC Δ to be Cohen Macaulay.

Theorem 5.1. *[11] Let Δ be a pure SC Δ of dimension $\mathcal{D}$ over a finite set $[n]$. If its facet ideal $I_{\mathcal{F}}(\Delta)$ has linear residuals, then the Stanley-Reisner ring $k[\Delta]$ is Cohen Macaulay.*

Theorem 5.2. *The SR-ring $K[\Delta_s(P(n,m))]$ associated to the SSC $\Delta_s(P(n,m))$ of the subdivided prism graph $P(n,m)$ is Cohen-Macaulay.*

Proof. To prove that SR-ring $K[\Delta_s(P(n,m))]$ is Cohen-Macaulay, we will prove that the facet ideal $I_{\mathcal{F}}\big(\Delta_s(P(n,m)\big)$ has linear residuals in $S = k[y_{11}, y_{12}, y_{13}, \dots, y_{1(m+3)}, y_{21}, y_{22}, y_{23}, \dots, y_{2(m+3)}, \dots, y_{n1}, y_{n2}, y_{n3}, \dots, y_{n(m+3)}]$ using Theorem 5.1. Since the spanning tress of the subdivided prism graph $P(n,m)$ are given by:

$$s(P(n,m)) = \Omega_{P1} \cup \Omega_{P2} \cup \Omega_{P3a} \cup \Omega_{P3b} \cup \Omega_{P3c} \cup \Omega_{P4}.$$

Therefore, the SSC of $P(n,m)$ is

$$\Delta_s(P(n,m)) = \left\langle \hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_n\lambda_n)} \right\rangle$$

where, $\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_n\lambda_n)} = \{\mathcal{E}(P(n,m))\backslash\{e_{\omega_1\lambda_1}, e_{\omega_2\lambda_2}, \dots, e_{\omega_n\lambda_n}\} \big| \hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_n\lambda_n)} \in s(P(n,m))\}$. Hence, the facet ideal of $\Delta_s(P(n,m))$ is

$$I_{\mathcal{F}}(\Delta_s(P(n,m))) = \Big(y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_n\lambda_n)}} \mid \hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_n\lambda_n)} \in s(P(n,m))\Big).$$

The facet ideal $I_{\mathcal{F}}(\Delta_s(P(n,m)))$ is a monomial ideal with degree of each monomial equal to $n(m+2)-1$. The product of all variables in S other than $y_{\omega_1\lambda_1}, y_{\omega_2\lambda_2}, \dots, y_{\omega_n\lambda_n}$ gives the monomials in $I_{\mathcal{F}}(\Delta_s(P(n,m)))$. Now we will prove that the facet ideal $I_{\mathcal{F}}(\Delta_s(P(n,m)))$ has linear residuals according to the orders of its monomials given as follows:

$$\Big\{y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_n\lambda_n)}} \mid \hat{\mathcal{E}} = \mathcal{E}(P(n,m)) \setminus \{e_{\alpha_1\lambda_1}, e_{\alpha_2\lambda_2}, \dots, e_{\alpha_n\lambda_n}\} \cup \{e_{\omega_1 2}\}\Big\}, \tag{5.1}$$

where $\alpha_1, \dots, \alpha_n, \omega_1 \in \{1, 2, \dots, n\}$ and $\lambda_1, \lambda_2, \dots, \lambda_n \in \{3, 4, \dots, (m+3)\}$.

$$\{y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_n\lambda_n)}} \mid \hat{\mathcal{E}} = \mathcal{E}(P(n,m)) \setminus \{e_{\beta_1\varsigma_1}, e_{\alpha_1\lambda_1}, \dots, e_{\alpha_n\lambda_n}\} \cup \{e_{\omega_1 2}, e_{\omega_2 2}\}\},$$

where $\alpha_1, \dots, \alpha_n, \omega_1, \omega_2 \in \{1, 2, \dots, n\}$, $\beta_1 \in \{1, \dots, \omega_n\}$ and $\varsigma_1, \lambda_1, \lambda_2, \dots, \lambda_n \in \{3, 4, \dots, (m+3)\}$.
$\dots,$

$$\{y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_n\lambda_n)}} \mid \hat{\mathcal{E}} = \mathcal{E}(P(n,m)) \setminus \{e_{\alpha_1\lambda_1}\} \cup \{e_{\omega_1 2}, e_{\omega_2 2}, \dots, e_{\omega_n 2}\}\},$$

where $\alpha_1, \omega_1, \omega_2, \dots, \omega_n \in \{1, 2, \dots, n\}$ and $\lambda_1 \in \{3, 4, \dots, (m+3)\}$.

$$\{y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_n\lambda_n)}} \mid \hat{\mathcal{E}} = \mathcal{E}(P(n,m)) \setminus \{e_{\beta_1\varsigma_1}, e_{\alpha_1\lambda_1}, \dots, e_{\alpha_n\lambda_n}\} \cup \{e_{\omega_1 1}\} \cup \{e_{\omega_1 2}\}\}$$

where $\alpha_1, \dots, \alpha_n, \omega_1 \in \{1, 2, \dots, n\}$, $\beta_1 \in \{1, 2, \dots, \omega_{n-1}, \omega_n\}$ and $\varsigma_1, \lambda_1, \lambda_2, \dots, \lambda_n \in \{3, 4, \dots, (m+3)\}$.

$$\{y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_n\lambda_n)}} \mid \hat{\mathcal{E}} = \mathcal{E}(P(n,m)) \setminus \{e_{\beta_1\varsigma_1}, e_{\alpha_1\lambda_1}, \dots, e_{\alpha_n\lambda_n}\} \cup \{e_{\omega_1 1}, e_{\omega_2 1}\} \cup \{e_{\omega_1 2}, e_{\omega_2 2}\}\},$$

where $\alpha_1, \dots, \alpha_n, \omega_1, \omega_2 \in \{1, 2, \dots, n\}$, $\beta_1 \in \{1, 2, \dots, \omega_{n-1}, \omega_n\}$ and $\varsigma_1, \lambda_1, \lambda_2, \dots, \lambda_n \in \{3, 4, \dots, (m+3)\}$.
$\dots,$

$$\{y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_n\lambda_n)}} \mid \hat{\mathcal{E}} = \mathcal{E}(P(n,m)) \setminus \{e_{\omega_1 1}, \dots, e_{\omega_{n-1} 1}\} \cup \{e_{\omega_1 2}, \dots, e_{\omega_{n-1} 2}\} \cup \{e_{\alpha_1\lambda_1}\}\},$$

where $\alpha_1, \omega_1, \omega_2, \dots, \omega_n \in \{1, 2, \dots, n\}$ and $\lambda_1 \in \{3, 4, \dots, (m+3)\}$.

In Eq. (5.1), the monomials $\big\{y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\dots,\omega_n\lambda_n)}} \mid \hat{\mathcal{E}} = \mathcal{E}(P(n,m)) \setminus \{e_{\alpha_1\lambda_1}, \dots, e_{\alpha_n\lambda_n}\} \cup \{e_{\omega_1 2}\}$, where $\alpha_1, \dots, \alpha_n, \omega_1 \in \{1, 2, \dots, n\}$ and $\lambda_1, \lambda_2, \dots, \lambda_n \in \{3, 4, \dots, (m+3)\}\big\}$ have the following pattern:

$$y_{\hat{\mathcal{E}}_{(11,21,\ldots,n1,12,22,\ldots,(n-1)2,\omega_n\lambda_n)}}, \ldots, y_{\hat{\mathcal{E}}_{(11,21,\ldots,n1,12,\omega_2\lambda_2,32,\ldots,(n-1)2,n2,\omega_n\lambda_n)}}, y_{\hat{\mathcal{E}}_{(11,21,\ldots,n1,\omega_1\lambda_1,22,\ldots,(n-1)2,n2,\omega_n\lambda_n)}}$$

where $\lambda_t \in \{3, 4, \ldots, (m+3)\}$ and $1 \leq \omega_t \leq n$. Similarly, the other monomials in 5.1. Now substituting

$$\text{Res}(y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)}}) = \left\{ \frac{y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)}}}{\gcd(m_t, y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)}})} \right\}$$

where m_t proceeds $y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)}}$ with respect to the order in Eq (5.1). In $\text{Res}(y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)}})$ substituting $\lambda = n$ gives,

$$\text{Res}(y_{\hat{\mathcal{E}}_{(11,21,\ldots,n1,12,22\ldots,(n-1)2,\omega_n\lambda_n)}}) = \left\{ \frac{y_{\hat{\mathcal{E}}_{(11,21,\ldots,n1,12,22\ldots,(n-1)2,\omega_n\lambda_n)}}}{\gcd(m_t, y_{\hat{\mathcal{E}}_{(11,21,\ldots,n1,12,22\ldots,(n-1)2,\omega_n\lambda_n)}})} \right\}.$$

Here m_t are all the monomials having the form $y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)}}$ in S, where $\lambda_t \neq n$ & $\omega_n = 3, 4, \ldots, (m+3)$. Since $y_{\hat{\mathcal{E}}_{(11,21,\ldots,n1,12,22\ldots,(n-1)2,\omega_n\lambda_n)}}$ and m_t have difference at only one point. Therefore, there are only linear terms in $\text{Res}(y_{\hat{\mathcal{E}}_{(11,21,\ldots,n1,12,22\ldots,(n-1)2,\omega_n\lambda_n)}})$. This shows that only the linear monomials generate the $\text{Res}(y_{\hat{\mathcal{E}}_{(11,21,\ldots,n1,12,22\ldots,(n-1)2,\omega_n\lambda_n)}})$ minimally. Following the similar procedure, the order of all the monomials in Eq. (5.1) of the facet ideal $I_{\mathcal{F}}(\Delta_s(P(n,m)))$ ensures that $\text{Res}(y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)}})$ has only linear monomials for all $y_{\hat{\mathcal{E}}_{(\omega_1\lambda_1,\omega_2\lambda_2,\ldots,\omega_n\lambda_n)}} \in I_{\mathcal{F}}(\Delta_s(P(n,m)))$. Hence, the facet $I_{\mathcal{F}}(\Delta_s(P(n,m)))$ has linear residuals, and by Theorem 5.1 $\Delta_s(P(n,m))$ is Cohen-Macaulay. □

6. Conclusions

The current paper debates on combinatorial properties related to spanning trees of a subdivided prism graph $P(n,m)$ and explores certain algebraic attributes of spanning simplicial complex associated to the prism graph $P(n,m)$. Here, we conclude the article with some future perspectives for study and related limits.

- The work done here can be discussed for some other classes of simple, finite and connected graphs.
- The computation of spanning trees of an arbitrary graph G is an NP-hard problem, and therefore, exploring the current work for a general class of simple, finite and connected graph is a hard problem to work on.

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