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Some Opial-type inequalities involving fractional integral operators

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Abstract

The core idea of this paper is to provide the Opial-type inequalities for Hadamard fractional integral operator and fractional integral of a function with respect to an increasing function g. Moreover, related extreme cases and counter part of our main results are also given in the paper.

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1. Introduction and Preliminaries

Mathematical inequalities play very significant role in development of all branches of mathematics. In 1960, Opial [1] presented a fascinating inequality which is called Opial's inequality afterwards. Opial-type inequalities have importance in applications in the theory of ordinary differential equations and boundary value problems. A large number of research papers have appeared in literature, for example, ([2, 3, 4, 5]). Willett [2] considered the Opial-type inequality which involves a higher order derivative. Later Beesak and Das [3] found a more general result but all generalization of Opial-type inequalities only involve a higher order in one order derivative at the left side of the inequality until now. Yang [6] has established some interesting generalization of the Opial's inequality by using the Mallows method [7].

The Opial's inequality is stated as:

Theorem 1.1. *Let $a > 0, \Phi \in C^1[0,a]$ with $\Phi(a) = \Phi(0) = 0$ and $\Phi(\xi) > 0$ on $(0,a)$, then*

$$\int_0^a |\Phi(\xi)\Phi'(\xi)|d\xi \leq \frac{a}{4}\int_0^a (\Phi'(\xi))^2 d\xi.$$

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the constant $\frac{a}{4}$ *is the best possible.*

Agarwal, Pang and Alzer ([8]-[10]) considered the wide range of Opial-type inequalities including ordinary derivatives with their application in differential equation and difference equation. Recently, Iqbal et al. [11] gave Opial-type inequalities for two functions with applications. Here our main interest is to give the Opial-type inequalities using Hadamard fractional integral operator and fractional integral of a function with respect to an increasing function g with related extreme and counter case of main results.

Next we give the definition of left-and right-sided Hadamard fractional integrals of order α see [12].

Definition 1.2. Suppose $(a, b), 0 \leq a < b \leq \infty$ be a finite or infinite interval of the half-axis $\mathbb{R}_+$ and $\alpha > 0$. The left-and right-sided Hadamard fractional integrals of order α are defined by

$$(J^\alpha_{a+}\Phi)(\xi) = \frac{1}{\Gamma(\alpha)} \int_a^\xi \left(\log \frac{\xi}{\eta}\right)^{\alpha-1} \frac{\Phi(\eta)d\eta}{\eta}, \xi > a, \tag{1.1}$$

$$(J^\alpha_{b-}\Phi)(\xi) = \frac{1}{\Gamma(\alpha)} \int_\xi^b \left(\log \frac{\eta}{\xi}\right)^{\alpha-1} \frac{\Phi(\eta)d\eta}{\eta}, \xi < b, \tag{1.2}$$

respectively. Here Γ represents usual Gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty \tau^{\alpha-1}e^{-\tau}d\tau, \ \mathbb{R}(\alpha) > 0.$$

Next we give the definition of the left-and right-sided fractional integrals of a function Φ with respect to an increasing function g in $[a, b]$, see [12].

Definition 1.3. Assume $(a, b), -\infty \leq a < b \leq \infty$ be a finite or infinite interval of the extended real line $\mathbb{R}$ and $\alpha \geq 0$. Also assume on $(a, b]$, g be the increasing function and on (a, b), g' is a continuous function. The left-and right-sided fractional integrals of a function Φ with respect to another increasing function g in $[a, b]$ are defined by

$$(I^\alpha_{a+;g}\Phi)(\xi) = \frac{1}{\Gamma(\alpha)} \int_a^\xi \frac{g'(\eta)\Phi(\eta)d\eta}{[g(\xi) - g(\eta)]^{1-\alpha}}, \xi > a, \tag{1.3}$$

$$(I^\alpha_{b-;g}\Phi)(\xi) = \frac{1}{\Gamma(\alpha)} \int_\xi^b \frac{g'(\eta)\Phi(\eta)d\eta}{[g(\eta) - g(\xi)]^{1-\alpha}}, \xi < b, \tag{1.4}$$

respectively with $g(\xi) \neq g(\eta)$.

2. Opial-type inequalities for Hadamard fractional integral oprator

First we shall give the Opial-type inequalities involving Hadamard fractional integral operator.

Theorem 2.1. *Let* $(J^\alpha_{a+}\Phi)$ *be the left side Hadamard fractional integral operator of* Φ *of order* α. *Let* $\varrho > 0$ *and a measurable function* $\lambda \geq 0$ *on* $[a, x]$. *Let* $s > 1, 0 < q < s$ *and* $p \geq 0$. *Assume* $\Phi, \Psi \in L_s[a, b]$. *Then*

$$\int_a^x \lambda(\xi) \left[|(J^\alpha_{a+}\Psi)(\xi)|^p|\Phi(\xi)|^q + |(J^\alpha_{a+}\Phi)(\xi)|^p|\Psi(\xi)|^q\right] d\xi$$

$$\leq 2^{1-\frac{q}{s}} \left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{s}} \left(\frac{q}{p+q}\right)^{\frac{q}{s}} \left(\int_a^x [U(\xi)]^{\frac{s}{s-q}} d\xi\right)^{\frac{s-q}{s}} \left(\int_a^x \varrho(\eta)[|\Phi(\eta)|^s + |\Psi(\eta)|^s]d\eta\right)^{\frac{p+q}{s}},$$

where

$$U(\xi)=\frac{1}{[\Gamma(\alpha)]^p}\lambda(\xi)[\varrho(\xi)]^{-\frac{q}{s}}[P(\xi)]^{\frac{p(s-1)}{s}}, \tag{2.1}$$

and

$$d_{\frac{p}{q}}=\begin{cases}2^{1-\frac{p}{q}}, & 0\le p\le q;\\ 1, & p\ge q.\end{cases} \tag{2.2}$$

Proof. Let $\xi\in[a,x]$, using the identity (1.1) and Hölder's inequality for conjugate exponent $\{\frac{s}{s-1},s\}$, we have

$$\begin{aligned}|(J^{\alpha}_{a+}\Psi)(\xi)| &\le \frac{1}{\Gamma(\alpha)}\int_a^{\xi}\left(\log\frac{\xi}{\eta}\right)^{\alpha-1}\frac{1}{\eta}[\varrho(\eta)]^{\frac{-1}{s}}[\varrho(\eta)]^{\frac{1}{s}}|\Psi(\eta)|d\eta\\ &\le \frac{1}{\Gamma(\alpha)}\left(\int_a^{\xi}\left[\left(\log\frac{\xi}{\eta}\right)\left(\frac{1}{\eta}\right)^{\frac{1}{\alpha-1}}[\varrho(\eta)]^{\frac{1}{-s(\alpha-1)}}\right]^{\frac{s(\alpha-1)}{s-1}}d\eta\right)^{\frac{s-1}{s}}\left(\int_a^{\xi}\varrho(\eta)|\Psi(\eta)|^s d\eta\right)^{\frac{1}{s}}\\ &= \frac{1}{\Gamma(\alpha)}[P(\xi)]^{\frac{s-1}{s}}[V(\xi)]^{\frac{1}{s}},\end{aligned} \tag{2.3}$$

where

$$V(\xi)=\int_a^{\xi}\varrho(\eta)|\Psi(\eta)|^s d\eta. \tag{2.4}$$

Take

$$W(\xi)=\int_a^{\xi}\varrho(\eta)|\Phi(\eta)|^s d\eta. \tag{2.5}$$

Then

$$|\Phi(\xi)|^q=[W'(\xi)]^{\frac{q}{s}}[\varrho(\xi)]^{\frac{-q}{s}}. \tag{2.6}$$

Now (2.3) and (2.6) imply that

$$\lambda(\xi)|(J^{\alpha}_{a+}\Psi)(\xi)|^p|\Phi(\xi)|^q\le U(\xi)[V(\xi)]^{\frac{p}{s}}[W'(\xi)]^{\frac{q}{s}}, \tag{2.7}$$

where

$$U(\xi)=\frac{1}{[\Gamma(\alpha)]^p}\lambda(\xi)[\varrho(\xi)]^{-\frac{q}{s}}[P(\xi)]^{\frac{p(s-1)}{s}}.$$

Integrating (2.7) and applying Hölder's inequality for conjugate exponent $\{\frac{s}{s-q},\frac{s}{q}\}$, we obtain

$$\int_a^x\lambda(\xi)|(J^{\alpha}_{a+}\Psi)(\xi)|^p|\Phi(\xi)|^q(d\xi)\le\left(\int_a^x[U(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}}\left(\int_a^x[V(\xi)]^{\frac{p}{q}}W'(\xi)d\xi\right)^{\frac{q}{s}}. \tag{2.8}$$

Similarly, we can write

$$\int_a^x\lambda(\xi)|(J^{\alpha}_{a+}\Phi)(\xi)|^p|\Psi(\xi)|^q d\xi\le\left(\int_a^x[U(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}}\left(\int_a^x[W(\xi)]^{\frac{p}{q}}V'(\xi)d\xi\right)^{\frac{q}{s}}. \tag{2.9}$$

Now, we use the inequality

$$c_{\epsilon}(\Delta+\Theta)^{\epsilon}\le\Delta^{\epsilon}+\Theta^{\epsilon}\le d_{\epsilon}(\Delta+\Theta)^{\epsilon},(\Delta,\Theta\ge 0), \tag{2.10}$$

where

$$c_\epsilon = \begin{cases} 1, & 0 \leq \epsilon \leq 1; \\ 2^{1-\epsilon}, & \epsilon \geq 1. \end{cases}$$

And

$$d_\epsilon = \begin{cases} 2^{1-\epsilon}, & 0 \leq \epsilon \leq 1; \\ 1, & \epsilon \geq 1. \end{cases}$$

Therefore from (2.8), (2.9) and (2.10) with $s > q$ we conclude that

$$\int_a^x \lambda(\xi) \left[|(J_{a+}^\alpha \Psi)(\xi)|^p |\Phi(\xi)|^q + |(J_{a+}^\alpha \Phi)(\xi)|^p |\Psi(\xi)|^q \right] d\xi$$

$$\leq \left(\int_a^x [U(\xi)]^{\frac{s}{s-q}} d\xi \right)^{\frac{s-q}{s}} 2^{1-\frac{q}{s}} \left(\int_a^x \left[[V(\xi)]^{\frac{p}{q}} W'(\xi) + [W(\xi)]^{\frac{p}{q}} V'(\xi) \right] d\xi \right)^{\frac{q}{s}}. \tag{2.11}$$

Since $V(a) = W(a) = 0$ then with (2.10) follows

$$\int_a^x \left[[V(\xi)]^{\frac{p}{q}} W'(\xi) + [W(\xi)]^{\frac{p}{q}} V'(\xi) \right] d\xi \leq \frac{q}{p+q} (d_{\frac{p}{q}} - 2^{\frac{-p}{q}}) [V(x) + W(x)]^{\frac{p}{q}+1}. \tag{2.12}$$

Hence from (2.11) and (2.12), we conclude that

$$\int_a^x \lambda(\xi) \left[|(J_{a+}^\alpha \Psi)(\xi)|^p |\Phi(\xi)|^q + |(J_{a+}^\alpha \Phi)(\xi)|^p |\Psi(\xi)|^q \right] d\xi$$

$$\leq 2^{1-\frac{q}{s}} \left(\frac{q}{p+q} \right)^{\frac{q}{s}} \left(d_{\frac{p}{q}} - 2^{\frac{-p}{q}} \right)^{\frac{q}{s}} \left(\int_a^x [U(\xi)]^{\frac{s}{s-q}} d\xi \right)^{\frac{s-q}{s}} \left(\int_a^x \varrho(\eta) [|\Phi(\eta)|^s + |\Psi(\eta)|^s] d\eta \right)^{\frac{p+q}{s}}.$$

The proof is complete. □

Next for $s = \infty$ we present the extreme case of Theorem 2.1.

Theorem 2.2. *Let $(J_{a+}^\alpha \Phi)$ be the left side Hadamard fractional integral operator of Φ of order α. Let $\lambda \geq 0$ be measurable function on $[a, x]$, and $p, l_1, l_2 \geq 0$ and $\Phi, \Psi \in L_\infty[a, b]$. Then*

$$\int_a^x \lambda(\xi) \left[|(J_{a+}^{\alpha_1} \Phi)(\xi)|^{l_1} |(J_{a+}^{\alpha_2} \Psi)(\xi)|^{l_2} |\Phi(\xi)|^p + |(J_{a+}^{\alpha_2} \Phi)(\xi)|^{l_2} |(J_{a+}^{\alpha_1} \Psi)(\xi)|^{l_1} |\Psi(\xi)|^p \right] d\xi$$

$$\leq N \left(\int_a^x \left(\log\left(\frac{\xi}{a} \right) \right)^{\alpha_1 l_1 + \alpha_2 l_2} d\xi \right) \frac{1}{2} \left[||\Phi||_\infty^{2(l_1+p)} + ||\Phi||_\infty^{2l_2} + ||\Psi||_\infty^{2l_2} + ||\Psi||_\infty^{2(l_1+p)} \right],$$

where $N = \frac{||\lambda||_\infty}{[\Gamma(\alpha_1+1)]^{l_1} [\Gamma(\alpha_2+1)]^{l_2}}$.

Proof. Let $\xi \in [a, x]$, using identity (1.1) the triangle inequality and Hölder's inequality, for $i = 1, 2$ we have

$$|(J_{a+}^{\alpha_i} \Phi)(\xi)|^{l_i} \leq \frac{1}{[\Gamma(\alpha_i)]^{l_i}} \left(\int_a^\xi \left(\log\left(\frac{\xi}{\eta} \right) \right)^{\alpha_i - 1} \frac{1}{\eta} d\eta \right)^{l_i} ||\Phi||_\infty^{l_i} = \frac{(\log(\frac{\xi}{a}))^{l_i \alpha_i}}{[\Gamma(\alpha_i + 1)]^{l_i}} ||\Phi||_\infty^{l_i}. \tag{2.13}$$

By analogy for $i = 1, 2$ we get

$$|(J_{a+}^{\alpha_i})\Psi(\xi)|^{l_i} \leq \frac{(\log(\frac{\xi}{a}))^{l_i \alpha_i}}{[\Gamma(\alpha_i + 1)]^{l_i}} ||\Psi||_\infty^{l_i}. \tag{2.14}$$

Also $|\Phi(\xi)|^p \leq ||\Phi||_\infty^p$, and $|\Psi(\xi)|^p \leq ||\Psi||_\infty^p$. Hence

$$|(J_{a+}^{\alpha_1}\Phi)(\xi)|^{l_1}|(J_{a+}^{\alpha_2}\Psi)(\xi)|^{l_2}|\Phi(\xi)|^p \leq \frac{(\log(\frac{\xi}{a}))^{l_1\alpha_1+l_2\alpha_2}}{[\Gamma(\alpha_1+1)]^{l_1}[\Gamma(\alpha_2+1)]^{l_2}}||\Phi||_\infty^{l_1+p}||\Psi||_\infty^{l_2}. \tag{2.15}$$

Likewise

$$|(J_{a+}^{\alpha_2}\Phi)(\xi)|^{l_2}|(J_{a+}^{\alpha_1}\Psi)(\xi)|^{l_1}|\Psi(\xi)|^p \leq \frac{(\log(\frac{\xi}{a}))^{l_1\alpha_1+l_2\alpha_2}}{[\Gamma(\alpha_1+1)]^{l_1}[\Gamma(\alpha_2+1)]^{l_2}}||\Phi||_\infty^{l_2}||\Psi||_\infty^{l_1+p}. \tag{2.16}$$

From (2.15) and (2.16) follows

$$\begin{aligned}
&\int_a^x \lambda(\xi)\left[|(J_{a+}^{\alpha_1}\Phi)(\xi)|^{l_1}|(J_{a+}^{\alpha_2}\Psi)(\xi)|^{l_2}|\Phi(\xi)|^p + |(J_{a+}^{\alpha_2}\Phi)(\xi)|^{l_2}|(J_{a+}^{\alpha_1}\Psi)(\xi)|^{l_1}|\Psi(\xi)|^p\right]d\xi \\
&\leq \frac{||\lambda||_\infty}{[\Gamma(\alpha_1+1)]^{l_1}[\Gamma(\alpha_2+1)]^{l_2}}\left(\int_a^x \left(\log\left(\frac{\xi}{a}\right)\right)^{\alpha_1 l_1+\alpha_2 l_2} d\xi\right) \\
&\quad \times\frac{1}{2}\left[||\Phi||_\infty^{2(l_1+p)} + ||\Phi||_\infty^{2l_2} + ||\Psi||_\infty^{2l_2} + ||\Psi||_\infty^{2(l_1+p)}\right], \\
&\leq N\left(\int_a^x \left(\log\left(\frac{\xi}{a}\right)\right)^{\alpha_1 l_1+\alpha_2 l_2} d\xi\right)\frac{1}{2}\left[||\Phi||_\infty^{2(l_1+p)} + ||\Phi||_\infty^{2l_2} + ||\Psi||_\infty^{2l_2} + ||\Psi||_\infty^{2(l_1+p)}\right].
\end{aligned}$$

This completes the proof. □

In the next theorem the counter part of Theorem 2.1 is given.

Theorem 2.3. *Let $(J_{a+}^{\alpha}\Phi)$ be the left side Hadamard fractional integral operator of Φ of order α . Assume $\lambda \geq 0, \varrho > 0$ be measurable functions on $[a, x]$. Let $s < 0, q > 0$ and $p \geq 0$. If $\Phi, \Psi \in L_s[a, b]$, each of which is of fixed sign a.e. On $[a, b]$, also $\frac{1}{\Phi}, \frac{1}{\Psi} \in L_s[a, b]$. Then*

$$\begin{aligned}
&\int_a^x \lambda(\xi)\left[|(J_{a+}^{\alpha}\Psi)(\xi)|^p|\Phi(\xi)|^q + |(J_{a+}^{\alpha}\Phi)(\xi)|^p|\Psi(\xi)|^q\right]d\xi \\
&\geq 2^{1-\frac{q}{s}}\left(c_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{s}}\left(\frac{q}{p+q}\right)^{\frac{q}{s}}\left(\int_a^x [U(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}}\left(\int_a^x \varrho(\eta)[|\Phi(\eta)|^s + |\Psi(\eta)|^s]d\eta\right)^{\frac{p+q}{s}},
\end{aligned}$$

where $U(\xi)$ is defined by (3.5).

Proof. Let $\xi \in [a, x]$ using the identity (1.1) the triangle inequality and reverse Hölder's inequality for $\{\frac{s}{s-1}, s\}$ we have

$$\begin{aligned}
|(J_{a+}^{\alpha}\Psi)(\xi)| &\geq \frac{1}{\Gamma(\alpha)}\int_a^\xi \left(\log\frac{\xi}{\eta}\right)^{\alpha-1}\frac{1}{\eta}[\varrho(\eta)]^{\frac{-1}{s}}[\varrho(\eta)]^{\frac{1}{s}}|\Psi(\eta)|d\eta \\
&\geq \frac{1}{\Gamma(\alpha)}\left(\int_a^\xi \left[\left(\log\frac{\xi}{\eta}\right)\left(\frac{1}{\eta}\right)^{\frac{1}{\alpha-1}}[\varrho(\eta)]^{\frac{1}{-s(\alpha-1)}}\right]^{\frac{s(\alpha-1)}{s-1}}d\eta\right)^{\frac{s-1}{s}}\left(\int_a^\xi \varrho(\eta)|\Psi(\eta)|^s d\eta\right)^{\frac{1}{s}} \\
&= \frac{1}{\Gamma(\alpha)}[P(\xi)]^{\frac{s-1}{s}}[V(\xi)]^{\frac{1}{s}}.
\end{aligned} \tag{2.17}$$

Let $V(\xi)$ and $W(\xi)$ be difened by (2.4) and (2.5) respectively. Then

$$|\Phi(\xi)|^q = [W'(\xi)]^{\frac{q}{s}}[\varrho(\xi)]^{\frac{-q}{s}}. \tag{2.18}$$

Now (2.17) and (2.18) imply that

$$\lambda(\xi)|(J^{\alpha}_{a+}\Psi)(\xi)|^p|\Phi(\xi)|^q \geq U(\xi)[V(\xi)]^{\frac{p}{s}}[W'(\xi)]^{\frac{q}{s}}, \tag{2.19}$$

where $U(\xi)$ defined by (3.5). Integrating (2.19) and applying reverse Hölder's inequality for $\{\frac{s}{s-q}, \frac{s}{q}\}$ we obtain

$$\int_a^x \lambda(\xi)|(J^{\alpha}_{a+}\Psi)(\xi)|^p|\Phi(\xi)|^q d\xi \geq \left(\int_a^x [U(\xi)]^{\frac{s}{s-q}} d\xi\right)^{\frac{s-q}{s}} \left(\int_a^x [V(\xi)]^{\frac{p}{q}} W'(\xi) d\xi\right)^{\frac{q}{s}}. \tag{2.20}$$

Similarly

$$\int_a^x \lambda(\xi)|(J^{\alpha}_{a+}\Phi)(\xi)|^p|\Psi(\xi)|^q d\xi \geq \left(\int_a^x [U(\xi)]^{\frac{s}{s-q}} d\xi\right)^{\frac{s-q}{s}} \left(\int_a^x [W(\xi)]^{\frac{p}{q}} V'(\xi) d\xi\right)^{\frac{q}{s}}. \tag{2.21}$$

Therefore from (2.20) (2.21) and (2.10) with $s > q$ we conclude that

$$\begin{aligned}
&\int_a^x \lambda(\xi)\left[|(J^{\alpha}_{a+}\Psi)(\xi)|^p|\Phi(\xi)|^q + |(J^{\alpha}_{a+}\Phi)(\xi)|^p|\Psi(\xi)|^q\right] d\xi \\
&\geq \left(\int_a^x [U(\xi)]^{\frac{s}{s-q}} d\xi\right)^{\frac{s-q}{s}} 2^{1-\frac{q}{s}} \left(\int_a^x \left[[V(\xi)]^{\frac{p}{q}} W'(\xi) + [W(\xi)]^{\frac{p}{q}} V'(\xi)\right] d\xi\right)^{\frac{q}{s}}.
\end{aligned} \tag{2.22}$$

Since $V(a) = W(a) = 0$ then with (2.10) follows

$$\int_a^x \left[[V(\xi)]^{\frac{p}{q}} W'(\xi) + [W(\xi)]^{\frac{p}{q}} V'(\xi)\right] d\xi \geq \frac{q}{p+q}(c_{\frac{p}{q}} - 2^{\frac{-p}{q}})[V(x) + W(x)]^{\frac{p}{q}+1}. \tag{2.23}$$

Hence from (2.22) and (2.23) we conclude that

$$\begin{aligned}
&\int_a^x \lambda(\xi)\left[|(J^{\alpha}_{a+}\Psi)(\xi)|^p|\Phi(\xi)|^q + |(J^{\alpha}_{a+}\Phi)(\xi)|^p|\Psi(\xi)|^q\right] d\xi \\
&\geq 2^{1-\frac{q}{s}} \left(\frac{q}{p+q}\right)^{\frac{q}{s}} \left(c_{\frac{p}{q}} - 2^{\frac{-p}{q}}\right)^{\frac{q}{s}} \left(\int_a^x [U(\xi)]^{\frac{s}{s-q}} d\xi\right)^{\frac{s-q}{s}} \left(\int_a^x \varrho(\eta)[|\Phi(\eta)|^s + |\Psi(\eta)|^s] d\eta\right)^{\frac{p+q}{s}}.
\end{aligned}$$

This complete the result. □

3. Opial-type inequalities for fractional integral of a function with respect to an increasing function

In this section we shall give the results for fractional integral of a function with respect to an increasing function g.

Theorem 3.1. *If $(I^{\alpha}_{a+;g}\Phi)$ be the left side fractional integral of a function Φ with respect to an increasing function g. Let $\varrho > 0, \lambda \geq 0$ be measurable functions on $[a, x]$. Assume $s > 1, s > q > 0$ also $p \geq 0$. If $\Phi, \Psi \in L_s[a, b]$. Then*

$$\int_a^x \lambda(\xi)\left[|(I^{\alpha}_{a+;g}\Psi)(\xi)|^p|\Phi(\xi)|^q + |(I^{\alpha}_{a+;g}\Phi)(\xi)|^p|\Psi(\xi)|^q\right]d\xi$$

$$\leq 2^{1-\frac{q}{s}}\left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{s}}\left(\frac{q}{p+q}\right)^{\frac{q}{s}}\left(\int_a^x [D(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}}\left(\int_a^x \varrho(\eta)[|\Phi(\eta)|^s + |\Psi(\eta)|^s]d\eta\right)^{\frac{p+q}{s}}, \tag{3.1}$$

where

$$D(\xi) = \frac{1}{[\Gamma(\alpha)]^p}\lambda(\xi)[\varrho(\xi)]^{-\frac{q}{s}}[Q(\xi)]^{\frac{p(s-1)}{s}}. \tag{3.2}$$

Proof. Let $\xi \in [a, x]$ using the identity (1.3) and Hölder's inequality for $\{\frac{s}{s-1}, s\}$ we have

$$\begin{aligned}
|(I^{\alpha}_{a+;g}\Psi)(\xi)| &\leq \frac{1}{\Gamma(\alpha)}\int_a^{\xi}[g(\xi) - g(\eta)]^{\alpha-1}g'(\eta)[\varrho(\eta)]^{\frac{-1}{s}}[\varrho(\eta)]^{\frac{1}{s}}|\Psi(\eta)|d\eta \\
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^{\xi}\left[[g(\xi) - g(\eta)](g'(\eta))^{\frac{1}{\alpha-1}}[\varrho(\eta)]^{\frac{1}{-s(\alpha-1)}}\right]^{\frac{s(\alpha-1)}{s-1}}d\eta\right)^{\frac{s-1}{s}}\left(\int_a^{\xi}\varrho(\eta)|\Psi(\eta)|^s d\eta\right)^{\frac{1}{s}} \\
&= \frac{1}{\Gamma(\alpha)}[Q(\xi)]^{\frac{s-1}{s}}[\Lambda(\xi)]^{\frac{1}{s}},
\end{aligned} \tag{3.3}$$

where

$$\Lambda(\xi) = \int_a^{\xi}\varrho(\eta)|\Psi(\eta)|^s d\eta. \tag{3.4}$$

Let us choose

$$\Upsilon(\xi) = \int_a^{\xi}\varrho(\eta)|\Phi(\eta)|^s d\eta. \tag{3.5}$$

Then

$$\Upsilon'(\xi) = \varrho(\xi)|\Phi(\xi)|^s,$$

that is

$$|\Phi(\xi)|^q = [\Upsilon'(\xi)]^{\frac{q}{s}}[\varrho(\xi)]^{\frac{-q}{s}}. \tag{3.6}$$

Now (3.3) and (3.6) imply that

$$\lambda(\xi)|(I^{\alpha}_{a+;g}\Psi)(\xi)|^p|\Phi(\xi)|^q \leq D(\xi)[\Lambda(\xi)]^{\frac{p}{s}}[\Upsilon'(\xi)]^{\frac{q}{s}}, \tag{3.7}$$

where

$$D(\xi) = \frac{1}{[\Gamma(\alpha)]^p}\lambda(\xi)[\varrho(\xi)]^{-\frac{q}{s}}[Q(\xi)]^{\frac{p(s-1)}{s}}.$$

Integrating (3.7) and applying Hölder's inequality for $\{\frac{s}{s-q}, \frac{s}{q}\}$ we obtain

$$\int_a^x \lambda(\xi)|(I^{\alpha}_{a+;g}\Psi)(\xi)|^p|\Phi(\xi)|^q d\xi \leq \left(\int_a^x [D(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}}\left(\int_a^x [\Lambda(\xi)]^{\frac{p}{q}}\Upsilon'(\xi)d\xi\right)^{\frac{q}{s}}. \tag{3.8}$$

Similarly

$$\int_a^x \lambda(\xi)|(I^{\alpha}_{a+;g}\Phi)(\xi)|^p|\Psi(\xi)|^q d\xi \leq \left(\int_a^x [D(\xi)]^{\frac{s}{s-q}} d\xi\right)^{\frac{s-q}{s}} \left(\int_a^x [\Upsilon(\xi)]^{\frac{p}{q}} \Lambda'(\xi) d\xi\right)^{\frac{q}{s}}. \tag{3.9}$$

Therefore from (3.8), (3.9) and (2.10) with $s > q$ we conclude that

$$\begin{aligned}&\int_a^x \lambda(\xi)\left[|(I^{\alpha}_{a+;g}\Psi)(\xi)|^p|\Phi(\xi)|^q + |(I^{\alpha}_{a+;g}\Phi)(\xi)|^p|\Psi(\xi)|^q\right] d\xi \\ &\leq \left(\int_a^x [D(\xi)]^{\frac{s}{s-q}} d\xi\right)^{\frac{s-q}{s}} 2^{1-\frac{q}{s}} \left(\int_a^x \left[[\Lambda(\xi)]^{\frac{p}{q}}\Upsilon'(\xi) + [\Upsilon(\xi)]^{\frac{p}{q}}\Lambda'(\xi)\right] d\xi\right)^{\frac{q}{s}}.\end{aligned} \tag{3.10}$$

Since $\Lambda(a) = \Upsilon(a) = 0$ then with (2.10) follows

$$\int_a^x \left[[\Lambda(\xi)]^{\frac{p}{q}}\Upsilon'(\xi) + [\Upsilon(\xi)]^{\frac{p}{q}}\Lambda'(\xi)\right] d\xi \leq \frac{q}{p+q}(d_{\frac{p}{q}} - 2^{\frac{-p}{q}})[\Lambda(x) + \Upsilon(x)]^{\frac{p}{q}+1}. \tag{3.11}$$

Hence from (3.10) and (3.11) we conclude that

$$\begin{aligned}&\int_a^x \lambda(\xi)\left[|(I^{\alpha}_{a+;g}\Psi)(\xi)|^p|\Phi(\xi)|^q + |(I^{\alpha}_{a+;g}\Phi)(\xi)|^p|\Psi(\xi)|^q\right] d\xi \\ &\leq 2^{1-\frac{q}{s}} \left(\frac{q}{p+q}\right)^{\frac{q}{s}} \left(d_{\frac{p}{q}} - 2^{\frac{-p}{q}}\right)^{\frac{q}{s}} \left(\int_a^x [D(\xi)]^{\frac{s}{s-q}} d\xi\right)^{\frac{s-q}{s}} \left(\int_a^x \varrho(\eta)[|\Phi(\eta)|^s + |\Psi(\eta)|^s] d\eta\right)^{\frac{p+q}{s}}.\end{aligned} \tag{3.12}$$

This complete the proof. □

Example 3.2. If we take $g(\xi) = \xi$ and $\lambda(\xi) = 1, \varrho(\xi) = 1, s = 2, a = 0$, in (3.1) we get

$$D(\xi) = \frac{(\xi)^{\frac{p(2\alpha-1)}{2}}}{[\Gamma(\alpha)]^p[2\alpha-1]^{\frac{p}{2}}},$$

and

$$\left(\int_0^x [D(\xi)]^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} = \frac{x^{\frac{p(2\alpha-1)-q+2}{2}}}{[\Gamma(\alpha)]^p[2\alpha-1]^{\frac{p}{2}}[\frac{p(2\alpha-1)-q+2}{2-q}]^{\frac{2-q}{2}}}.$$

Then (3.1) take the form

$$\begin{aligned}&\int_0^x \left[|(I^{\alpha}_{a+}\Psi)(\xi)|^p|\Phi(\xi)|^q + |(I^{\alpha}_{a+}\Phi)(\xi)|^p|\Psi(\xi)|^q\right] d\xi \\ &\leq 2^{1-\frac{q}{2}} \left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{2}} \left(\frac{q}{p+q}\right)^{\frac{q}{2}} \frac{x^{\frac{p(2\alpha-1)-q+2}{2}}}{[\Gamma(\alpha)]^p[2\alpha-1]^{\frac{p}{2}}[\frac{p(2\alpha-1)-q+2}{2-q}]^{\frac{2-q}{2}}} \left(\int_0^x [|\Phi(\eta)|^2 + |\Psi(\eta)|^2] d\eta\right)^{\frac{p+q}{2}},\end{aligned}$$

where $(I^{\alpha}_{a+}\Psi)(\xi)$ denote the Riemann-Liouville fractional integral of order α.

Now we give the extreme case of Theorem 3.1 for $s = \infty$.

Theorem 3.3. *Let $(I^{\alpha}_{a+;g}\Phi)$ be the left side fractional integral operator. Let $\lambda \geq 0$ be measurable function on $[a,x]$, and $p, l_1, l_2 \geq 0$ and $\Phi, \Psi \in L_\infty[a,b]$. Then*

$$\int_a^x \lambda(\xi)\left[|(I^{\alpha_1}_{a+;g}\Phi)(\xi)|^{l_1}|(I^{\alpha_2}_{a+;g}\Psi)(\xi)|^{l_2}|\Phi(\xi)|^p + |(I^{\alpha_2}_{a+;g}\Phi)(\xi)|^{l_2}|(I^{\alpha_1}_{a+;g}\Psi)(\xi)|^{l_1}|\Psi(\xi)|^p\right]d\xi$$
$$\leq N\left(\int_a^x [g(\xi)-g(a)]^{\alpha_1 l_1+\alpha_2 l_2}\right)d\xi\frac{1}{2}\left[||\Phi||_\infty^{2(l_1+p)} + ||\Phi||_\infty^{2l_2} + ||\Psi||_\infty^{2l_2} + ||\Psi||_\infty^{2(l_1+p)}\right],$$

where $N = \frac{||\lambda||_\infty}{[\Gamma(\alpha_1+1)]^{l_1}[\Gamma(\alpha_2+1)]^{l_2}}$.

Proof. Let $\xi \in [a,x]$ using identity (1.3) the triangle inequality and Hölder's inequality, for $i = 1,2$ we have

$$|(I^{\alpha_i}_{a+;g}\Phi)(\xi)|^{l_i} \leq \frac{1}{[\Gamma(\alpha_i)]^{l_i}}\left(\int_a^\xi [g(\xi)-g(\eta)]^{\alpha_i-1}(g'(\eta))d\eta\right)^{l_i} ||\Phi||_\infty^{l_i} = \frac{[g(\xi)-g(a)]^{l_i\alpha_i}}{[\Gamma(\alpha_i+1)]^{l_i}}||\Phi||_\infty^{l_i}. \quad (3.13)$$

By analogy for $i = 1,2$ we get

$$|(I^{\alpha_i}_{a+;g}\Psi)(\xi)|^{l_i} \leq \frac{[g(\xi)-g(a)]^{l_i\alpha_i}}{[\Gamma(\alpha_i+1)]^{l_i}}||\Psi||_\infty^{l_i}. \quad (3.14)$$

Also $|\Phi(\xi)|^p \leq ||\Phi||_\infty^p$, and $|\Psi(\xi)|^p \leq ||\Psi||_\infty^p$. Hence

$$|(I^{\alpha_1}_{a+;g}\Phi)(\xi)|^{l_1}|(I^{\alpha_2}_{a+;g}\Psi)(\xi)|^{l_2}|\Phi(\xi)|^p \leq \frac{[g(\xi)-g(a)]^{l_1\alpha_1+l_2\alpha_2}}{[\Gamma(\alpha_1+1)]^{l_1}[\Gamma(\alpha_2+1)]^{l_2}}||\Phi||_\infty^{l_1+p}||\Psi||_\infty^{l_2}. \quad (3.15)$$

Likewise we can write

$$|(I^{\alpha_2}_{a+;g}\Phi)(\xi)|^{l_2}|(I^{\alpha_1}_{a+;g}\Psi)(\xi)|^{l_1}|\Psi(\xi)|^p \leq \frac{[g(\xi)-g(a)]^{l_1\alpha_1+l_2\alpha_2}}{[\Gamma(\alpha_1+1)]^{l_1}[\Gamma(\alpha_2+1)]^{l_2}}||\Phi||_\infty^{l_2}||\Psi||_\infty^{l_1+p}. \quad (3.16)$$

From (3.15) and (3.16) follows

$$\int_a^x \lambda(\xi)\left[|(I^{\alpha_1}_{a+;g}\Phi)(\xi)|^{l_1}|(I^{\alpha_2}_{a+;g}\Psi)(\xi)|^{l_2}|\Phi(\xi)|^p + |(I^{\alpha_2}_{a+;g}\Phi)(\xi)|^{l_2}|(I^{\alpha_1}_{a+;g}\Psi)(\xi)|^{l_1}|\Psi(\xi)|^p\right]d\xi$$
$$\leq \frac{||\lambda||_\infty}{[\Gamma(\alpha_1+1)]^{l_1}[\Gamma(\alpha_2+1)]^{l_2}}\left(\int_a^x [g(\xi)-g(a)]^{\alpha_1 l_1+\alpha_2 l_2}\right)d\xi\frac{1}{2}\left[||\Phi||_\infty^{2(l_1+p)} + ||\Phi||_\infty^{2l_2} + ||\Psi||_\infty^{2l_2} + ||\Psi||_\infty^{2(l_1+p)}\right],$$
$$\leq N\left(\int_a^x [g(\xi)-g(a)]^{\alpha_1 l_1+\alpha_2 l_2}\right)d\xi\frac{1}{2}\left[||\Phi||_\infty^{2(l_1+p)} + ||\Phi||_\infty^{2l_2} + ||\Psi||_\infty^{2l_2} + ||\Psi||_\infty^{2(l_1+p)}\right].$$

This complete the result. □

Specially for $l_1, l_2 = 0$, and $g(\xi) = \xi$ we get the next corollary.

Corollary 3.4. *Let $(I^{\alpha}_{a+;g}\Phi)$ be the left side fractional integral of a function with respect to an increasing function g. Let $\lambda \geq 0$ be measurable function on $[a,x]$, also $p \geq 0$ and $\Phi, \Psi \in L_\infty[a,b]$.*

$$\int_a^x \lambda(\xi)\left[|\Phi(\xi)|^p + |\Psi(\xi)|^p\right]d\xi \leq ||\lambda||_\infty \int_a^x (1)d\xi \times \frac{1}{2}\left[||\Phi||_\infty^{2p} + ||\Psi||_\infty^{2p}\right] \leq \frac{||\lambda||_\infty(x-a)\left[||\Phi||_\infty^{2p} + ||\Psi||_\infty^{2p}\right]}{2}.$$

Now we present the counter part of Theorem 3.1 for the condition $s < 0$.

Theorem 3.5. *If $(I^{\alpha}_{a+;g}\Phi)$ be the left side fractional integral operator of a function Φ with respect to an increasing function g. Suppose $\varrho > 0, \lambda \geq 0$ be measurable functions on $[a,x]$. Assume $s < 0, q > 0$ also $p \geq 0$. If $\Phi, \Psi \in L_s[a,b]$, each of which is of fixed sign a.e. On $[a,b]$, also $\frac{1}{\Phi}, \frac{1}{\Psi} \in L_s[a,b]$. Then*

$$\int_a^x \lambda(\xi)\left[|(I^{\alpha}_{a+;g}\Psi)(\xi)|^p|\Phi(\xi)|^q + |(I^{\alpha}_{a+;g}\Phi)(\xi)|^p|\Psi(\xi)|^q\right]d\xi$$

$$\geq 2^{1-\frac{q}{s}}\left(c_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{s}}\left(\frac{q}{p+q}\right)^{\frac{q}{s}}\left(\int_a^x [D(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}}\left(\int_a^x \varrho(\eta)[|\Phi(\eta)|^s + |\Psi(\eta)|^s]d\eta\right)^{\frac{p+q}{s}},$$

where $D(\xi) = \frac{1}{[\Gamma(\alpha)]^p}\lambda(\xi)[\varrho(\xi)]^{-\frac{q}{s}}[Q(\xi)]^{\frac{p(s-1)}{s}}$.

Proof. Let $\xi \in [a,x]$, using identity (1.3) and reverse Hölder's inequality for $\{\frac{s}{s-1}, s\}$ we have

$$\begin{aligned}
|(I^{\alpha}_{a+;g}\Psi)(\xi)| &\geq \frac{1}{\Gamma(\alpha)}\int_a^{\xi}[g(\xi)-g(\eta)]^{\alpha-1}g'(\eta)[\varrho(\eta)]^{\frac{-1}{s}}[\varrho(\eta)]^{\frac{1}{s}}|\Psi(\eta)|d\eta \\
&\geq \frac{1}{\Gamma(\alpha)}\left(\int_a^{\xi}\left[[g(\xi)-g(\eta)](g'(\eta))^{\frac{1}{\alpha-1}}[\varrho(\eta)]^{\frac{1}{-s(\alpha-1)}}\right]^{\frac{s(\alpha-1)}{s-1}}d\eta\right)^{\frac{s-1}{s}}\left(\int_a^{\xi}\varrho(\eta)|\Psi(\eta)|^s d\eta\right)^{\frac{1}{s}} \\
&= \frac{1}{\Gamma(\alpha)}[Q(\xi)]^{\frac{s-1}{s}}[\Lambda(\xi)]^{\frac{1}{s}}, \qquad (3.17)
\end{aligned}$$

where $\Lambda(\xi)$ defined by (3.4). Choose $\Upsilon(\xi)$ defined by (3.5), then

$$\Upsilon'(\xi) = \varrho(\xi)|\Phi(\xi)|^s,$$

that is

$$|\Phi(\xi)|^q = [\Upsilon'(\xi)]^{\frac{q}{s}}[\varrho(\xi)]^{\frac{-q}{s}}. \qquad (3.18)$$

Now (3.17) and (3.18) imply that

$$\lambda(\xi)|(I^{\alpha}_{a+;g}\Psi)(\xi)|^p|\Phi(\xi)|^q \geq D(\xi)[\Lambda(\xi)]^{\frac{p}{s}}[\Upsilon'(\xi)]^{\frac{q}{s}}, \qquad (3.19)$$

where $D(\xi)$ defined by (3.2). Integrating (3.19) and applying Hölder's inequality for $\{\frac{s}{s-q}, \frac{s}{q}\}$ we obtain

$$\int_a^x \lambda(\xi)|(I^{\alpha}_{a+;g}\Psi)(\xi)|^p|\Phi(\xi)|^q d\xi \geq \left(\int_a^x [D(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}}\left(\int_a^x [\Lambda(\xi)]^{\frac{p}{q}}\Upsilon'(\xi)d\xi\right)^{\frac{q}{s}}. \qquad (3.20)$$

Similarly

$$\int_a^x \lambda(\xi)|(I^{\alpha}_{a+;g}\Phi)(\xi)|^p|\Psi(\xi)|^q d\xi \geq \left(\int_a^x [D(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}}\left(\int_a^x [\Upsilon(\xi)]^{\frac{p}{q}}\Lambda'(\xi)d\xi\right)^{\frac{q}{s}}. \qquad (3.21)$$

Therefore from (3.20) (3.21) and (2.10) with $s > q$ we conclude that

$$\int_a^x \lambda(\xi)\left[|(I^{\alpha}_{a+;g}\Psi)(\xi)|^p|\Phi(\xi)|^q + |(I^{\alpha}_{a+;g}\Phi)(\xi)|^p|\Psi(\xi)|^q\right]d\xi$$
$$\geq \left(\int_a^x [D(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}} 2^{1-\frac{q}{s}}\left(\int_a^x \left[[\Lambda(\xi)]^{\frac{p}{q}}\Upsilon'(\xi) + [\Upsilon(\xi)]^{\frac{p}{q}}\Lambda'(\xi)\right]d\xi\right)^{\frac{q}{s}}. \tag{3.22}$$

Since $\Lambda(a) = \Upsilon(a) = 0$ then with (2.10) follows

$$\int_a^x \left[[\Lambda(\xi)]^{\frac{p}{q}}W'(\xi) + [\Upsilon(\xi)]^{\frac{p}{q}}\Lambda'(\xi)\right]d\xi \geq \frac{q}{p+q}(c_{\frac{p}{q}} - 2^{\frac{-p}{q}})[\Lambda(x)+\Upsilon(x)]^{\frac{p}{q}+1}. \tag{3.23}$$

Hence from (3.22) and (3.23) we conclude that

$$\int_a^x \lambda(\xi)\left[|(I^{\alpha}_{a+;g}\Psi)(\xi)|^p|\Phi(\xi)|^q + |(I^{\alpha}_{a+;g}\Phi)(\xi)|^p|\Psi(\xi)|^q\right]d\xi$$
$$\geq 2^{1-\frac{q}{s}}\left(\frac{q}{p+q}\right)^{\frac{q}{s}}\left(c_{\frac{p}{q}} - 2^{\frac{-p}{q}}\right)^{\frac{q}{s}}\left(\int_a^x [D(\xi)]^{\frac{s}{s-q}}d\xi\right)^{\frac{s-q}{s}}\left(\int_a^x \varrho(\eta)[|\Phi(\eta)|^s + |\Psi(\eta)|^s]d\eta\right)^{\frac{p+q}{s}}.$$

The result is complete. □

Remark 3.6. If we choose $g(x) = \log x$ in results of Section 3, we the results for Hadamard fractional integrals.

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