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A Conceptual Framework of Convex and Concave Sets under Refined Intuitionistic Fuzzy Set Environment

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Abstract

Intuitionistic fuzzy set deals with membership and non-membership of a certain element of universe of discourse whereas these are further partitioned into their sub-membership degrees in refined intuitionistic fuzzy set. This study aims to introduce the notions of convex and concave refined intuitionistic fuzzy sets. Moreover, some of its important properties e.g. complement, union, intersection etc. and results are discussed.

Keywords: Ortho-convexity, Ortho-Concavity, Sub-membership degree, Sub non-membership degree, Infimum projection, Supremum projection..
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1. Introduction

In 1965, Zadeh [1] introduced fuzzy set theory for dealing with uncertainties. In fuzzy set, every member of the universe under consideration, is assigned a membership degree from the interval $[0, 1]$. Zadeh [2] employed his own concept of fuzzy sets as a basis for a theory of possibility. Dubois et al. [4, 5] discussed relationship between fuzzy sets and probability theories. They derived monotonicity property for algebraic operations performed between random set-valued variables. Ranking fuzzy numbers in the setting of possibility theory was done by Dubois et al. [3]. Beg et al. [6, 7, 8] calculated similarities between fuzzy sets under certain implications. Osman et al. [9] calculated the solution of nonlinear partial differential equations under fuzzy environment. Khan et al. [10] conceptualized some semigroups under the settings

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of intuitionistic fuzzy interior ideals. Rahman et al. [11] developed the conceptual framework of (m, n)-Convexity-cum-Concavity on fuzzy soft set with applications in first and second senses.
In certain real life situations, only membership grade is not sufficient. In 1986, Atanassov [12, 13] conceptualized intuitionistic fuzzy sets to adequate fuzzy sets for non-membership grade. Both membership value and non-membership values are allocated from unit closed interval to all the elements of universe of discourse. The development of intuitionistic fuzzy set (as the generalization of the fuzzy set) proved very useful tool for researchers. Ejegwa et al. [14] expanded the concept with the discussion on operations, algebra, model operators and normalization on Intuitionistic fuzzy sets. Montes et al. [15] investigated divergence measures for intuitionistic fuzzy sets and also discussed their properties. Boran et al. [16] characterized a biparametric similarity measure on intuitionistic fuzzy sets and discussed its applications to pattern recognition. Edalatpanah [17, 18] discussed data envelopment analysis model with triangular intuitionistic fuzzy numbers and neutrosophic structured element respectively. Yaqoob et al. [19] investigated complex intuitionistic fuzzy graphs and discussed their application in cellular network provider companies. Gulistan et al. [20] characterized direct product of general intuitionistic fuzzy sets of subtraction algebras and generalized their properties. Khan et al. [21] made valuable discussion on $(\epsilon, \epsilon V q_k)$-intuitionistic (fuzzy ideals, fuzzy soft ideals) of subtraction algebras. Xu et al. [22] conceptualized star-shaped intuitionistic fuzzy sets and discussed their properties.
Chaudhuri [23, 24] introduced the concept of concave fuzzy set and discussed some useful properties of concave fuzzy sets and also described some associated concepts of concave fuzzy set with their computational approaches. This concept is useful in development of fuzzy geometry and fuzzy structures. Yu-Ru Syau [25] enhanced this concept to convex and concave fuzzy mappings. Sarkar [26] introduced concavo-convex fuzzy set and established some interesting properties of this special type of fuzzy set. Ban [27, 28] made valuable discussion on convex intuitionistic fuzzy sets and developed Convex temporal intuitionistic fuzzy sets. D$\acute{\imath}$az et al. [29] characterized the aggregation of convex intuitionistic fuzzy sets and discussed its generalized properties in detail. Smarandache [30] refined the definition of fuzzy sets and enhanced it to refined Intuitionistic fuzzy set, refined inconsistent intuitionistic fuzzy set, refined picture fuzzy set, refined ternary fuzzy set. This concept of refinement was very useful and helpful for more refinements in fuzzy sets such as refined pythagorean fuzzy set, refined atanassov's intuitionistic fuzzy set of type 2, refined spherical fuzzy set, refined n-hyper spherical fuzzy set, refined q-rung orthopair fuzzy set etc. Rahman et al. [31] studied the fundamental properties, operations and results of refined intuitionistic fuzzy sets with examples. Motivating from the concept discussed in [27, 28, 30], convex and concave sets are characterized under refined intuitionistic fuzzy set. Moreover, some important essential properties and results are discussed in this context.
The rest of the paper is organized as: section 2 recalls basic definitions and terms relating to main result from literature, section 3 presents novel concept of convexity cum concavity on refined intuitionistic fuzzy set along with some properties and results and section 4 finally concludes the paper.

2. Preliminaries

In this section, some basic definitions and terms are recalled from literature for supporting main study of this work. Here Z , G and I will play the role of universal set, R^n and $[0, 1]$ respectively.

Definition 2.1. [1, 2] A set A is *fuzzy set* if A is characterized by a membership function Γ_A and defined

$$A = \{(\Gamma_A(\alpha)), \alpha \in Z\}, \Gamma_A : Z \to P(I)$$

where $A \subseteq Z$.

- The *union* of A and B where both are fuzzy sets with respective membership functions $f_A(\alpha)$ and $f_B(\alpha)$ is another fuzzy set C, written as $C = A \cup B$, whose membership function is is given by

$$f_C(\alpha) = Max(f_A(\alpha), f_B(\alpha)), \alpha \in Z$$

it can also be written as

$$f_C = f_A \vee f_B.$$

- The *intersection* of A and B where both are fuzzy sets with respective membership functions $f_A(\alpha)$ and $f_B(\alpha)$ is another fuzzy set C, written as $C = A \cap B$, whose membership function is given by

$$f_C(\alpha) = Min(f_A(\alpha), f_B(\alpha)), \alpha \in Z$$

it can also be written as

$$f_C = f_A \wedge f_B.$$

Definition 2.2. [1] A fuzzy set A is *convex* iff

$$f_A(\lambda\alpha_1 + (1-\lambda)\alpha_2) \geq Min(f_A(\alpha_1), f_A(\alpha_2))$$

for all $\alpha_1, \alpha_2 \in Z$ and λ in I.

Definition 2.3. [23] A fuzzy set A is *concave* iff

$$f_A(\lambda\alpha_1 + (1-\lambda)\alpha_2) \leq Max(f_A(\alpha_1), f_A(\alpha_2))$$

for all $\alpha_1, \alpha_2 \in Z$ and λ in I.

Definition 2.4. [12] A set B is an *intuitionistic fuzzy set* if B is characterized by a membership function T_B and non-membership function F_B, and defined as

$$B = \{(T_B(\alpha), F_B(\alpha))), \alpha \in Z\}, T_B(\alpha), F_B(\alpha) : Z \to P(I)$$

where $B \subseteq Z$.

Definition 2.5. [27] An intuitionistic fuzzy set B is *convex* if

(i) $T_B(\lambda\alpha_1 + (1-\lambda)\alpha_2) \geq Min(T_B(\alpha_1), t_B(\alpha_2))$

(ii) $F_B(\lambda\alpha_1 + (1-\lambda)\alpha_2) \leq Max(F_B(\alpha_1), F_B(\alpha_2))$

for all $\alpha_1, \alpha_2 \in Z$ and λ in I.

Definition 2.6. [30] A *refined fuzzy set* A is defined as

$$A_{RFS} = \left\{\left(\Gamma_A^1(\alpha), \Gamma_A^2(\alpha), ..., \Gamma_A^p(\alpha), p \geq 2, \alpha \in A\right)\right\}$$

where Γ_A^j is sub-membership of degree j^{th}-type of elements of Z w.r.t. A, and is subset of I for $1 \leq j \leq p$ and $\sum_{j=1}^{p} \sup \Gamma_\alpha^j \leq 1, \forall \alpha \in A$.

Definition 2.7. [30] When membership degree $\Gamma_A(\alpha)$ and non-membership degree $\mathcal{F}_A(\alpha)$ are refined/splitted to sub-membership degrees, then *refined intuitionistic fuzzy set* can be defined as

$$A_{RIFS} = \left\{\alpha\left(\Gamma_A^1(\alpha), \Gamma_A^2(\alpha), ..., \Gamma_A^p(\alpha); \mathcal{F}_A^1(\alpha), \mathcal{F}_A^2(\alpha), ..., \mathcal{F}_A^s(\alpha)\right), p+s \geqslant 3, \alpha \in A\right\}$$

with p, s are positive nonzero integers and $\sum_{k=1}^{p} \sup \Gamma^k + \sum_{l=1}^{s} \sup \mathcal{F}^l \leq 1$, and $\Gamma_{A_{RIFS}}^k, \mathcal{F}_{A_{RIFS}}^l \subseteq [0,1]$ for $1 \leq k \leq p$ and $1 \leq l \leq s$ where $\Gamma_{A_{RIFS}}^k$ is sub-membership of degree k^{th}-type of the elements w.r.t A, and is subset of I for $1 \leq k \leq p$ and $\mathcal{F}_{A_{RIFS}}^l$ is sub non-membership of degree l^{th}-type of the elements w.r.t A, and is subset of I for $1 \leq l \leq s$

3. Convex and Concave Refined Intuitionistic Fuzzy Sets

In this section, convex and concave refined intuitionistic fuzzy sets are defined. Some important results are discussed.

Definition 3.1. A refined intuitionistic fuzzy set A_{RIFS} in G is *Convex* if for all $u, v \in G$ and all w on the line segment $\overline{uv}$

$$\Gamma^k_{A_{RIFS}}(w) \geq \min\left(\Gamma^k_{A_{RIFS}}(u), \Gamma^k_{A_{RIFS}}(v)\right), 1 \leq k \leq p$$

$$\mathcal{F}^l_{A_{RIFS}}(w) \leq \max\left(\mathcal{F}^l_{A_{RIFS}}(u), \mathcal{F}^l_{A_{RIFS}}(v)\right), 1 \leq l \leq s$$

where
$\Gamma^k_{A_{RIFS}}$ is sub-membership of degree k^{th}-type of the elements w.r.t A, and is subset of I for $1 \leq k \leq p$ and $\sum_{k=1}^{p} \sup \Gamma^k \leq 1$. $\mathcal{F}^l_{A_{RIFS}}$ is sub non-membership of degree l^{th}-type of the elements w.r.t A, and is subset of I for $1 \leq l \leq s$ and $\sum_{l=1}^{s} \sup \mathcal{F}^l \leq 1$ with condition $\sum_{k=1}^{p} \sup \Gamma^k + \sum_{l=1}^{s} \sup \mathcal{F}^l \leq 1$.

Definition 3.2. A refined intuitionistic fuzzy set A_{RIFS} in G is *Ortho-Convex* if for all $u, v \in G$ and all w on the line segment $\overline{uv}$ lie on line which is parallel to co-ordinate axis

$$\Gamma^k_{A_{RIFS}}(w') \geq \min\left(\Gamma^k_{A_{RIFS}}(u'), \Gamma^k_{A_{RIFS}}(v')\right), 1 \leq k \leq p.$$

$$\mathcal{F}^l_{A_{RIFS}}(w') \leq \max\left(\mathcal{F}^l_{A_{RIFS}}(u'), \mathcal{F}^l_{A_{RIFS}}(v')\right), 1 \leq l \leq s.$$

where
$\Gamma^k_{A_{RIFS}}$ is sub-membership of degree k^{th}-type of the elements w.r.t A, and is subset of I for $1 \leq k \leq p$ and $\sum_{k=1}^{p} \sup \Gamma^k \leq 1$. $\mathcal{F}^l_{A_{RIFS}}$ is sub non-membership of degree l^{th}-type of the elements w.r.t A, and is subset of I for $1 \leq l \leq s$ and $\sum_{l=1}^{s} \sup \mathcal{F}^l \leq 1$ with condition $\sum_{k=1}^{p} \sup \Gamma^k + \sum_{l=1}^{s} \sup \mathcal{F}^l \leq 1$.

Remark 3.3. An ortho-convex RIFS is convex RIFS but its converse may or may not true.

Definition 3.4. A refined intuitionistic fuzzy set A_{RIFS} in G is *Concave* if for all $u, v \in G$ and all w on the line segment $\overline{uv}$

$$\Gamma^k_{A_{RIFS}}(w) \leq \max\left(\Gamma^k_{A_{RIFS}}(u), \Gamma^k_{A_{RIFS}}(v)\right), 1 \leq k \leq p.$$

$$\mathcal{F}^l_{A_{RIFS}}(w) \geq \min\left(\mathcal{F}^l_{A_{RIFS}}(u), \mathcal{F}^l_{A_{RIFS}}(v)\right), 1 \leq l \leq s.$$

where
$\Gamma^k_{A_{RIFS}}$ is sub-membership of degree k^{th}-type of the elements w.r.t A, and is subset of I for $1 \leq k \leq p$ and $\sum_{k=1}^{p} \sup \Gamma^k \leq 1$. $\mathcal{F}^l_{A_{RIFS}}$ is sub non-membership of degree l^{th}-type of the elements w.r.t A, and is subset of I for $1 \leq l \leq s$ and $\sum_{l=1}^{s} \sup \mathcal{F}^l \leq 1$ with condition $\sum_{k=1}^{p} \sup \Gamma^k + \sum_{l=1}^{s} \sup \mathcal{F}^l \leq 1$.

Definition 3.5. A refined intuitionistic fuzzy set A_{RIFS} in G is *Ortho-Concave* if for all $u, v \in G$ and all w on the line segment $\overline{uv}$ lie on line which is parallel to co-ordinate axis

$$\Gamma^k_{A_{RIFS}}(w') \leq \max\left(\Gamma^k_{A_{RIFS}}(u'), \Gamma^k_{A_{RIFS}}(v')\right), 1 \leq k \leq p.$$

$$\mathcal{F}^l_{A_{RIFS}}(w') \geq \min\left(\mathcal{F}^l_{A_{RIFS}}(u'), \mathcal{F}^l_{A_{RIFS}}(v')\right), 1 \leq l \leq s.$$

where
$\Gamma^k_{A_{RIFS}}$ is sub-membership of degree k^{th}-type of the elements w.r.t A, and is subset of I for $1 \leq k \leq p$ and $\sum_{k=1}^{p} \sup \Gamma^k \leq 1$. $\mathcal{F}^l_{A_{RIFS}}$ is sub non-membership of degree l^{th}-type of the elements w.r.t A, and is subset of I for $1 \leq l \leq s$ and $\sum_{l=1}^{s} \sup \mathcal{F}^l \leq 1$ with condition $\sum_{k=1}^{p} \sup \Gamma^k + \sum_{l=1}^{s} \sup \mathcal{F}^l \leq 1$.

Remark 3.6. An ortho-concave RIFS is concave RIFS but its converse may or may not be true.

Theorem 3.7. *The Complement of convex A_{RIFS} is concave RIFS.*

Proof. If A is convex refined intuitionistic fuzzy then for any two points u and v and another point w which lies on $\overline{uv}$

$$\Gamma^k_{A_{RIFS}}(w) \geq \min\left(\Gamma^k_{A_{RIFS}}(u), \Gamma^k_{A_{RIFS}}(v)\right), 1 \leq k \leq p$$

so

$$\overline{\Gamma}^k_A(w) \leq 1 - \min\left(1 - \overline{\Gamma}^k_A(u), 1 - \overline{\Gamma}^k_A(v)\right), 1 \leq k \leq p \tag{3.1}$$

now if

$$1 - \overline{\Gamma}^k_A(u) \leq 1 - \overline{\Gamma}^k_A(v)$$

then

$$\min\left(1 - \overline{\Gamma}^k_A(u), 1 - \overline{\Gamma}^k_A(v)\right) = 1 - \overline{\Gamma}^k_A(u)$$

and there from (3.1)

$$\overline{\Gamma}^k_A(w) \leq \overline{\Gamma}^k_A(u)$$

similarly if

$$1 - \overline{\Gamma}^k_A(v) \leq 1 - \overline{\Gamma}^k_A(u)$$

then

$$\min\left(1 - \overline{\Gamma}^k_A(u), 1 - \overline{\Gamma}^k_A(v)\right) = 1 - \overline{\Gamma}^k_A(v)$$

so from (3.1)

$$\overline{\Gamma}^k_A(w) \leq \overline{\Gamma}^k_A(v)$$

Hence,

$$\overline{\Gamma}^k_A(w) \leq \max\left(\overline{\Gamma}^k_A(u), \overline{\Gamma}^k_A(v)\right), 1 \leq k \leq p$$

Similarly If A is convex refined intuitionistic fuzzy then for any two points u and v and another point w which lies on $\overline{uv}$

$$\mathcal{F}^l_{A_{RIFS}}(w) \leq \max\left(\mathcal{F}^l_{A_{RIFS}}(u), \mathcal{F}^l_{A_{RIFS}}(v)\right), 1 \leq l \leq s$$

then

$$\overline{\mathcal{F}}^l_A(w) \geq 1 - \max\left(1 - \overline{\mathcal{F}}^l_A(u), 1 - \overline{\mathcal{F}}^l_A(v)\right), 1 \leq l \leq s \tag{3.2}$$

now if

$$1 - \overline{\mathcal{F}}^l_A(u) \geq 1 - \overline{\mathcal{F}}^l_A(v)$$

then

$$\max\left(1 - \overline{\mathcal{F}}^l_A(u), 1 - \overline{\mathcal{F}}^l_A(v)\right) = 1 - \overline{\mathcal{F}}^l_A(u)$$

and from (3.2)

$$\overline{\mathcal{F}}^l_A(w) \geq \overline{\mathcal{F}}^l_A(u)$$

similarly if

$$1 - \overline{\mathcal{F}}^l_A(v) \geq 1 - \overline{\mathcal{F}}^l_A(u)$$

then

$$\max\left(1-\overline{\mathcal{F}}^{l}_{A}(u), 1-\overline{\mathcal{F}}^{l}_{A}(v)\right) = 1-\overline{\mathcal{F}}^{l}_{A}(v)$$

so from (3.2)

$$\overline{\mathcal{F}}^{l}_{A}(w) \geq \overline{\mathcal{F}}^{l}_{A}(v)$$

Hence

$$\overline{\mathcal{F}}^{l}_{A}(w) \geq \min\left(\overline{\mathcal{F}}^{l}_{A}(u), \overline{\mathcal{F}}^{l}_{A}(v)\right), 1 \leq l \leq s$$

which means compliment of A_{RIFS} is concave RIFS. □

Remark 3.8. The Complement of ortho-convex A_{RIFS} is ortho-concave and hence concave RIFS.

Theorem 3.9. *The union of two convex refined intuitionistic fuzzy sets is a convex refined intuitionistic fuzzy set.*

Proof. Let A_{RIFS} and B_{RIFS} be two convex refined intuitionistic fuzzy sets and $H = A_{RIFS} \cup B_{RIFS}$. Consider two points u and v and w which lie on $\overline{uv}$, Now

$$\Gamma^{k}_{H_{RIFS}}(u) = \min\left(\Gamma^{k}_{A_{RIFS}}(u), \Gamma^{k}_{B_{RIFS}}(u)\right), 1 \leq k \leq p$$

$$\Gamma^{k}_{H_{RIFS}}(v) = \min\left(\Gamma^{k}_{A_{RIFS}}(v), \Gamma^{k}_{B_{RIFS}}(v)\right), 1 \leq k \leq p$$

$$\Gamma^{k}_{H_{RIFS}}(w) = \min\left(\Gamma^{k}_{A_{RIFS}}(w), \Gamma^{k}_{B_{RIFS}}(w)\right), 1 \leq k \leq p$$

Now

$$\min\left(\Gamma^{k}_{H_{RIFS}}(u), \Gamma^{k}_{H_{RIFS}}(v)\right)$$

$$= \min\left(\min\left(\Gamma^{k}_{A_{RIFS}}(u), \Gamma^{k}_{B_{RIFS}}(u)\right), \min\left(\Gamma^{k}_{A_{RIFS}}(v), \Gamma^{k}_{B_{RIFS}}(v)\right)\right)$$

$$= \min\left(\Gamma^{k}_{A_{RIFS}}(u), \Gamma^{k}_{B_{RIFS}}(u), \Gamma^{k}_{A_{RIFS}}(v), \Gamma^{k}_{B_{RIFS}}(v)\right) \tag{3.3}$$

let

$$\Gamma^{k}_{A_{RIFS}}(w) \leq \Gamma^{k}_{B_{RIFS}}(w)$$

in (3.3) so that

$$\Gamma^{k}_{H_{RIFS}}(w) = \Gamma^{k}_{A_{RIFS}}(w)$$

as A is convex refined intuitionistic fuzzy set so

$$\Gamma^{k}_{A_{RIFS}}(w) \geq \min\left(\Gamma^{k}_{A_{RIFS}}(u), \Gamma^{k}_{A_{RIFS}}(v)\right)$$

$$\geq \min\left(\Gamma^{k}_{A_{RIFS}}(u), \Gamma^{k}_{B_{RIFS}}(u), \Gamma^{k}_{A_{RIFS}}(v), \Gamma^{k}_{B_{RIFS}}(v)\right)$$

i.e.

$$\Gamma^{k}_{A_{RIFS}}(w) = \Gamma^{k}_{H_{RIFS}}(w) \geq \min\left(\Gamma^{k}_{H_{RIFS}}(u)\, \Gamma^{k}_{H_{RIFS}}(v)\right)$$

similarly for $\Gamma^{k}_{B_{RIFS}}(w) \leq \Gamma^{k}_{A_{RIFS}}(w)$ in equation (3.3) so that

$$\Gamma^{k}_{H_{RIFS}}(w) = \Gamma^{k}_{B_{RIFS}}(w)$$

as B is convex refined intuitionistic fuzzy set so (3.3) becomes

$$\Gamma^{k}_{B_{RIFS}}(w) \geq \min\left(\Gamma^{k}_{B_{RIFS}}(u), \Gamma^{k}_{B_{RIFS}}(v)\right)$$

$$\geq \min\left(\Gamma^k_{A_{RIFS}}(u), \Gamma^k_{B_{RIFS}}(u), \Gamma^k_{A_{RIFS}}(v), \Gamma^k_{B_{RIFS}}(v)\right)$$

i.e.

$$\Gamma^k_{H_{RIFS}}(w) \geq \min\left(\Gamma^k_{H_{RIFS}}(u)\, \Gamma^k_{H_{RIFS}}(v)\right)$$

Similarly
Let A_{RIFS} and B_{RIFS} be two convex refined intuitionistic fuzzy sets and $H = A_{RIFS} \cup B_{RIFS}$. Consider two points u and v and w which lie on $\overline{uv}$, Now

$$\mathcal{F}^l_{H_{RIFS}}(u) = \max\left(\mathcal{F}^l_{A_{RIFS}}(u), \mathcal{F}^l_{B_{RIFS}}(u)\right), 1 \leq l \leq s$$

$$\mathcal{F}^l_{H_{RIFS}}(v) = \max\left(\mathcal{F}^l_{A_{RIFS}}(v), \mathcal{F}^l_{B_{RIFS}}(v)\right), 1 \leq l \leq s$$

$$\mathcal{F}^l_{H_{RIFS}}(w) = \max\left(\mathcal{F}^l_{A_{RIFS}}(w), \mathcal{F}^l_{B_{RIFS}}(w)\right), 1 \leq l \leq s$$

Now

$$\max\left(\mathcal{F}^l_{H_{RIFS}}(u), \mathcal{F}^l_{H_{RIFS}}(v)\right)$$

$$= \max\left(\max\left(\mathcal{F}^l_{A_{RIFS}}(u), \mathcal{F}^l_{B_{RIFS}}(u)\right), \max\left(\mathcal{F}^l_{A_{RIFS}}(v), \mathcal{F}^l_{B_{RIFS}}(v)\right)\right)$$

$$= \max\left(\mathcal{F}^l_{A_{RIFS}}(u), \mathcal{F}^l_{B_{RIFS}}(u), \mathcal{F}^l_{A_{RIFS}}(v), \mathcal{F}^l_{B_{RIFS}}(v)\right) \tag{3.4}$$

let

$$\mathcal{F}^l_{A_{RIFS}}(w) \leq \mathcal{F}^l_{B_{RIFS}}(w)$$

in (3.4) so that

$$\mathcal{F}^l_{H_{RIFS}}(w) = \mathcal{F}^l_{A_{RIFS}}(w)$$

as A is convex refined intuitionistic fuzzy set so

$$\mathcal{F}^l_{A_{RIFS}}(w) \leq \max\left(\mathcal{F}^l_{A_{RIFS}}(u), \mathcal{F}^l_{A_{RIFS}}(v)\right)$$

$$\leq \max\left(\mathcal{F}^l_{A_{RIFS}}(u), \mathcal{F}^l_{B_{RIFS}}(u), \mathcal{F}^l_{A_{RIFS}}(v), \mathcal{F}^l_{B_{RIFS}}(v)\right)$$

i.e.

$$\mathcal{F}^l_{A_{RIFS}}(w) = \mathcal{F}^l_{H_{RIFS}}(w) \leq \max\left(\mathcal{F}^l_{H_{RIFS}}(u)\, \mathcal{F}^l_{H_{RIFS}}(v)\right)$$

similarly for $\mathcal{F}^l_{B_{RIFS}}(w) \geq \mathcal{F}^l_{A_{RIFS}}(w)$ in equation (3.4) so that

$$\mathcal{F}^l_{H_{RIFS}}(w) = \mathcal{F}^l_{B_{RIFS}}(w)$$

as B is convex refined intuitionistic fuzzy set so (3.4) becomes

$$\mathcal{F}^l_{B_{RIFS}}(w) \leq \max\left(\mathcal{F}^l_{B_{RIFS}}(u), \mathcal{F}^l_{B_{RIFS}}(v)\right)$$

$$\leq \max\left(\mathcal{F}^l_{A_{RIFS}}(u), \mathcal{F}^l_{B_{RIFS}}(u), \mathcal{F}^l_{A_{RIFS}}(v), \mathcal{F}^l_{B_{RIFS}}(v)\right)$$

i.e.

$$\mathcal{F}^l_{H_{RIFS}}(w) \leq \max\left(\mathcal{F}^l_{H_{RIFS}}(u)\, \mathcal{F}^l_{H_{RIFS}}(v)\right)$$

hence the proof. □

Theorem 3.10. *The union of two ortho-convex RIFS is a ortho-convex RIFS and hence convex RIFS.*

Proof. Let A_{RIFS} and B_{RIFS} be two convex refined intuitionistic fuzzy sets and $H = A_{RIFS} \cup A_{RIFS}$ Consider two points u' and v' and and another point w' which lie on $\overline{u'v'}$ which is parallel to coordinate axis Now

$$\Gamma^k_{H_{RIFS}}(u') = \min\left(\Gamma^k_{A_{RIFS}}(u'), \Gamma^k_{B_{RIFS}}(u')\right), 1 \leq k \leq p$$

$$\Gamma^k_{H_{RIFS}}(v') = \min\left(\Gamma^k_{A_{RIFS}}(v'), \Gamma^k_{B_{RIFS}}(v')\right), 1 \leq k \leq p$$

$$\Gamma^k_{H_{RIFS}}(w') = \min\left(\Gamma^k_{A_{RIFS}}(w'), \Gamma^k_{B_{RIFS}}(w')\right), 1 \leq k \leq p$$

Now

$$\min\left(\Gamma^k_{H_{RIFS}}(u'), \Gamma^k_{H_{RIFS}}(v')\right)$$

$$= \min\left(\min\left(\Gamma^k_{A_{RIFS}}(u'), \Gamma^k_{B_{RIFS}}(u')\right), \min\left(\Gamma^k_{A_{RIFS}}(v'), \Gamma^k_{B_{RIFS}}(v')\right)\right)$$

$$= \min\left(\Gamma^k_{A_{RIFS}}(u'), \Gamma^k_{B_{RIFS}}(u'), \Gamma^k_{A_{RIFS}}(v'), \Gamma^k_{B_{RIFS}}(v')\right) \tag{3.5}$$

let

$$\Gamma^k_{A_{RIFS}}(w') \leq \Gamma^k_{B_{RIFS}}(w')$$

in (3.5) so that

$$\Gamma^k_{H_{RIFS}}(w') = \Gamma^k_{A_{RIFS}}(w')$$

as A is ortho-convex refined intuitionistic fuzzy set so

$$\Gamma^k_{A_{RIFS}}(w') \geq \min\left(\Gamma^k_{A_{RIFS}}(u'), \Gamma^k_{A_{RIFS}}(v')\right)$$

$$\geq \min\left(\Gamma^k_{A_{RIFS}}(u'), \Gamma^k_{B_{RIFS}}(u'), \Gamma^k_{A_{RIFS}}(v'), \Gamma^k_{B_{RIFS}}(v')\right)$$

i.e.

$$\Gamma^k_{A_{RIFS}}(w') = \Gamma^k_{H_{RIFS}}(w') \geq \min\left(\Gamma^k_{H_{RIFS}}(u')\, \Gamma^k_{H_{RIFS}}(v')\right)$$

similarly for $\Gamma^k_{B_{RIFS}}(w') \leq \Gamma^k_{A_{RIFS}}(w')$ in equation (3.5) so that

$$\Gamma^k_{H_{RIFS}}(w') = \Gamma^k_{B_{RIFS}}(w')$$

as B is ortho-convex refined intuitionistic fuzzy set so

$$\Gamma^k_{B_{RIFS}}(w') \geq \min\left(\Gamma^k_{B_{RIFS}}(u'), \Gamma^k_{B_{RIFS}}(v')\right)$$

$$\geq \min\left(\Gamma^k_{A_{RIFS}}(u'), \Gamma^k_{B_{RIFS}}(u'), \Gamma^k_{A_{RIFS}}(v'), \Gamma^k_{B_{RIFS}}(v')\right)$$

i.e.

$$\Gamma^k_{B_{RIFS}}(w') = \Gamma^k_{H_{RIFS}}(w') \geq \min\left(\Gamma^k_{H_{RIFS}}(u')\, \Gamma^k_{H_{RIFS}}(v')\right)$$

Similarly
Consider two points u' and v' and and another point w' which lie on $\overline{u'v'}$ which is parallel to coordinate axis Now

$$\mathcal{F}^l_{H_{RIFS}}(u') = \max\left(\mathcal{F}^l_{A_{RIFS}}(u'), \mathcal{F}^l_{B_{RIFS}}(u')\right), 1 \leq l \leq s$$

$$\mathcal{F}^l_{H_{RIFS}}(v') = \max\left(\mathcal{F}^l_{A_{RIFS}}(v'), \mathcal{F}^l_{B_{RIFS}}(v')\right), 1 \leq l \leq s$$

$$\mathcal{F}^l_{H_{RIFS}}(w') = \max\left(\mathcal{F}^l_{A_{RIFS}}(w'), \mathcal{F}^l_{B_{RIFS}}(w')\right), 1 \leq l \leq s$$

Now

$$\max\left(\mathcal{F}^l_{H_{RIFS}}(u'),\mathcal{F}^l_{H_{RIFS}}(v')\right)$$
$$=\max\left(\max\left(\mathcal{F}^l_{A_{RIFS}}(u'),\mathcal{F}^l_{B_{RIFS}}(u')\right),\max\left(\mathcal{F}^l_{A_{RIFS}}(v'),\mathcal{F}^l_{B_{RIFS}}(v')\right)\right)$$
$$=\max\left(\mathcal{F}^l_{A_{RIFS}}(u'),\mathcal{F}^l_{B_{RIFS}}(u'),\mathcal{F}^l_{A_{RIFS}}(v'),\mathcal{F}^l_{B_{RIFS}}(v')\right) \tag{3.6}$$

let

$$\mathcal{F}^l_{A_{RIFS}}(w')\geq\mathcal{F}^l_{B_{RIFS}}(w')$$

in (3.6) so that

$$\mathcal{F}^l_{H_{RIFS}}(w')=\mathcal{F}^l_{A_{RIFS}}(w')$$

as A is ortho-convex refined intuitionistic fuzzy set so

$$\mathcal{F}^l_{A_{RIFS}}(w')\leq\max\left(\mathcal{F}^l_{A_{RIFS}}(u'),\mathcal{F}^l_{A_{RIFS}}(v')\right)$$

$$\leq\max\left(\mathcal{F}^l_{A_{RIFS}}(u'),\mathcal{F}^l_{B_{RIFS}}(u'),\mathcal{F}^l_{A_{RIFS}}(v'),\mathcal{F}^l_{B_{RIFS}}(v')\right)$$

i.e.

$$\mathcal{F}^l_{A_{RIFS}}(w')=\mathcal{F}^l_{H_{RIFS}}(w')\leq\max\left(\mathcal{F}^l_{H_{RIFS}}(u')\,\mathcal{F}^l_{H_{RIFS}}(v')\right)$$

similarly for $\mathcal{F}^l_{B_{RIFS}}(w')\geq\mathcal{F}^l_{A_{RIFS}}(w')$ in equation (3.6) so that

$$\mathcal{F}^l_{H_{RIFS}}(w')=\mathcal{F}^l_{B_{RIFS}}(w')$$

as B is ortho-convex refined intuitionistic fuzzy set so

$$\mathcal{F}^l_{B_{RIFS}}(w')\leq\max\left(\mathcal{F}^l_{B_{RIFS}}(u'),\mathcal{F}^l_{B_{RIFS}}(v')\right)$$

$$\leq\max\left(\mathcal{F}^l_{A_{RIFS}}(u'),\mathcal{F}^l_{B_{RIFS}}(u'),\mathcal{F}^l_{A_{RIFS}}(v'),\mathcal{F}^l_{B_{RIFS}}(v')\right)$$

i.e.

$$\mathcal{F}^l_{B_{RIFS}}(w')=\mathcal{F}^l_{H_{RIFS}}(w')\leq\max\left(\mathcal{F}^l_{H_{RIFS}}(u')\,\mathcal{F}^l_{H_{RIFS}}(v')\right)$$

since every ortho-convex refined intuitionistic fuzzy set is also convex refined intuitionistic fuzzy set which leads to completion of proof. □

Remark 3.11. The union of family of convex refined intuitionistic fuzzy sets is a convex refined intuitionistic fuzzy set.

Remark 3.12. The union of family of ortho-convex refined intuitionistic fuzzy sets is ortho-convex refined intuitionistic fuzzy and hence convex refined intuitionistic fuzzy set.

Theorem 3.13. *The Complement of concave A_{RIFS} is convex RIFS.*

Proof. If A is concave refined intuitionistic fuzzy set then for any two points u and v and another point w which lies on $\overline{uv}$, then

$$\Gamma^k_{A_{RIFS}}(w)\leq\max\left(\Gamma^k_{A_{RIFS}}(u),\Gamma^k_{A_{RIFS}}(v)\right),1\leq k\leq p$$

so

$$\overline{\Gamma}^k_A(w)\geq1-\max\left(1-\overline{\Gamma}^k_A(u),1-\overline{\Gamma}^k_A(v)\right),1\leq k\leq p \tag{3.7}$$

now if

$$1-\overline{\Gamma}^k_A(u)\leq1-\overline{\Gamma}^k_A(v)$$

then

$$\max\left(1-\overline{\Gamma}_A^k(u),1-\overline{\Gamma}_A^k(v)\right)=1-\overline{\Gamma}_A^k(v)$$

and from (3.7)

$$\overline{\Gamma}_A^k(w)\geq\overline{\Gamma}_A^k(v)$$

similarly if

$$1-\overline{\Gamma}_A^k(v)\leq 1-\overline{\Gamma}_A^k(u)$$

then

$$\max\left(1-\overline{\Gamma}_A^k(u),1-\overline{\Gamma}_A^k(v)\right)=1-\overline{\Gamma}_A^k(u)$$

so from (3.7)

$$\overline{\Gamma}_A^k(w)\geq\overline{\Gamma}_A^k(u)$$

Hence

$$\overline{\Gamma}_A^k(w)\geq\min\left(\overline{\Gamma}_A^k(u),\overline{\Gamma}_A^k(v)\right),1\leq k\leq p$$

which means compliment of A_{RIFS} is convex RIFS.
Similarly
If A is concave refined intuitionistic fuzzy set then for any two points u and v and another point w which lies on $\overline{uv}$, then

$$\mathcal{F}_{A_{RIFS}}^l(w)\geq\min\left(\mathcal{F}_{A_{RIFS}}^l(u),\mathcal{F}_{A_{RIFS}}^l(v)\right),1\leq l\leq s$$

so we have

$$\overline{\mathcal{F}}_A^l(w)\leq 1-\min\left(1-\overline{\mathcal{F}}_A^l(u),1-\overline{\mathcal{F}}_A^l(v)\right),1\leq l\leq s \tag{3.8}$$

now if

$$1-\overline{\mathcal{F}}_A^l(u)\geq 1-\overline{\mathcal{F}}_A^l(v)$$

then

$$\min\left(1-\overline{\mathcal{F}}_A^l(u),1-\overline{\mathcal{F}}_A^l(v)\right)=1-\overline{\mathcal{F}}_A^l(v)$$

and there from (3.8)

$$\overline{\mathcal{F}}_A^l(w)\leq\overline{\mathcal{F}}_A^l(v)$$

similarly if

$$1-\overline{\mathcal{F}}_A^l(v)\geq 1-\overline{\mathcal{F}}_A^l(u)$$

then

$$\min\left(1-\overline{\mathcal{F}}_A^l(u),1-\overline{\mathcal{F}}_A^l(v)\right)=1-\overline{\mathcal{F}}_A^l(u)$$

so from (3.8)

$$\overline{\mathcal{F}}_A^l(w)\leq\overline{\mathcal{F}}_A^l(u)$$

Hence

$$\overline{\mathcal{F}}_A^l(w)\leq\max\left(\overline{\mathcal{F}}_A^l(u),\overline{\mathcal{F}}_A^l(v)\right),1\leq l\leq s$$

which means compliment of A_{RIFS} is convex RIFS. □

Remark 3.14. The Complement of ortho-concave A_{RIFS} is ortho-convex and hence convex RIFS.

Theorem 3.15. *The union of two concave RIF-sets is a concave refined intuitionistic fuzzy set.*

Proof. Let A_{RIFS} and B_{RIFS} be two concave refined intuitionistic fuzzy sets and $H_{RIFS} = A_{RIFS} \cup B_{RIFS}$. Consider two points u and v and another point w on $\overline{uv}$
now

$$\Gamma^k_{H_{RIFS}}(u) = \max\left(\Gamma^k_{A_{RIFS}}(u), \Gamma^k_{B_{RIFS}}(u)\right), 1 \leq k \leq p$$

$$\Gamma^k_{H_{RIFS}}(v) = \max\left(\Gamma^k_{A_{RIFS}}(v), \Gamma^k_{B_{RIFS}}(v)\right), 1 \leq k \leq p$$

$$\Gamma^k_{H_{RIFS}}(w) = \max\left(\Gamma^k_{A_{RIFS}}(w), \Gamma^k_{B_{RIFS}}(w)\right), 1 \leq k \leq p$$

now

$$\max\left(\Gamma^k_{H_{RIFS}}(u), \Gamma^k_{H_{RIFS}}(v)\right)$$

$$= \max\left(\max\left(\Gamma^k_{A_{RIFS}}(u), \Gamma^k_{B_{RIFS}}(u)\right), \max\left(\Gamma^k_{A_{RIFS}}(v), \Gamma^k_{B_{RIFS}}(v)\right)\right)$$

$$= \max\left(\Gamma^k_{A_{RIFS}}(u), \Gamma^k_{B_{RIFS}}(u), \Gamma^k_{A_{RIFS}}(v), \Gamma^k_{B_{RIFS}}(v)\right) \tag{3.9}$$

let

$$\Gamma^k_{A_{RIFS}}(w) \geq \Gamma^k_{B_{RIFS}}(w)$$

in equation (3.9) so that

$$\Gamma^k_{H_{RIFS}}(w) = \Gamma^k_{A_{RIFS}}(w)$$

as A is concave refined intuitionistic fuzzy set so equation (3.9) becomes

$$\Gamma^k_{H_{RIFS}}(w) = \Gamma^k_{A_{RIFS}}(w) \leq \max\left(\Gamma^k_{A_{RIFS}}(u), \Gamma^k_{A_{RIFS}}(v)\right)$$

$$\Gamma^k_{H_{RIFS}}(w) \leq \max\left(\Gamma^k_{A_{RIFS}}(u), \Gamma^k_{B_{RIFS}}(u), \Gamma^k_{A_{RIFS}}(v)\, \Gamma^k_{B_{RIFS}}(v)\right)$$

i.e.

$$\Gamma^k_{H_{RIFS}}(w) \leq \max\left(\Gamma^k_{H_{RIFS}}(u)\, \Gamma^k_{H_{RIFS}}(v)\right)$$

similarly for $\Gamma^k_{B_{RIFS}}(w) \geq \Gamma^k_{A_{RIFS}}(w)$,in equation (3.9) so that

$$\Gamma^k_{H_{RIFS}}(w) = \Gamma^k_{B_{RIFS}}(w)$$

as B is concave refined intuitionistic fuzzy set so equation (3.9) becomes

$$\Gamma^k_{H_{RIFS}}(w) = \Gamma^k_{B_{RIFS}}(w) \leq \max\left(\Gamma^k_{B_{RIFS}}(u), \Gamma^k_{A_{RIFS}}(v)\right)$$

$$\Gamma^k_{H_{RIFS}}(w) \leq \max\left(\Gamma^k_{A_{RIFS}}(u), \Gamma^k_{B_{RIFS}}(u), \Gamma^k_{A_{RIFS}}(v)\, \Gamma^k_{B_{RIFS}}(v)\right)$$

i.e.

$$\Gamma^k_{H_{RIFS}}(w) \leq \max\left(\Gamma^k_{H_{RIFS}}(u)\, \Gamma^k_{H_{RIFS}}(v)\right)$$

Similarly
Consider two points u and v and another point w on $\overline{uv}$
now

$$\mathcal{F}^l_{H_{RIFS}}(u) = \max\left(\mathcal{F}^l_{A_{RIFS}}(u), \mathcal{F}^l_{B_{RIFS}}(u)\right), 1 \leq l \leq s$$

$$\mathcal{F}^l_{H_{RIFS}}(v) = \max\left(\mathcal{F}^l_{A_{RIFS}}(v), \mathcal{F}^l_{B_{RIFS}}(v)\right), 1 \leq l \leq s \tag{3.10}$$

$$\mathcal{F}^l_{H_{RIFS}}(w) = \max\left(\mathcal{F}^l_{A_{RIFS}}(w), \mathcal{F}^l_{B_{RIFS}}(w)\right), 1 \leq l \leq s$$

now

$$\max\left(\mathcal{F}^l_{H_{RIFS}}(u),\mathcal{F}^l_{H_{RIFS}}(v)\right)$$

$$=\max\left(\max\left(\mathcal{F}^l_{A_{RIFS}}(u),\mathcal{F}^l_{B_{RIFS}}(u)\right),\max\left(\mathcal{F}^l_{A_{RIFS}}(v),\mathcal{F}^l_{B_{RIFS}}(v)\right)\right)$$

$$=\max\left(\mathcal{F}^l_{A_{RIFS}}(u),\mathcal{F}^l_{B_{RIFS}}(u),\mathcal{F}^l_{A_{RIFS}}(v),\mathcal{F}^l_{B_{RIFS}}(v)\right)$$

let

$$\mathcal{F}^l_{A_{RIFS}}(w)\geq\mathcal{F}^l_{B_{RIFS}}(w)$$

in equation (3.10) so that

$$\mathcal{F}^l_{H_{RIFS}}(w)=\mathcal{F}^l_{A_{RIFS}}(w)$$

as A is concave refined intuitionistic fuzzy set so equation (3.10) becomes

$$\mathcal{F}^l_{H_{RIFS}}(w)=\mathcal{F}^l_{A_{RIFS}}(w)\leq\max\left(\mathcal{F}^l_{A_{RIFS}}(u),\mathcal{F}^l_{A_{RIFS}}(v)\right)$$

$$\mathcal{F}^l_{H_{RIFS}}(w)\leq\max\left(\mathcal{F}^l_{A_{RIFS}}(u),\mathcal{F}^l_{B_{RIFS}}(u),\mathcal{F}^l_{A_{RIFS}}(v)\,\mathcal{F}^l_{B_{RIFS}}(v)\right)$$

i.e.

$$\mathcal{F}^l_{H_{RIFS}}(w)\leq\max\left(\mathcal{F}^l_{H_{RIFS}}(u)\,\mathcal{F}^l_{H_{RIFS}}(v)\right)$$

similarly for $\mathcal{F}^l_{B_{RIFS}}(w)\geq\mathcal{F}^l_{A_{RIFS}}(w)$,in equation (3.10) so that

$$\mathcal{F}^l_{H_{RIFS}}(w)=\mathcal{F}^l_{B_{RIFS}}(w)$$

as B is concave refined intuitionistic fuzzy set so equation (3.10) becomes

$$\mathcal{F}^l_{H_{RIFS}}(w)=\mathcal{F}^l_{B_{RIFS}}(w)\leq\max\left(\mathcal{F}^l_{B_{RIFS}}(u),\mathcal{F}^l_{A_{RIFS}}(v)\right)$$

$$\mathcal{F}^l_{H_{RIFS}}(w)\leq\max\left(\mathcal{F}^l_{A_{RIFS}}(u),\mathcal{F}^l_{B_{RIFS}}(u),\mathcal{F}^l_{A_{RIFS}}(v)\,\mathcal{F}^l_{B_{RIFS}}(v)\right)$$

i.e.

$$\mathcal{F}^l_{H_{RIFS}}(w)\leq\max\left(\mathcal{F}^l_{H_{RIFS}}(u)\,\mathcal{F}^l_{H_{RIFS}}(v)\right)$$

hence the proof. □

Theorem 3.16. *The union of two ortho-concave RIFS is a ortho-concave RIFS and hence concave RIFS.*

Proof. Let A and B be two ortho-concave RIFS and $H=A\cup B$
Consider two points u' and v' and another point w' on $\overline{u'v'}$ with condition that $\overline{u'v'}$ is parallel to coordinate axis.
Now

$$\Gamma^k_{H_{RIFS}}(u')=\max\left(\Gamma^k_{A_{RIFS}}(u'),\Gamma^k_{B_{RIFS}}(u')\right),1\leq k\leq p$$

$$\Gamma^k_{H_{RIFS}}(v')=\max\left(\Gamma^k_{A_{RIFS}}(v'),\Gamma^k_{B_{RIFS}}(v')\right),1\leq k\leq p$$

$$\Gamma^k_{H_{RIFS}}(w')=\max\left(\Gamma^k_{A_{RIFS}}(w'),\Gamma^k_{B_{RIFS}}(w')\right),1\leq k\leq p$$

now

$$\max\left(\Gamma^k_{H_{RIFS}}(u'),\Gamma^k_{H_{RIFS}}(v')\right)$$

$$=\max\left(\max\left(\Gamma^k_{A_{RIFS}}(u'),\Gamma^k_{B_{RIFS}}(u')\right),\max\left(\Gamma^k_{A_{RIFS}}(v'),\Gamma^k_{B_{RIFS}}(v')\right)\right)$$

$$= \max\left(\Gamma^k_{A_{RIFS}}(u'), \Gamma^k_{B_{RIFS}}(u'), \Gamma^k_{A_{RIFS}}(v'), \Gamma^k_{B_{RIFS}}(v')\right) \quad (3.11)$$

let

$$\Gamma^k_{A_{RIFS}}(w') \geq \Gamma^k_{B_{RIFS}}(w')$$

in (3.11) so that

$$\Gamma^k_{H_{RIFS}}(w') = \Gamma^k_{A_{RIFS}}(w')$$

as A is ortho-concave refined intuitionistic fuzzy set so

$$\Gamma^k_{H_{RIFS}}(w') = \Gamma^k_{A_{RIFS}}(w') \leq \max\left(\Gamma^k_{A_{RIFS}}(u'), \Gamma^k_{A_{RIFS}}(v')\right)$$

$$\Gamma^k_{H_{RIFS}}(w') \leq \max\left(\Gamma^k_{A_{RIFS}}(u'), \Gamma^k_{B_{RIFS}}(u'), \Gamma^k_{A_{RIFS}}(v') \Gamma^k_{B_{RIFS}}(v')\right)$$

i.e.

$$\Gamma^k_{H_{RIFS}}(w') \leq \max\left(\Gamma^k_{H_{RIFS}}(u') \Gamma^k_{H_{RIFS}}(v')\right)$$

similarly for $\Gamma^k_{B_{RIFS}}(w') \geq \Gamma^k_{A_{RIFS}}(w')$,in equation (3.11) so that

$$\Gamma^k_{H_{RIFS}}(w') = \Gamma^k_{B_{RIFS}}(w')$$

as B is ortho-concave refined intuitionistic fuzzy set so equation (3.11) becomes

$$\Gamma^k_{H_{RIFS}}(w') = \Gamma^k_{B_{RIFS}}(w') \leq \max\left(\Gamma^k_{B_{RIFS}}(u'), \Gamma^k_{A_{RIFS}}(v')\right)$$

$$\Gamma^k_{H_{RIFS}}(w') \leq \max\left(\Gamma^k_{A_{RIFS}}(u'), \Gamma^k_{B_{RIFS}}(u'), \Gamma^k_{A_{RIFS}}(v') \Gamma^k_{B_{RIFS}}(v')\right)$$

i.e.

$$\Gamma^k_{H_{RIFS}}(w') \leq \max\left(\Gamma^k_{H_{RIFS}}(u') \Gamma^k_{H_{RIFS}}(v')\right)$$

Similarly
Consider two points u' and v' and another point w' on $\overline{u'v'}$ with condition that $\overline{u'v'}$ is parallel to coordinate axis.
Now

$$\mathcal{F}^l_{H_{RIFS}}(u') = \min\left(\mathcal{F}^l_{A_{RIFS}}(u'), \mathcal{F}^l_{B_{RIFS}}(u')\right), 1 \leq l \leq s$$

$$\mathcal{F}^l_{H_{RIFS}}(v') = \min\left(\mathcal{F}^l_{A_{RIFS}}(v'), \mathcal{F}^l_{B_{RIFS}}(v')\right), 1 \leq l \leq s$$

$$\mathcal{F}^l_{H_{RIFS}}(w') = \min\left(\mathcal{F}^l_{A_{RIFS}}(w'), \mathcal{F}^l_{B_{RIFS}}(w')\right), 1 \leq l \leq s$$

now

$$\min\left(\mathcal{F}^l_{H_{RIFS}}(u'), \mathcal{F}^l_{H_{RIFS}}(v')\right)$$

$$= \min\left(\min\left(\mathcal{F}^l_{A_{RIFS}}(u'), \mathcal{F}^l_{B_{RIFS}}(u')\right), \min\left(\mathcal{F}^l_{A_{RIFS}}(v'), \mathcal{F}^l_{B_{RIFS}}(v')\right)\right)$$

$$= \min\left(\mathcal{F}^l_{A_{RIFS}}(u'), \mathcal{F}^l_{B_{RIFS}}(u'), \mathcal{F}^l_{A_{RIFS}}(v'), \mathcal{F}^l_{B_{RIFS}}(v')\right) \quad (3.12)$$

let

$$\mathcal{F}^l_{A_{RIFS}}(w') \leq \mathcal{F}^l_{B_{RIFS}}(w')$$

in (3.12) so that

$$\mathcal{F}^l_{H_{RIFS}}(w') = \mathcal{F}^l_{A_{RIFS}}(w')$$

as A is ortho-concave refined intuitionistic fuzzy set so

$$\mathcal{F}^l_{H_{RIFS}}(w') = \mathcal{F}^l_{A_{RIFS}}(w') \geq \min\left(\mathcal{F}^l_{A_{RIFS}}(u'), \mathcal{F}^l_{A_{RIFS}}(v')\right)$$

$$\mathcal{F}^{l}_{H_{RIFS}}\left(w'\right) \geq \min\left(\mathcal{F}^{l}_{A_{RIFS}}\left(u'\right), \mathcal{F}^{l}_{B_{RIFS}}\left(u'\right), \mathcal{F}^{l}_{A_{RIFS}}\left(v'\right) \mathcal{F}^{l}_{B_{RIFS}}\left(v'\right)\right)$$

i.e.

$$\mathcal{F}^{l}_{H_{RIFS}}\left(w'\right) \geq \min\left(\mathcal{F}^{l}_{H_{RIFS}}\left(u'\right) \mathcal{F}^{l}_{H_{RIFS}}\left(v'\right)\right)$$

similarly for $\mathcal{F}^{l}_{B_{RIFS}}\left(w'\right) \leq \mathcal{F}^{l}_{A_{RIFS}}\left(w'\right)$,in equation (3.12) so that

$$\mathcal{F}^{l}_{H_{RIFS}}\left(w'\right) = \mathcal{F}^{l}_{B_{RIFS}}\left(w'\right)$$

as B is ortho-concave refined intuitionistic fuzzy set so equation (3.12) becomes

$$\mathcal{F}^{l}_{H_{RIFS}}\left(w'\right) = \mathcal{F}^{l}_{B_{RIFS}}\left(w'\right) \geq \min\left(\mathcal{F}^{l}_{B_{RIFS}}\left(u'\right), \mathcal{F}^{l}_{A_{RIFS}}\left(v'\right)\right)$$

$$\mathcal{F}^{l}_{H_{RIFS}}\left(w'\right) \geq \min\left(\mathcal{F}^{l}_{A_{RIFS}}\left(u'\right), \mathcal{F}^{l}_{B_{RIFS}}\left(u'\right), \mathcal{F}^{l}_{A_{RIFS}}\left(v'\right) \mathcal{F}^{l}_{B_{RIFS}}\left(v'\right)\right)$$

i.e.

$$\mathcal{F}^{l}_{H_{RIFS}}\left(w'\right) \geq \min\left(\mathcal{F}^{l}_{H_{RIFS}}\left(u'\right) \mathcal{F}^{l}_{H_{RIFS}}\left(v'\right)\right)$$

since every ortho-concave refined intuitionistic fuzzy set is also concave refined intuitionistic fuzzy set which leads to completion of proof. □

Remark 3.17. The union of family of concave refined intuitionistic fuzzy sets is a concave refined intuitionistic fuzzy set.

Remark 3.18. The union of family of ortho-concave refined intuitionistic fuzzy sets is ortho-concave refined intuitionistic fuzzy and hence concave refined intuitionistic fuzzy set.

Definition 3.19. For any point $p \in L$ where L is a line, L_p is perpendicular to L at A, the *inf projection* A_L of concave refined intuitionistic fuzzy set A in R^2 is mapping of each point $p \in L$ into $\inf\{A(r), r \in L_p\}$.

Definition 3.20. For any point $p \in L$ where L is a line, L_p is perpendicular to L at A, the *sup projection* A_L of concave refined intuitionistic fuzzy set A in R^2 is mapping of each point $p \in L$ into $\sup\{A(r), r \in L_p\}$.

Theorem 3.21. *If A is concave RIFS, so is A_L.*

Proof. If u, v, w are three points of L such that w lies on $\overline{uv}$, given any $\epsilon > 0$, let u' and v' be points on L_u and L_v so that $\Gamma^k_{A_L}(u) > \Gamma^k_A(u') - \epsilon$ and $\Gamma^k_{A_K}(v) > \Gamma^k_A(v') - \epsilon$. Let w' be the intersection of line segment $\overline{u'v'}$ with L_w. Since A is concave and $w' \in \overline{u'v'}$, then we have

$$\Gamma^k_A\left(w'\right) \leq \max\left(\Gamma^k_A\left(u'\right), \Gamma^k_A\left(v'\right)\right), 1 \leq k \leq p,$$

$$< \max\left(\Gamma^k_{A_L}\left(u\right) + \epsilon, \Gamma^k_{A_L}\left(v\right) + \epsilon\right)$$

$$= \max\left(\Gamma^k_{A_L}\left(u\right), \Gamma^k_{A_L}\left(v\right)\right) + \epsilon$$

but by definition of inf projection

$$\Gamma^k_A\left(w'\right) \geq \Gamma^k_{A_L}\left(w\right)$$

hence

$$\Gamma^k_{A_L}\left(w\right) < \max\left(\Gamma^k_{A_L}\left(u\right), \Gamma^k_{A_L}\left(v\right)\right) + \epsilon$$

since $\epsilon > 0$ is arbitrary, we have

$$\Gamma^k_{A_L}\left(w\right) \leq \max\left(\Gamma^k_{A_L}\left(u\right), \Gamma^k_{A_L}\left(v\right)\right)$$

Similarly
If u, v, w are three points of L such that w lies on $\overline{uv}$, given any $\epsilon > 0$, let u' and v' be points on L_u and L_v so that $\mathcal{F}^k_{A_L}(u) < \mathcal{F}^k_A(u') - \epsilon$ and $\mathcal{F}^k_{A_L}(v) < \mathcal{F}^k_A(v') - \epsilon$. Let w' be the intersection of line segment $\overline{u'v'}$ with L_w. Since A is concave and $w' \in \overline{u'v'}$, then we have

$$\mathcal{F}^k_A(w') \geq \min\left(\mathcal{F}^k_A(u'), \mathcal{F}^k_A(v')\right), 1 \leq k \leq n,$$

$$> \min\left(\mathcal{F}^k_{A_L}(u) + \epsilon, \mathcal{F}^k_{A_L}(v) + \epsilon\right)$$

$$= \min\left(\mathcal{F}^k_{A_L}(u), \mathcal{F}^k_{A_L}(v)\right) + \epsilon$$

but by definition of inf projection

$$\mathcal{F}^k_A(w') \leq \mathcal{F}^k_{A_L}(w)$$

hence

$$\mathcal{F}^k_{A_L}(w) > \min\left(\mathcal{F}^k_{A_L}(u), \mathcal{F}^k_{A_L}(v)\right) + \epsilon$$

since $\epsilon > 0$ is arbitrary, we have

$$\mathcal{F}^k_{A_L}(w) \geq \min\left(\mathcal{F}^k_{A_L}(u), \mathcal{F}^k_{A_L}(v)\right)$$

so A_L is concave. □

Remark 3.22. If A is convex RIFS, so is A_L.

4. Conclusion

In this paper, convexity cum concavity is defined on refined intuitionistic fuzzy set and some of useful results are established. This work can further be extended by developing certain variants of convexity like s-convex, (s,m)-convex, strongly-convexity, strictly-convexity, concavo convex, graded convexity, triangular convexity etc., under refined intuitionistic fuzzy set environment.

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