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Partition dimension of generalized Peterson and Harary graphs

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Abstract

The distance of a connected, simple graph $\mathbb{P}$ is denoted by $d(\alpha_1, \alpha_2)$, which is the length of a shortest path between the vertices $\alpha_1, \alpha_2 \in V(\mathbb{P})$, where $V(\mathbb{P})$ is the vertex set of $\mathbb{P}$. The l-ordered partition of $V(\mathbb{P})$ is $K = \{K_1, K_2, \ldots, K_l\}$. A vertex $\alpha \in V(\mathbb{P})$, and $r(\alpha|K) = \{d(\alpha, K_1), d(\alpha, K_2), \ldots, d(\alpha, K_l)\}$ be a l-tuple distances, where $r(\alpha|K)$ is the representation of a vertex α with respect to set K. If $r(\alpha|K)$ of α is unique, for every pair of vertices, then K is the resolving partition set of $V(\mathbb{P})$. The minimum number l in the resolving partition set K is known as partition dimension ($pd(\mathbb{P})$). In this paper, we studied the generalized families of Peterson graph, $P_{\lambda,\chi}$ and proved that these families have bounded partition dimension.

Keywords: Generalized Peterson graph, Harary Graph, partition dimension, partition resolving set, sharp bounds of partition dimension.
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1. Introduction

Let $\mathbb{G} = (\mathbb{P}_k)_{k\geq 1}$ be a family with connected graphs $\mathbb{P}_k$ depending on the number of graphs in the family which is denoted by k as follows: $|V(\mathbb{P}_k)| = O(k)$ is the order with condition $\lim_{k\to\infty} O(k) = \infty$. If $\exists$ a constant number $\alpha \geq 0$ such that $pd(\mathbb{P}_k) \leq \alpha$ for each possible value $k \geq 1$, then we can deduce the result that $\mathbb{G}$ has bounded partition dimension and the upper bound is the constant number α, otherwise the $\mathbb{G}$ does

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not have bounded partition dimension. If every graph in the family $\mathbb{G}$ have the equal partition dimension (which mean that the partition dimension do not depends on the number k), $\mathbb{G}$ is said to be a family with constant partition dimension [16].

In 1975 the idea delivered by Slater had a background in networking, usually referred to as locating set or beacons set. The entire network or a graph is controlled by specifically chosen vertices from the vertex set in this concept. These vertices have to choose with a specific condition that each vertex of a graph has a unique position in terms of representations, we refer to the Definition 1.2 for this concept. Later Melter and Harary rename this concept as resolving set [12]. In the graph's theoretical study, this concept is called a metric basis or basis set of a graph. The count of vertices in a resolving set or metric basis is referred as the metric dimension of a graph [27]. Instead of choosing particular nodes into a subset with the defined condition, it is possible to arrange the entire vertex set into subsets keeping the defined condition of $r(\gamma|F)$, which is actually came from the idea of the unique position of each vertex in a graph. This concept is called as the partition resolving set, and the least number of subsets is called the partition dimension, introduced by Chartrand et al. in 2000 [5]. To better understand this concept, we refer to the mathematical Definition 1.3 and 1.4 and for the latest ideas related to this concept, see [10].

Representing a graph with each of its vertex has unique position is falling in different real-world applications, such as for the strategies, coding, and decoding of mastermind games brief in [9], the popular relation which is named as Djokovic-Winkler linked to this concept [4], the piloting or the guidance of a robot also associated with this unique idea [17], the procedure of verifying and discovering a network related to this concept [3]. There are many applications to explore those, we refer to see [12, 21]. Finding of a resolving set is NP-hard problem [13, 18] and the partition resolving set is the generalization of resolving set it also falls in the category of NP-hard [5].

The concept of resolving partition set and partition dimension extensively appeared in the literature. For example, the graph with partition dimension $|V| - 3$ discussed [2], the graph obtained by few graph operations and its corresponding partition dimension studied in [28], bounds on the partition dimension for convex polytopes in [6, 8, 14], bounds of partition on the circulant and multipartite discussed in [11, 19], chemical structure partitioning discussed in [20], on the bounded partition dimension of the Cartesian product of graphs are studied in [30], [1] gave bounds for the subdivision of different graphs, [25] provide the bounds on tree graph, [26] discussed bounds of unicyclic graphs in the form of subgraphs. For the resolving set and metric dimension of Peterson and generalized Peterson graph, we refer to the articles [23, 15]. For more recent literature and results, we refer to see [25, 22, 24].

Following are basic mathematical definitions of the concepts used in this research work.

Definition 1.1. Suppose P be an undirected, simple graph with the set of vertices named as $V(P)$ and edge set $E(P)$, the distance which also known as geodesics, between $\alpha_1, \alpha_2 \in V(P)$ two vertices is the count of minimum edges between $\alpha_1 - \alpha_2$ path. It is denoted by $d(\alpha_1, \alpha_2)$.

Definition 1.2. Suppose an ordered set of vertices from $V(P)$ labeled as $R = \{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ and $\alpha \in V(P)$. The representations $r(\alpha|R)$ of α-vertex with respect to an ordered subset R is the $s-$tuple distances $(d(\alpha, \alpha_1), d(\alpha, \alpha_2), \ldots, d(\alpha, \alpha_s))$. If each vertex from $V(P)$ have unique representations according to R, then R is called a resolving set of graph P, and minimum count of the elements in R is called the metric dimension of graph P and it is represented by $dim(P)$.

Definition 1.3. Let χ is the s-ordered partition set and $r(\alpha|\chi) = \{d(\alpha, \chi_1),\ d(\alpha, \chi_2),\ \ldots,\ d(\alpha, \chi_s)\}$, is the s-tuple distance representations of a vertex α with respect to χ. If the representations of α with respect to χ are unique, then B is the partition resolving set of the vertex set of a graph P.

Definition 1.4. The minimum count of subsets in the partition resolving set of $V(P)$ is defined as the partition dimension $(pd(P))$ of P.

Following theorems are very helpful in finding the partition dimension of a graph.

Theorem 1.5. [5] *Let χ be a partition resolving set of $V(P)$ and $\alpha_1, \alpha_2 \in V(P)$. If $d(\alpha_1, \alpha) = d(\alpha_2, \alpha)$ for all vertices $\alpha \in V(P)\backslash(\alpha_1, \alpha_2)$, then α_1, α_2 belongs to different subsets of χ.*

$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4	$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4
a_0	1	4	3	0	a_1	2	5	4	0
a_2	3	5	5	0	a_3	4	4	5	0
a_4	5	3	4	0	a_5	5	2	3	0
a_6	4	1	2	0	a_7	3	2	1	0
a_9	2	4	1	0	a_{10}	3	5	2	0
a_{11}	4	5	3	0	a_{12}	4	2	5	0
a_{13}	5	3	5	0	a_{14}	4	2	5	0
a_{15}	3	2	4	0	a_{16}	2	3	3	0

Table 1: Representations of outer vertices

$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4	$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4
b_1	2	6	3	0	b_2	4	6	4	0
b_3	5	5	5	0	b_4	6	4	5	0
b_5	6	2	4	0	b_7	3	2	2	0
b_8	1	4	1	0	b_9	1	5	2	0
b_{10}	3	6	3	0	b_{11}	5	6	4	0
b_{12}	6	5	5	0	b_{13}	6	3	5	0
b_{14}	5	1	4	0	b_{15}	4	1	3	0
b_{16}	2	3	2	0					

Table 2: Representations of inner vertices

2. Generalized Peterson Graph $P_{\alpha,\beta}$

For each odd integer $\alpha = 2\beta+1 \geq 3$, the generalized Petersen graph $P_{\alpha,\beta}$ is a graph with vertex set $a \cup b$ where $\{a_\eta,\ b_\eta:\ \eta = 0, 1, \ldots, \alpha-1\}$ and edge set $E = \{a_\eta a_{\eta+1},\ b_\eta b_{\eta+\beta},\ a_\eta b_\eta:\ \eta = 0, 1, \ldots, \alpha-1\}$. For our purpose, we call the vertices $a_0,\ a_1, \ldots, a_{\alpha-1}$ outer vertices and $b_0,\ b_1, \ldots, b_{\alpha-1}$ inner vertices. Here and throughout the paper, the subscripts are to be taken as integers modulo α. Following are some bounds on the partition dimension of generalized Peterson graph $P_{\alpha,\beta}$.

Theorem 2.1. *Let $P_{\alpha,\beta}$ be generalized Petersen graphs with $\alpha = 2\beta+1$ and $\beta \equiv 0(mod\ 4)$, then $pd\,(P_{\alpha,\beta}) \leq 4$.*

Proof. For $\beta = 4$, it is easy to see that $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ where $\chi_1 = \{b_0\}$, $\chi_2 = \{b_4\}$, $\chi_3 = \{a_5\}$, $\chi_4 = V\,(P_{\alpha,\beta}) \setminus \{b_0, b_4, a_5\}$ is a resolving partitioning for $V\,(P_{9,4})$.

For $\beta \geq 8$, $\beta \equiv 0(mod\ 4)$ and for the chosen index η such that $0 \leq \eta \leq \alpha-1$, we shall show that $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ where $\chi_1 = \{b_\eta\}$, $\chi_2 = \{b_{\eta+2\xi+2}\}$, $\chi_3 = \{a_{\eta+\beta}\}$, $\chi_4 = V\,(P_{\alpha,\beta}) \setminus \{b_\eta, b_{\eta+2\xi+2}, a_{\eta+\beta}\}$ is a resolving partitioning for $V\,(P_{\alpha,\beta})$, where $\xi = \frac{\beta}{4}$.

For $\beta = 8$, the representations of the vertices of $V\,(P_{17,8})$, are in the Tables 1 and 2.

It can be seen that all the vertices in $V\,(P_{17,8})$, have distinct representations with respect to χ. Now for $\beta \geq 8$ the representations of the vertices of $V\,(P_{\alpha,\beta})$, are: $r\,(b_{\eta+1}|\chi) = (2,\ 2\xi+2,\ 3,\ 0)$, $r\,(b_{\eta+2\beta}|\chi) = (2,\ 2\xi,\ 2,\ 0)$, and remaining in Tables 3 and 4.

From these tables, one can see that all the vertices of $P_{\alpha,\beta}$ lying in column 1 of Table 3 and Table 4 have distinct representations with respect to χ. Thus χ is resolving partitioning for $P_{\alpha,\beta}$. Hence, for all $\alpha = 2\beta+1$, $\beta \geq 8$ and $\beta \equiv 0(mod\ 4)$, $pd\,(P_{\alpha,\beta}) \leq 4$. □

Theorem 2.2. *Let $P_{\alpha,\beta}$ be generalized Petersen graphs with $\alpha = 2\beta+1 \geq 5$ and $\beta \equiv 1(mod\ 4)$, then $pd\,(P_{\alpha,\beta}) \leq 4$.*

$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4	
$a_{\eta+\zeta}$	$\zeta+1$	$2\xi+\zeta$	$\zeta+3$	0	$0\le\zeta\le1$
$a_{\eta+\zeta+2}$	$\zeta+3$	$2\xi-\zeta+1$	$\zeta+5$	0	$0\le\zeta\le2\xi-4$
$a_{\eta+2\xi+\zeta-1}$	$2\xi+\zeta$	$4-\zeta$	$2\xi-\zeta+1$	0	$0\le\zeta\le1$
$a_{\eta+2\xi+\zeta+1}$	$2\xi-\zeta+1$	$2-\zeta$	$2\xi-\zeta-1$	0	$0\le\zeta\le1$
$a_{\eta+\beta-\zeta-1}$	$\zeta+3$	$2\xi-\zeta-2$	$\zeta+1$	0	$0\le\zeta\le2\xi-4$
$a_{\eta+\beta+\zeta+1}$	$\zeta+2$	$2\xi+\zeta$	$\zeta+1$	0	$0\le\zeta\le1$
$a_{\eta+\beta+\zeta+3}$	$\zeta+4$	$2\xi-\zeta+1$	$\zeta+3$	0	$0\le\zeta\le2\xi-3$
$a_{\eta+2\xi+\zeta+\beta+1}$	$2\xi-\zeta+1$	$3-\zeta$	$2\xi+1$	0	$0\le\zeta\le1$
$a_{\eta+2\beta-\zeta}$	$\zeta+2$	$2\xi-\zeta-1$	$\zeta+3$	0	$0\le\zeta\le2\xi-3$

Table 3: Representations of outer vertices

$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4	
$b_{\eta+\zeta+2}$	$\zeta+4$	$2\xi-\zeta+2$	$\zeta+4$	0	$0\le\zeta\le2\xi-3$
$b_{\eta+\zeta+2\xi}$	$2\xi+2$	$4-2\zeta$	$2\xi-\zeta+1$	0	$0\le\zeta\le1$
$b_{\eta+2\xi+\zeta+3}$	$2\xi-\zeta$	$2\zeta+2$	$2\xi-\zeta-2$	0	$0\le\zeta\le1$
$b_{\eta+2\xi+\zeta+5}$	$2\xi-\zeta-2$	$\zeta+5$	$2\xi-\zeta-4$	0	$0\le\zeta\le2\xi-7$
$b_{\eta+\beta-\zeta}$	$2\zeta+1$	$2\xi+\zeta+1$	$\zeta+2$	0	$0\le\zeta\le1$
$b_{\eta+\beta+\zeta+1}$	$2\zeta+1$	$2\xi+\zeta+1$	$\zeta+2$	0	$0\le\zeta\le1$
$b_{\eta+\beta+\zeta+3}$	$\zeta+5$	$2\xi-\zeta+2$	$\zeta+4$	0	$0\le\zeta\le2\xi-3$
$b_{\eta+2\xi+\zeta+\beta+1}$	$2\xi-\zeta+2$	$3-2\zeta$	$2\xi-\zeta+1$	0	$0\le\zeta\le1$
$b_{\eta+2\xi+\zeta+\beta+3}$	$2\xi-\zeta$	$2\zeta+1$	$2\xi-\zeta-1$	0	$0\le\zeta\le1$
$b_{\eta+2\beta-\zeta-1}$	$\zeta+4$	$2\xi-\zeta-1$	$\zeta+3$	0	$0\le\zeta\le2\xi-6$

Table 4: Representations of inner vertices

$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4	
$a_{\eta+\zeta}$	$\zeta+3$	$2\xi+\zeta+1$	$\zeta+3$	0	$0\leq\zeta\leq 1$
$a_{\eta+\zeta+2}$	$\zeta+1$	$2\xi-\zeta+1$	$\zeta+5$	0	$0\leq\zeta\leq 2\xi-3$
$a_{\eta+2\xi+\zeta}$	$2\xi+\zeta-1$	$3-\zeta$	$2\xi-\zeta+1$	0	$0\leq\zeta\leq 2$
$a_{\eta+\beta-\zeta-1}$	$\zeta+5$	$2\xi-\zeta-1$	$\zeta+1$	0	$0\leq\zeta\leq 2\xi-3$
$a_{\eta+\beta+\zeta+1}$	$3-\zeta$	$2\xi+\zeta+1$	$\zeta+1$	0	$0\leq\zeta\leq 1$
$a_{\eta+\beta+\zeta+3}$	$\zeta+2$	$2\xi-\zeta+1$	$\zeta+3$	0	$0\leq\zeta\leq 2\xi-1$
$a_{\eta+2\beta-\zeta}$	$\zeta+4$	$2\xi-\zeta$	$\zeta+3$	0	$0\leq\zeta\leq 2\xi-2$

Table 5: Representations of outer vertices

$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4	
$b_{\eta+\zeta}$	$4-2\zeta$	$2\xi+\zeta+2$	$\zeta+2$	0	$0\leq\zeta\leq 1$
$b_{\eta+\zeta+4}$	$\zeta+4$	$2\xi-\zeta$	$\zeta+6$	0	$0\leq\zeta\leq 2\xi-4$
$b_{\eta-\beta+\zeta}$	$\zeta+5$	$2\xi-\zeta+1$	$\zeta+1$	0	$0\leq\zeta\leq 2\xi-3$
$b_{\eta+\beta+\zeta+1}$	$3-2\zeta$	$2\xi+\zeta+2$	$\zeta+2$	0	$0\leq\zeta\leq 1$
$b_{\eta+\beta+\zeta+3}$	$2\zeta+1$	$2\xi-\zeta+2$	$\zeta+4$	0	$0\leq\zeta\leq 1$
$b_{\eta+\beta+\zeta+5}$	$\zeta+5$	$2\xi-\zeta$	$\zeta+6$	0	$0\leq\zeta\leq 2\xi-5$
$b_{\eta+\beta+2\xi+\zeta+1}$	$2\xi+\zeta+1$	$3-2\zeta$	$2\xi-\zeta+2$	0	$0\leq\zeta\leq 1$
$b_{\eta+\beta+2\xi+\zeta+3}$	$2\xi-\zeta+3$	$2\zeta+1$	$2\xi-\zeta$	0	$0\leq\zeta\leq 1$
$b_{\eta+2\beta-\zeta}$	$\zeta+5$	$2\xi+\zeta+1$	$\zeta+2$	0	$0\leq\zeta\leq 2\xi-4$

Table 6: Representations of inner vertices

Proof. For $\beta=5$, it is easy to see that $\chi=\{\chi_1,\chi_2,\chi_3,\chi_4\}$ where $\chi_1=\{b_2\}$, $\chi_2=\{b_4\}$, $\chi_3=\{a_5\}$, $\chi_4=V\left(P_{\alpha,\beta}\right)\backslash\{b_2,b_4,a_5\}$ is a resolving partitioning for $V\left(P_{11,5}\right)$.

For $\beta\geq 5$, $\beta\equiv 1(mod\ 4)$ and for the chosen index η such that $0\leq\eta\leq\alpha-1$, we shall show that $\chi=\{\chi_1,\chi_2,\chi_3,\chi_4\}$, where $\chi_1=\{b_{\eta+2}\}$, $\chi_2=\{b_{\eta+2\xi+2}\}$, $\chi_3=\{a_{\eta+\beta}\}$, $\chi_4=V\left(P_{\alpha,\beta}\right)\backslash\{b_{\eta+2},b_{\eta+2\xi+2},a_{\eta+\beta}\}$ is a resolving partitioning for $V\left(P_{\alpha,\beta}\right)$.

The representations of $V\left(P_{\alpha,\beta}\right)$, are: $r\left(b_{\eta+3}|\chi\right)=(2,\ 2\xi+1,\ 5,\ 0)$, $r\left(b_{\eta+2\xi+1}|\chi\right)=(2\xi+1,\ 2,\ 2\xi+1,\ 0)$, $r\left(b_{\eta+2\xi+3}|\chi\right)=(2\xi+3,\ 2,\ 2\xi-1,\ 0)$ and in the Tables 5 and 6.

From these tables, one can see that all the vertices of $P_{\alpha,\beta}$ lying in column 1 of Table 5 and Table 6 have distinct representations with respect to χ. Thus χ is resolving partitioning for $P_{\alpha,\beta}$. Hence, for all $\alpha=2\beta+1$, $\beta\geq 5$ and $\beta\equiv 1(mod\ 4)$, $pd\left(P_{\alpha,\beta}\right)\leq 4$. □

Theorem 2.3. *Let $P_{\alpha,\beta}$ be generalized Petersen graphs with $\alpha=2\beta+1\geq 2$ and $\beta\equiv 2(mod\ 4)$, then $pd\left(P_{\alpha,\beta}\right)\leq 4$.*

Proof. For $\beta=6$, it is easy to see from the Tables 7 and 7 that $\chi=\{\chi_1,\chi_2,\chi_3,\chi_4\}$ where $\chi_1=\{b_3\}$, $\chi_2=\{b_4\}$, $\chi_3=\{a_7\}$, $\chi_4=V\left(P_{\alpha,\beta}\right)\backslash\{b_3,b_4,a_7\}$ is a resolving partitioning for $V\left(P_{13,6}\right)$.

For $\beta=10$, it is easy to see from the Tables 9 and 10 that $\chi=\{\chi_1,\chi_2,\chi_3,\chi_4\}$ where $\chi_1=\{b_3\}$, $\chi_2=\{b_6\}$, $\chi_3=\{a_{11}\}$, $\chi_4=V\left(P_{\alpha,\beta}\right)\backslash\{b_3,b_6,a_{11}\}$ is a resolving partitioning for $V\left(P_{21,10}\right)$.

For $\beta\geq 10$, $\beta\equiv 2(mod\ 4)$ and for the chosen index η such that $0\leq\eta\leq\alpha-1$, we shall show that $\chi=\{\chi_1,\chi_2,\chi_3,\chi_4\}$ where $\chi_1=\{b_{\eta+3}\}$, $\chi_2=\{b_{\eta+2\xi+2}\}$, $\chi_3=\{a_{\eta+\beta+1}\}$, $\chi_4=V\left(P_{\alpha,\beta}\right)\backslash\{b_{\eta+3},b_{\eta+2\xi+2},a_{\eta+\beta+1}\}$ is a resolving partitioning for $V\left(P_{\alpha,\beta}\right)$, where $\xi=\frac{\beta-2}{4}$.

For $\beta\geq 11$, $\beta\equiv 2(mod\ 4)$ the representations of the vertices of $V\left(P_{\alpha,\beta}\right)$, are: $r\left(a_\eta|\chi\right)=(4,\ 2\xi+2,\ 3,\ 0)$, $r\left(b_\eta|\chi\right)=(5,\ 2\xi+3,\ 2,\ 0)$, $r\left(b_{\eta+4}|\chi\right)=(2,\ 2\xi,\ 5,\ 0)$ $r\left(b_{\eta+2\xi+1}|\chi\right)=(2\xi,\ 2,\ 2\xi+2,\ 0)$ $r\left(b_{\eta+2\xi+3}|\chi\right)=(2\xi+2,\ 2,\ 2\xi+1,\ 0)$ and in the Tables 11 and 12.

From these tables, one can see that all the vertices of $P_{\alpha,\beta}$ lying in column 1 of Table 1 and Table 12 have distinct representations with respect to χ. Thus χ is resolving partitioning for $P_{\alpha,\beta}$. Hence, for all

$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4	$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4	$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4
a_0	4	4	3	0	a_1	3	4	3	0	a_2	2	3	4	0
a_3	1	2	4	0	a_4	2	1	3	0	a_5	3	2	2	0
a_6	4	3	1	0	a_8	3	4	1	0	a_9	2	3	2	0
a_{10}	2	2	3	0	a_{11}	3	2	4	0	a_{12}	4	3	4	0

Table 7: Representations of outer vertices

$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4	$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4	$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4
b_0	5	5	2	0	b_1	4	5	2	0	b_2	2	4	3	0
b_5	4	2	3	0	b_6	5	4	2	0	b_7	5	5	1	0
b_8	3	5	2	0	b_9	1	3	3	0	b_{10}	1	1	4	0
b_{11}	3	1	4	0	b_{12}	5	3	3	0					

Table 8: Representations of inner vertices

$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4	$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4
a_0	4	6	3	0	a_1	3	6	3	0
a_2	2	5	4	0	a_3	1	4	5	0
a_4	2	3	6	0	a_5	3	2	6	0
a_6	4	1	5	0	a_7	5	2	4	0
a_8	6	3	3	0	a_9	6	4	2	0
a_{10}	5	5	1	0	a_{12}	3	6	1	0
a_{13}	2	5	2	0	a_{14}	2	4	3	0
a_{15}	3	3	4	0	a_{16}	4	2	5	0
a_{17}	5	2	6	0	a_{18}	6	3	6	0
a_{19}	6	4	5	0	a_{20}	5	5	4	0

Table 9: Representations of outer vertices

$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4	$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4
b_0	5	7	2	0	b_1	4	7	2	0
b_2	2	6	3	0	b_4	2	4	5	0
b_5	4	2	6	0	b_7	6	2	5	0
b_8	7	4	4	0	b_9	7	5	3	0
b_{10}	6	6	2	0	b_{11}	5	7	1	0
b_{12}	3	7	2	0	b_{13}	1	6	3	0
b_{14}	1	5	4	0	b_{15}	3	3	5	0
b_{16}	5	1	6	0	b_{17}	6	1	6	0
b_{18}	7	3	5	0	b_{19}	7	5	4	0
b_{20}	6	6	3	0					

Table 10: Representations of inner vertices

$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4	
$a_{\eta+\zeta+1}$	$\zeta+3$	$2\xi-\zeta+2$	$\zeta+3$	0	$0\leq\zeta\leq 1$
$a_{\eta+\zeta+3}$	$\zeta+1$	$2\xi-\zeta$	$\zeta+5$	0	$0\leq\zeta\leq 2\xi-3$
$a_{\eta+2\xi+\zeta+1}$	$2\xi+\zeta-1$	$2-\zeta$	$2\xi-\zeta+2$	0	$0\leq\zeta\leq 1$
$a_{\eta+2\xi+\zeta+3}$	$2\xi+\zeta+1$	$\zeta+2$	$2\xi-\zeta$	0	$0\leq\zeta\leq 1$
$a_{\eta+\beta-\zeta}$	$\zeta+5$	$2\xi-\zeta+1$	$\zeta+1$	0	$0\leq\zeta\leq 2\xi-3$
$a_{\eta+\beta+\zeta+2}$	$3-\zeta$	$2\xi-\zeta+2$	$\zeta+1$	0	$0\leq\zeta\leq 1$
$a_{\eta+\beta+\zeta+4}$	$\zeta+2$	$2\xi-\zeta$	$\zeta+3$	0	$0\leq\zeta\leq 2\xi-2$
$a_{\eta+\beta+\zeta+2\xi+3}$	$2\xi+\zeta+1$	$\zeta+2$	$2\xi+2$	0	$0\leq\zeta\leq 1$
$a_{\eta+2\beta-\zeta}$	$\zeta+5$	$2\xi-\zeta+1$	$\zeta+4$	0	$0\leq\zeta\leq 2\xi-3$

Table 11: Representations of outer vertices

$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4	
$b_{\eta+\zeta+1}$	$4-2\zeta$	$2\xi-\zeta+3$	$\zeta+2$	0	$0\leq\zeta\leq 1$
$b_{\eta+\zeta+5}$	$\zeta+4$	$2\xi-\zeta-1$	$\zeta+6$	0	$0\leq\zeta\leq 2\xi-5$
$b_{\eta+2\xi+\zeta+4}$	$2\xi+3$	$\zeta+4$	$2\xi-\zeta$	0	$0\leq\zeta\leq 1$
$b_{\eta+\beta-\zeta+1}$	$\zeta+5$	$2\xi-\zeta+3$	$\zeta+1$	0	$0\leq\zeta\leq 2\xi-3$
$b_{\eta+\beta+\zeta+2}$	$3-2\zeta$	$2\xi-\zeta+3$	$\zeta+2$	0	$0\leq\zeta\leq 1$
$b_{\eta+\beta+\zeta+4}$	$2\zeta+15$	$2\xi-\zeta+1$	$\zeta+4$	0	$0\leq\zeta\leq 1$
$b_{\eta+\beta+6}$	$\zeta+5$	$2\xi-\zeta-1$	$\zeta+6$	0	$0\leq\zeta\leq 2\xi-6$
$b_{\eta+\beta+\zeta+1+2\xi}$	$2\xi+\zeta$	$3-2\zeta$	$2\xi+\zeta+1$	0	$0\leq\zeta\leq 1$
$b_{\eta+\beta+\zeta+3+2\xi}$	$2\xi+\zeta+2$	$2\zeta+1$	$2\xi-\zeta+2$	0	$0\leq\zeta\leq 1$
$b_{\eta+2\beta-\zeta}$	$\zeta+6$	$2\xi-\zeta+2$	$2\xi-\zeta+3$	0	$0\leq\zeta\leq 2\xi-3$

Table 12: Representations of inner vertices

$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4	$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4	$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4
a_0	2	5	4	0	a_1	3	4	5	0	a_2	4	3	4	0
a_3	5	2	3	0	a_4	4	1	2	0	a_5	3	2	1	0
a_7	2	4	1	0	a_8	3	5	2	0	a_9	4	1	3	0
a_{10}	5	3	4	0	a_{11}	4	2	5	0	a_{12}	3	2	4	0
a_{13}	2	3	3	0	a_{14}	1	4	3	0					

Table 13: Representations of outer vertices

$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4	$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4	$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4
b_0	2	6	3	0	b_1	4	5	4	0	b_2	5	4	5	0
b_3	6	2	4	0	b_5	3	2	2	0	b_6	1	4	1	0
b_7	1	5	2	0	b_8	3	6	3	0	b_9	5	5	4	0
b_{10}	6	3	5	0	b_{11}	5	1	4	0	b_{12}	4	1	3	0
b_{13}	2	3	4	0										

Table 14: Representations of inner vertices

$\alpha = 2\beta + 1$, $\beta \geq 2$ and $\beta \equiv 2(mod\ 4)$, $pd\,(P_{\alpha,\beta}) \leq 4$. □

Theorem 2.4. *Let $P_{\alpha,\beta}$ be generalized Petersen graphs with $\alpha = 2\beta + 1 \geq 3$ and $\beta \equiv 3(mod\ 4)$, then $pd\,(P_{\alpha,\beta}) \leq 4$.*

Proof. For $\beta = 6$, it is easy to see that $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ where $\chi_1 = \{b_2\}$, $\chi_2 = \{b_6\}$, $\chi_3 = \{a_2\}$, $\chi_4 = V\,(P_{\alpha,\beta}) \setminus \{b_2, b_6, a_2\}$ is a resolving partitioning for $V\,(P_{7,3})$.

For $\beta \geq 7$, $\beta \equiv 3(mod\ 4)$ and for the chosen index η such that $0 \leq \eta \leq \alpha - 1$, we shall show that $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ where $\chi_1 = \{b_{\eta+2\beta}\}$, $\chi_2 = \{b_{\eta+2\xi+2}\}$, $\chi_3 = \{a_{\eta+\beta-1}\}$, $\chi_4 = V\,(P_{\alpha,\beta}) \setminus \{b_{\eta+2\beta}, b_{\eta+2\xi+2}, a_{\eta+\beta-1}\}$ is a resolving partitioning for $V\,(P_{\alpha,\beta})$, where $\xi = \frac{\beta-3}{4}$.

For $\beta = 7$ the representations of the vertices of $V\,(P_{15,7})$, are in the Tables 13 and 14. For $\beta > 7$, $\beta \equiv 3(mod\ 4)$ the representations of the vertices of $V\,(P_{\alpha,\beta})$, are: $r\,(a_{\eta+\beta+2\xi+2}|\chi) = (2\xi + 2,\ 2,\ 2\xi + 3,\ 0)$, $r\,(a_{\eta+2\beta}|\chi) = (1,\ 2\xi + 2,\ 3,\ 0)$, $r\,(b_\eta|\chi) = (2,\ 2\xi + 4,\ 3,\ 0)$ $r\,(b_{\eta+2\xi+1}|\chi) = (2\xi + 4,\ 2,\ 2\xi + 2,\ 0)$ $r\,(b_{\eta+2\xi+3}|\chi) = (2\xi + 2,\ 2,\ 2\xi,\ 0)$, $r\,(b_{\eta+2\beta-1}|\chi) = (2,\ 2\xi + 2,\ 2,\ 0)$ and in the Tables 15 and 16.

From these tables, one can see that all the vertices of $P_{\alpha,\beta}$ lying in column 1 of Table 15 and Table 16 have distinct representations with respect to χ. Thus χ is resolving partitioning for $P_{\alpha,\beta}$. Hence, for all $\alpha = 2\beta + 1$, $\beta \geq 7$ and $\beta \equiv 3(mod\ 4)$, $pd\,(P_{\alpha,\beta}) \leq 4$. □

$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4	
$a_{\eta+\zeta}$	$\zeta + 2$	$2\xi - \zeta + 3$	$\zeta + 4$	0	$0 \leq \zeta \leq 2\xi - 1$
$a_{\eta+2\xi+\zeta}$	$2\xi + \zeta + 2$	$3 - \zeta$	$2\xi - \zeta + 4$	0	$0 \leq \zeta \leq 1$
$a_{\eta+2\xi+\zeta+2}$	$2\xi - \zeta + 2$	$\zeta + 1$	$2\xi - \zeta$	0	$0 \leq \zeta \leq 2\xi - 1$
$a_{\eta+\beta+\zeta}$	$\zeta + 2$	$2\xi + \zeta + 2$	$\zeta + 1$	0	$0 \leq \zeta \leq 1$
$a_{\eta+\beta+\zeta+2}$	$\zeta + 4$	$2\xi - \zeta + 2$	$\zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 1$
$a_{\eta+2\beta-\zeta-1}$	$\zeta + 2$	$2\xi - \zeta + 1$	$\zeta + 3$	0	$0 \leq \zeta \leq 2\xi - 1$

Table 15: Representations of outer vertices

$r(.\|\chi)$	χ_1	χ_2	χ_3	χ_4	
$b_{\eta+\zeta+1}$	$\zeta+1$	$2\xi-\zeta+3$	$\zeta+4$	0	$0\le\zeta\le 2\xi-1$
$b_{\eta+2\xi+\zeta+4}$	$2\xi-\zeta+1$	$\zeta+4$	$2\xi-\zeta-1$	0	$0\le\zeta\le 2\xi-4$
$b_{\eta+\beta-\zeta-1}$	$2\zeta+1$	$2\xi-\zeta+2$	$\zeta+1$	0	$0\le\zeta\le 1$
$b_{\eta+\beta+\zeta}$	$2\zeta+1$	$2\xi+\zeta+3$	$\zeta+2$	0	$0\le\zeta\le 1$
$b_{\eta+\beta+\zeta+2}$	$\zeta+5$	$2\xi-\zeta+3$	$\zeta+4$	0	$0\le\zeta\le 2\xi-2$
$b_{\eta+2\xi+\beta+\zeta+1}$	$2\xi-\zeta+4$	$3-2\zeta$	$2\xi-\zeta+3$	0	$0\le\zeta\le 1$
$b_{\eta+2\xi+\beta+\zeta+3}$	$2\xi-\zeta+2$	$2\zeta+1$	$2\xi-\zeta+1$	0	$0\le\zeta\le 1$
$b_{\eta+2\beta-\zeta-2}$	$\zeta+4$	$2\xi-\zeta+1$	$\zeta+3$	0	$0\le\zeta\le 2\xi-4$

Table 16: Representations of inner vertices

3. Harary Graph $H_{\psi,\lambda}$

Harary $H_{\psi,\lambda}$, is an ψ-regular graph with order λ the vertex set $V(H_{\psi,\lambda})=\{\beta_\epsilon:\ \epsilon=1,2,\ldots,\lambda\}$, if ψ is even then $\psi=2\xi\le\lambda-1$ for some integer $\xi\le\frac{\lambda-1}{2}$. For each $\epsilon\,(1\le\epsilon\le\lambda)$, we join β_ϵ to $\beta_{\epsilon+1},\ \beta_{\epsilon+2},\ldots,\ \beta_{\epsilon+\xi}$ and to $\beta_{\epsilon-1},\ \beta_{\epsilon-2}.\ldots,\ \beta_{\epsilon-\xi}$. If we think of arranging of the vertices $\beta_1,\ \beta_2,\ldots,\ \beta_\lambda$ cyclically then each vertex β_ϵ is adjacent to the ξ vertices that immediately follow β_ϵ and the ξ vertices that immediately proceed β_ϵ. Following are some bounds on the partition dimension of Harary graph $H_{\psi,\lambda}$.

Theorem 3.1. *Let $H_{\psi,\lambda}$ is a Harary graph with $\lambda\ge5$, $\lambda\equiv 0,2,3(mod\ \psi)$ and $\psi=4$. Then the partition dimension of $H_{\psi,\lambda}$ is ≤ 4.*

Proof. To prove $pd\,(H_{\psi,\lambda})\le 4$ we split the proof into following cases;
Case 1: $\lambda\equiv 0(mod\ 4)$, $\lambda=4\xi$, $\xi(\ge 2)\in\mathbb{Z}^+$.

Assume the partition resolving set $\chi=\{\chi_1,\chi_2,\chi_3,\chi_4\}$ where $\chi_1=\{\beta_1\}$, $\chi_2=\{\beta_3\}$, $\chi_3=\{\beta_{\lambda-2}\}$, $\chi_4=V(H_{\psi,\lambda})\backslash\{\beta_1,\beta_3,\beta_{\lambda-2}\}$ following are the representations of the entire vertex set of $H_{\psi,\lambda}$ with respect to χ.

$$r(\beta_{2\epsilon}|\chi)=\begin{cases}(\epsilon,\ \epsilon-1,\ \epsilon-1,\ 0) & \text{if } \epsilon=2,3,\ldots,\xi;\\ (\xi,\ \xi,\ \xi,\ 0) & \text{if } \epsilon=\xi+1;\\ (2\xi-\epsilon+1,\ 2\xi-\epsilon+1,\ 2\xi-\epsilon+1,\ 0) & \text{if } \epsilon=\xi+2,\ldots,2\xi.\end{cases}$$

$$r(\beta_{2\epsilon+1}|\chi)=\begin{cases}(\epsilon,\ \epsilon,\ \epsilon-1,\ 0) & \text{if } \epsilon=2,3,\ldots,\xi;\\ (2\xi-\epsilon,\ 2\xi-\epsilon+1,\ 2\xi-\epsilon+1,\ 0) & \text{if } \epsilon=\xi+1,\ldots,2\xi-1.\end{cases}$$

Case 2: $\lambda\equiv 2(mod\ 4)$, $\lambda=4\xi+2$, $\xi\in\mathbb{Z}^+$.

Assume the partition resolving set $\chi=\{\chi_1,\chi_2,\chi_3,\chi_4\}$ where $\chi_1=\{\beta_1\}$, $\chi_2=\{\beta_2\}$, $\chi_3=\{\beta_3\}$, $\chi_4=V(H_{\psi,\lambda})\backslash\{\beta_1,\beta_2,\beta_3\}$ following are the representations of the entire vertex set of $H_{\psi,\lambda}$ with respect to χ.

$$r(\beta_{2\epsilon}|\chi)=\begin{cases}(\epsilon,\ \epsilon-1,\ \epsilon-1,\ 0) & \text{if } \epsilon=2,3,\ldots,\xi+1;\\ (2\xi-\epsilon+2,\ 2\xi-\epsilon+2,\ 2\xi-\epsilon+3,\ 0) & \text{if } \epsilon=\xi+2,\ldots,2\xi+1.\end{cases}$$

$$r(\beta_{2\epsilon+1}|\chi)=\begin{cases}(\epsilon,\ \epsilon,\ \epsilon-1,\ 0) & \text{if } \epsilon=2,3,\ldots,\xi;\\ (\xi,\ \xi+1,\ \xi,\ 0) & \text{if } \epsilon=\xi+1;\\ (2\xi-\epsilon+1,\ 2\xi-\epsilon+2,\ 2\xi-\epsilon+2,\ 0) & \text{if } \epsilon=\xi+2,\ldots,2\xi.\end{cases}$$

Case 3: $\lambda \equiv 3(mod\ 4)$, $\lambda = 4\xi + 3$, $\xi \in \mathbb{Z}^+$.

Assume the partition resolving set $\chi = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ where $\chi_1 = \{\beta_1\}$, $\chi_2 = \{\beta_2\}$, $\chi_3 = \{\beta_3\}$, $\chi_4 = V(H_{\psi,\lambda}) \setminus \{\beta_1, \beta_2, \beta_3\}$ following are the representations of the entire vertex set of $H_{\psi,\lambda}$ with respect to χ.

$$r(\beta_{2\epsilon}|\chi) = \begin{cases} (\epsilon,\ \epsilon-1,\ \epsilon-1,\ 0) & \text{if } \epsilon = 2,3,\ldots,\xi+1; \\ (2\xi-\epsilon+2,\ 2\xi-\epsilon+3,\ 2\xi-\epsilon+3,\ 0) & \text{if } \epsilon = \xi+2,\ldots,2\xi+1. \end{cases}$$

$$r(\beta_{2\epsilon+1}|\chi) = \begin{cases} (\epsilon,\ \epsilon,\ \epsilon-1,\ 0) & \text{if } \epsilon = 2,3,\ldots,\xi+1; \\ (2\xi-\epsilon+2,\ 2\xi-\epsilon+2,\ 2\xi-\epsilon+3,\ 0) & \text{if } \epsilon = \xi+2,\ldots,2\xi+1. \end{cases}$$

The entire vertex set of $H_{\psi,\lambda}$ w.r.t. to the partition resolving set χ have distinct representations hence, $pd(H_{\psi,\lambda}) \leq 4$. □

4. Conclusion

In this paper provide the sharp bounds of the partition dimension for the generalized Peterson graph $P_{\alpha,\beta}$, and we also studied the partition dimension of Harary graph $H_{\psi,\lambda}$, and we concluded that

$$pd(P_{\alpha,\beta}) = pd(H_{\psi,\lambda}) \leq 4.$$

Theorem 2.1 to 2.4 discussed the partition dimension of the generalized Peterson graph when $\beta \equiv 0,1,2,3(mod\ 4)$, respectively and the partitioning of Harary graph studied in the Theorem 3.1.

References

[1] Azmi, S., Soeprianto, H., Turmuzi, M., & Anwar, Y. S. (2019, November). The partition dimension of subdivision graph on the star. In *Journal of Physics: Conference Series* (Vol. 1280, No. 2, p. 022037). IOP Publishing.

[2] Baskoro, E. T., & Haryeni, D. O. (2020). All graphs of order $n = 11$ and diameter 2 with partition dimension $n - 3$. *Heliyon*, 6(4), e03694. 1

[3] Beerliova, Z., Eberhard, F., Erlebach, T., Hall, A., Hoffmann, M., Mihal'ak, M., & Ram, L. S. (2006). Network discovery and verification. *IEEE Journal on Selected Areas in Communications, 24*(12), 2168-2181. 1

[4] Cceres, J., Hernando, C., Mora, M., Pelayo, I. M., Puertas, M. L., Seara, C., & Wood, D. R. (2007). On the metric dimension of cartesian products of graphs. *SIAM Journal on Discrete Mathematics, 21*(2), 423-441. 1

[5] Chartrand, G., Salehi, E., & Zhang, P. (2000). The partition dimension of a graph. *Aequationes Mathematicae, 59*(1), 45-54. 1

[6] Chu, Y. M., Nadeem, M. F., Azeem, M., & Siddiqui, M. K. (2020). On sharp bounds on partition dimension of Convex polytopes. *IEEE Access, 8*, 224781-224790. 1, 1.5

[7] Coxeter, H. S. (1950). Self-dual configurations and regular graphs. *Bulletin of the American Mathematical Society, 56*(5), 413-455. 1

[8] Liu, J. B., Nadeem, M. F., & Azeem, M. (2020). Bounds on the partition dimension of convex polytopes. *Combinatorial chemistry & high throughput screening,* 10.2174/1386207323666201204144422.

[9] Chvtal, V. (1983). Mastermind. *Combinatorica, 3*(3-4), 325-329. 1

[10] Nadeem, M. F., Azeem, M., & Khalil, A. (2021). The locating number of hexagonal Mbius ladder network. *Journal of Applied Mathematics and Computing, 66*(1), 149-165. 1

[11] Grigorious, C., Stephen, S., Rajan, B., & Miller, M. (2017). On the partition dimension of circulant graphs. *The Computer Journal, 60*(2), 180-184. 1

[12] Harary, F., & Melter, R. A. (1976). On the metric dimension of a graph. *Ars Combinatoria, 2*, 191-195,. 1

[13] Hauptmann, M., Schmied, R., & Viehmann, C. (2012). Approximation complexity of metric dimension problem. *Journal of Discrete Algorithms, 14*, 214-222. 1

[14] Hussain, Z., Kang, S. M., Rafique, M., Munir, M., Ali, U., Zahid, A., & Saleem, M. S. (2019). Bounds for partition dimension of M-wheels. *Open Physics, 17*(1), 340-344. 1

[15] Imran, M., Siddiqui, M. K., & Naeem, R. (2018). On the metric dimension of generalized petersen multigraphs. *IEEE Access, 6*, 74328-74338. 1

[16] Javaid, I., Rahim, M. T., & Ali, K. (2008). Families of regular graphs with constant metric dimension. *Utilitas Mathematica, 75*, 21-34. 1

[17] Khuller, S., Raghavachari, B., & Rosenfeld, A. (1996). Landmarks in graphs. *Discrete Applied Mathematics, 70*(3), 217-229. 1

[18] Hartmanis, J. (1982). Computers and intractability: a guide to the theory of np-completeness (Michael R. Garey and David S. Johnson). *Siam Review, 24*(1), 90. 1

[19] Maritz, E. C., & Vetrk, T. (2018). The partition dimension of circulant graphs. *Quaestiones Mathematicae, 41*(1), 49-63. 1

[20] Mehreen, N., Farooq, R., & Akhter, S. (2018). On partition dimension of fullerene graphs. *AIMS Mathematics, 3*(3), 343-352. 1

[21] Melter, R. A., & Tomescu, I. (1984). Metric bases in digital geometry. *Computer Vision, Graphics, and Image Processing, 25*(1), 113-121. 1

[22] Chris Monica, M., & Santhakumar, S. (2016). Partition dimension of honeycomb derived networks. *International Journal of Pure and Applied Mathematics, 108*(4), 809-818. 1

[23] Naz, S., Salman, M., Ali, U., Javaid, I., & Bokhary, S. A. U. H. (2014). On the constant metric dimension of generalized Petersen graphs $P(n, 4)$. *Acta Mathematica Sinica, English Series, 30*(7), 1145-1160. 1

[24] Rehman, T. U., & Mehreen, N. (2020). Partition dimension and strong metric dimension of chain cycle. *Jordan Journal of Mathematics and Statistics, 13*(2), 305-325. 1

[25] Rodr`łuez-Velàzquez, J. A., Yero, I. G., & Lemanska, M. (2014). On the partition dimension of trees. *Discrete Applied Mathematics, 166*, 204-209 1

[26] Fernau, H., Rodr`łuez-Velàzquez, J. A., & Yero, I. G. (2014). On the partition dimension of unicyclic graphs. *Bulletin mathmatique de la Société des Sciences Mathématiques de Roumanie*, 381-391. 1

[27] Slater, P. J. (1975). Leaves of trees. *Congress Numerentium, 14*, 549-559. 1

[28] Vertana, H., & Kusmayadi, T. A. (2016). On the partition dimension of $C_m + P_n$ graph. *Journal of Physics: Conference Series, 855*. 1

[29] Watkins, M. E. (1969). A theorem on Tait colorings with an application to the generalized Petersen graphs. *Journal of Combinatorial Theory, 6*(2), 152-164. 1

[30] Yero, I. G., & Rodr`łuez-Velàzquez, J. A. (2010). A note on the partition dimension of Cartesian product graphs. *Applied Mathematics and Computation, 217*(7), 3571-3574.
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