Available online at http://jprm.sms.edu.pk/
Journal of Prime Research in Mathematics, 17(2) (2021), 168–181

Journal of Prime Research in Mathematics

The nonlinear time-fractional differential equations with integral conditions

H. Merad[a,*], F. Merghadi[b], A. Merad[c]

[a]Laboratory of Mathematics, Informatics and Systems (LAMIS), Larbi Tebessi University - Tebessa, Algeria.

[b]Department of Mathematics and Informatics, Laboratory of Mathematics, Informatics and Systems (LAMIS), Larbi TebessiUniversity - Tebessa, Algeria.

[c]Department of Mathematics and Informatics, Laboratory of Dynamical Systems and Control, Larbi Ben M'hidi University of Oum El Bouaghi, Algeria.

Abstract

In this paper, we present a nonlinear equation modeling a time-fractional pseudoparabolic problem, involving fractional Caputo derivative where the fractional order is $0 < \alpha < 1$. We first started with the associated linear problem, we establish the energy inequalities to obtaine a priori estimate,and demonstrate the density of the operator's range generated. Accordingly, the existence and uniqueness of the weak solutions are given, then we use the preceding results to handle the nonlinear case via an iterative process.

Keywords: Energy inequality, fractional evolution equations, integral condition, priori estimate.
2010 MSC: 34A12, 35B45.

1. Introduction

Fractional Differential equations, started with the ideas of G. Leibniz, by extending the order of derivatives and integrals with any number irrational, fractional or complex, this initial spark motivated many mathematicians, physicists and engineers to develop it gradually up to now, and it has been recognized as one of the leading instruments to portray long-memory processes, radioactive nuclear decay in fluid flows, plasma of physics, population dynamics, semiconductor modeling, transmission theory and certain biological processes, underground-water flow. Such models were curiously for engineers and physicists but moreover for pure mathematicians, and it was introduced in various different ways, most evident formulations is based on the fundamental definition such Riemann-Liouville, Grunwald-Leitnikov, Weyl, Riesz, Caputo.

For all models such as those used, the solution may not necessarily be directly solvable, or do not have explicit form, their only often unclear, but whether solutions are unique or exist, are also a notable

*Corresponding author
Email address: `meradhad@gmail.com` (H. Merad)

Received : 4 February 2022; Accepted: 11 May 2022; Published Online: June 2022.

subjects of interest. However there are a various results concerning the existence and the uniqueness of solution have been established, mentioning : energy inequality[1] and Adomian decomposition [2],Variational Iteration Method [3] and Fractional Difference Method [4], Rothe time-discretization [5], Power Series [6], Laplace Transform [7], Fractional Green's Function [8], Mellin Transform.[9] Or restricted on it existence and uniqueness only as fixed point of a sum operator, fixed point of Schauder [10], upper and lower solutions, perturbation method [11, 12].

In this paper, we focus on providing the existence and uniqueness of the solutions to non-linear time-fractional differential equations equation, where the highest order derivative may be greater than one, and the fractional order is $0 < \alpha < 1$, depending on Caputo approach, by the energy inequality method. Firstly the statement of the problem where the equation is posed, then in Sec.3 introduce some useful function spaces, and obtain some results for the associated linear problem, thereafter in the Sec.4 we examine an iterative process based on the result obtained on the previous sections for validating the existence and uniqueness of the non–linear problem

2. Formulation of the problem

We considering the fractional nonlinear partial differential equation as follows

$$\mathcal{L}\theta = {}_0^C D_t^\alpha \theta - k\frac{\partial^2\theta}{\partial x^2} - \eta\frac{\partial^3\theta}{\partial t\partial x^2} = h(x,t,\theta,\frac{\partial\theta}{\partial x}), \tag{2.1}$$

in the bounded domain $Q_T = \left\{(x,t) \in \mathbb{R}^2, T > 0 : 0 \le x \le 1, \quad 0 < t \le T \right\}$, associating with initial conditions

$$q\theta = \frac{\partial\theta(x,0)}{\partial x} = \theta_0'(x) \tag{2.2}$$

$$\frac{\partial\theta(0,t)}{\partial t} = \mu(t) \tag{2.3}$$

and the integral condition

$$\int_0^1 \theta(x,t)dx = E(t), \tag{2.4}$$

where k and η are positive real constants, and the operator ${}_0^C D_t^\alpha$ is the Caputo left fractional derivative of order $0 < \alpha < 1$ defined as

$${}_0^C D_t^\alpha \theta = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\theta_t}{(t-\tau)^\alpha}d\tau, t > 0.$$

For more details we refer to [15].

The nonlinear equation (2.1) has some known importance because of its applications in describing certain heat diffusion phenomena with source terms. The obvious example is given by the theory of thermal conduction related to a deformable body [16]. Here θ plays the role of the conductive temperature, k is the report of the conductivity upon the specific heat, η is positive real constant denotes the temperature discrepancy factor (the conductive temperature and thermodynamic temperature) and $h(x,t,\theta,\frac{\partial\theta}{\partial x})$ is intensity of heat source.

Moreover, this equation is accompanied by some conditions that lead to a description of the physical properties of the system: the integral and Neumann conditions (2.4, 2.3), which can be interpreted respectively as the average values and the flow of quantities physical respectively. Then $E(t)$ in (2.4) will be taken as the heat energy removed and condition (2.3) means that the heat flow across the boundary is equal to $\mu(t)$.

Besides, this equation (2.1) can describe other physical processes, among which: infiltration of homogeneous fluids through fissured rocks [17], non-stationary flows of second-order fluids [18], diffusion of resonant radiation trapped through a gas [19] and unidirectional propagation of nonlinear dispersive long waves [20] etc.

Since the integral boundary conditions are inhomogeneous, we introducing the new function $u(x,t) = \theta(x,t) - U(x,t)$ defined as

$$U(x,t) = 3x^2E(t) + (1-2x)\int_0^t \mu(t)dt. \tag{2.5}$$

$$\begin{cases} {}_0^C D_t^{\alpha} u - k\frac{\partial^2 u}{\partial x^2} - \eta\frac{\partial^3 u}{\partial t\partial x^2} = f(x,t,u,\frac{\partial u}{\partial x}) \\ \frac{\partial u(x,0)}{\partial x} = u_0'(x) \\ \frac{\partial u(0,t)}{\partial t} = 0 \\ \int_0^1 u(x,t)dx = 0 \end{cases}, \tag{2.6}$$

where

$$f(x,t,u,\frac{\partial u}{\partial x}) = h(x,t,u+U,\frac{\partial u}{\partial x}+\frac{\partial U}{\partial x}) - \mathcal{L}U$$
$$u_0'(x) = \theta_0'(x) - qU(x,t)$$

In addition we assume that f is a lipschitz function for all $(x,t) \in Q_T$.

Problems with non-stationary conditions are little known and require some precautions. Consequently, in order to convert it to the usual problem of stationary conditions, we introduce an appropriate transformation (2.5) and thus we reduce the problem (2.1) – (2.4) with non-homogeneous and non-stationary integral conditions to problem (2.6) with homogeneous and stationary conditions. Obviously, the two problems are equivalent and specify the same behavior of physical phenomena

3. The linear problem associated

In the first part, in this section, we restrict on the linear problem (3.1), which presents a particular restriction where the effect of the nonlinear terms is considered negligible and manifests itself only weakly and in a smooth way in the dynamics of the system.

$$\begin{cases} {}_0^C D_t^{\alpha} u - k\frac{\partial^2 u}{\partial x^2} - \eta\frac{\partial^3 u}{\partial t\partial x^2} = f(x,t) \\ \frac{\partial u(x,0)}{\partial x} = u_0'(x) \\ \frac{\partial u(0,t)}{\partial t} = 0 \\ \int_0^1 u(x,t)dx = 0. \end{cases}, \tag{3.1}$$

For investigating the posed problem (3.1) , we introduce some function spaces that we used .

Starting by $L^2(0,1)$, $L^2(0,T;H)$ where H is a Hilbert space, and $C(0,T;H)$ is the standard functional space, we denoted by $L^2_\sigma(0,1)$ the weighted L^2-space normed with

$$\|u\|_{L^2_\sigma(0,1)} = \left(\int_0^1 (1-x)u^2dx\right)^{1/2}, \tag{3.2}$$

and $H^1_\sigma(0,1)$ is the weighted Sobolev space normed with

$$\|u\|_{H^1_\sigma(0,1)} = \left(\|u\|^2_{L^2_\sigma(0,1)} + \left\|\frac{\partial u}{\partial x}\right\|^2_{L^2_\sigma(0,1)}\right)^{1/2}, \tag{3.3}$$

as well let E being the Banach space with the finite norm

$$\|u\|_B^2 = {}_0^C D_t^{\alpha} \left\| (\int_0^x (1-\zeta)\frac{\partial u}{\partial \zeta} d\zeta \right\|_{L^2(Qt)}^2 + \|u\|_{C(0,T;H_\sigma^1(0,1))}^2 , \tag{3.4}$$

also F is the Hilbert space with the finite norm

$$\|\mathcal{F}\|_F^2 = \|f\|_{L^2(Qt)}^2 + \|u_0'\|_{L_\sigma^2(0,1)}^2 . \tag{3.5}$$

Now refer to (2.6) considering the operator $Lu = \mathcal{F}$, where $L = (\mathcal{L}, q), \mathcal{F} = (f, u_0')$, acting from E into F, and the domain of definition of the operator L is $D(L)$, The solution of the problem (2.6) can be considered as a solution of the problem in the operational form $Lu = \mathcal{F}$, moreover set all functions $u \in L^2(0,T; L_\sigma^2(0,1))$, $\frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial x \partial t} \frac{\partial^3 u}{\partial x^2 \partial t} \in L^2(0,T; L_\sigma^2(0,1))$, and satisfying integral conditions in problem (2.4).

3.1. Uniqueness of solution (a priori estimate)

Lemma 3.1. *For any function $u \in D(L)$, there exist a positive constant C such that*

$$\|u\|_B \le C \|Lu\|_F . \tag{3.6}$$

Proof. We take the scalar product in $L^2(0,1)$ of the equation in (3.1) and the integro-differential operator $Mu = \int_0^x (1-\zeta)\frac{\partial u}{\partial \zeta} d\zeta$, and integrating over (Q_τ) with $\tau \le T$, we have

$$\int_0^\tau \int_0^1 \begin{pmatrix} {}_0^C D_t^{\alpha+1} u \cdot Mu - k\frac{\partial^2 u}{\partial x^2} \cdot Mu - \eta \frac{\partial^3 u}{\partial t \partial x^2} \cdot Mu \\ = f(x,t) \cdot Mu \end{pmatrix} dxdt. \tag{3.7}$$

Taking into account the initial and boundary conditions(3.1), and integrating by parts each terms of (3.7), we obtain

$$-k \int_{Qt} \frac{\partial^2 u}{\partial x^2} \left(\int_0^x (1-\zeta)\frac{\partial u}{\partial \zeta} d\zeta \right) dxdt$$

$$= k \int_0^\tau \left\| \frac{\partial u}{\partial x} \right\|_{L_\sigma^2(0,1)}^2 . \tag{3.8}$$

Further,

$$-\eta \int_{Qt} \frac{\partial^3 u}{\partial t \partial x^2} \left(\int_0^x (1-\zeta)\frac{\partial u}{\partial \zeta} d\zeta \right) dxdt$$

$$= \frac{\eta}{2} \left(\left\| \frac{\partial u(x,\tau)}{\partial x} \right\|_{L_\sigma^2(0,1)}^2 - \|u_0'\|_{L_\sigma^2(0,1)}^2 \right). \tag{3.9}$$

For the first term of (3.7) we use Alikhanov's inequality [14]

$$\int_{Qt} {}_0^C D_t^{\alpha} u \left(\int_0^x (1-\zeta)\frac{\partial u}{\partial \zeta} d\zeta \right) dxdt$$

$$= -\int_{Qt} {}_0^C D_t^{\alpha} \frac{\partial u}{\partial x} \left(\int_0^x \int_0^n (1-\zeta) \frac{\partial u}{\partial \zeta} d\zeta dn \right) dxdt$$

$$\geqslant \frac{1}{2} \int_{Qt} {}_0^C D_t^{\alpha} (\int_0^x (1-\zeta) \frac{\partial u}{\partial \zeta} d\zeta)^2 dxdt = \frac{1}{2} {}_0^C D_t^{\alpha} \left\| (\int_0^x (1-\zeta) \frac{\partial u}{\partial \zeta} d\zeta \right\|^2_{L^2(Qt)}. \tag{3.10}$$

For the first term on the right side of (3.7) ,we applying Cauchy ϵ inequality [21], then we use Poincare type inequalities [21], this yields

$$\int_{Qt} f \left(\int_0^x (1-\zeta) \frac{\partial u}{\partial \zeta} d\zeta \right) dxdt$$

$$\leqslant \frac{\epsilon}{2} \int_0^{\tau} \int_0^1 \left(f^2\right) dxdt + \frac{1}{2\epsilon} \int_0^{\tau} \int_0^1 \left(\int_0^x (1-\zeta) \frac{\partial u}{\partial \zeta} d\zeta \right)^2 dxdt$$

$$= \frac{\epsilon}{2} \int_0^{\tau} \|f\|^2_{L^2(0,1)} dt + \frac{1}{2\epsilon} \frac{1}{\sqrt{2}} \int_0^{\tau} \left\| \frac{\partial u}{\partial x} \right\|^2_{L^2_{\sigma}(0,1)} dt. \tag{3.11}$$

Now substituting all the precedent results(3.8) − (3.10) into (3.7), we find

$$\frac{1}{2} {}_0^C D_t^{\alpha+1} \left\| (\int_0^x (1-\zeta) \frac{\partial u(\zeta,t)}{\partial \zeta} d\zeta \right\|_{L^2(Qt)} + \frac{\eta}{2} \left\| \frac{\partial u(x,\tau)}{\partial x} \right\|^2_{L^2_{\sigma}(0,1)}$$

$$\leq \left(\frac{\epsilon}{2} \int_0^{\tau} \|f\|^2_{L^2(0,1)} dt + \frac{\eta}{2} \|u_0'\|^2_{L^2_{\sigma}(0,1)} \right) + \left(\frac{1}{2\epsilon} \frac{1}{\sqrt{2}} - k \right) \int_0^{\tau} \left\| \frac{\partial u(x,t)}{\partial x} \right\|^2_{L^2_{\sigma}(0,1)} dt. \tag{3.12}$$

We simplifying (3.12), by using the Gronwall's lemma [13], we need $\left(\frac{1}{2\epsilon}\frac{1}{\sqrt{2}} - k\right) > 0$, let's suppose is realized, we get

$${}_0^C D_t^{\alpha} \left\| (\int_0^x (1-\zeta) \frac{\partial u}{\partial \zeta} d\zeta \right\|_{L^2(Qt)} + \left\| \frac{\partial u(x,\tau)}{\partial x} \right\|^2_{L^2_{\sigma}(0,1)}$$

$$\leq C \left(\int_0^{\tau} \|f\|^2_{L^2(0,1)} dt + \|u_0'\|^2_{L^2_{\sigma}(0,1)} \right), \tag{3.13}$$

where

$$C = \frac{\max\left(\frac{\epsilon}{2}, \frac{\eta}{2}, \left(\frac{1}{2\epsilon}\frac{1}{\sqrt{2}} - k\right)\right)}{\min\left(\frac{1}{2}, \frac{\eta}{2}\right)} e^{\left(\frac{1}{2\epsilon}\frac{1}{\sqrt{2}} - k\right)T}. \tag{3.14}$$

Now adding this following elementary inequality to (3.13)

$$\|u(x,\tau)\|^2_{L^2_{\sigma}(0,1)} \leq \int_0^{\tau} \|u(x,t)\|^2_{L^2_{\sigma}(0,1)} dt + \int_0^{\tau} \left\| \frac{\partial u(x,t)}{\partial x} \right\|^2_{L^2_{\sigma}(0,1)} dt, \tag{3.15}$$

it yields

$${}_0^C D_t^{\alpha} \left\| (\int_0^x (1-\zeta) \frac{\partial u}{\partial \zeta} d\zeta \right\|_{L^2(Qt)} + \|u(x,\tau)\|^2_{H^1_{\sigma}(0,1)}$$

$$\leq C\left(\int_0^\tau \|f\|^2_{L^2(0,1)}\,dt+\left\|u_0'\right\|^2_{L^2_\sigma(0,1)}\right)+\int_0^\tau \|u(x,\tau)\|^2_{H^1_\sigma(0,1)}\,dt. \tag{3.16}$$

Inequality (3.16) is equivalent to

$$\begin{aligned}&{}_0^CD_t^\alpha\left\|(\int_0^x(1-\zeta)\frac{\partial u}{\partial\zeta}d\zeta\right\|_{L^2(Qt)}+\|u(x,\tau)\|^2_{H^1_\sigma(0,1)}\\ &\leq C^\circ\left(\int_0^T \|f\|^2_{L^2(0,1)}\,dt+\left\|u_0'\right\|^2_{L^2_\sigma(0,1)}\right)+\int_0^\tau \|u(x,\tau)\|^2_{H^1_\sigma(0,1)}\,dt,\end{aligned} \tag{3.17}$$

where

$$C^\circ=\max\left(C,1\right). \tag{3.18}$$

Again we applying the Gronwall's lemma to (3.17)

$${}_0^CD_t^\alpha\left\|(\int_0^x(1-\zeta)\frac{\partial u}{\partial\zeta}d\zeta\right\|_{L^2(Qt)}+\|u(x,\tau)\|^2_{H^1_\sigma(0,1)}\leq C^\circ\left(\int_0^T \|f\|^2_{L^2(0,1)}\,dt+\left\|u_0'\right\|^2_{L^2_\sigma(0,1)}\right), \tag{3.19}$$

$$C'=C^\circ e^T. \tag{3.20}$$

Since the right-hand side is independent of τ, we take the upper boundary on the left-hand side of (3.19)

$${}_0^CD_t^\alpha\left\|(\int_0^x(1-\zeta)\frac{\partial u}{\partial\zeta}d\zeta\right\|^2_{L^2(Qt)}+\|u\|^2_{C(0,T;H^1_\sigma(0,1))}\leq C^\circ\left(\|f\|^2_{L^2(Qt)}+\left\|u_0'\right\|^2_{L^2_\sigma(0,1)}\right). \tag{3.21}$$

□

3.2. Existence of solution of the linear problem

The proof based on two steps (result of functional analysis), The operator $L\colon B\longrightarrow F$ is closable, (a linear operator is called closable if it has a closed extension) and $R(L)$ is dense in F

Corollary 3.2. *The operator $L\colon B\longrightarrow F$ admit a closure*

Proof. Let $(u_n)_n\in D\left(L\right)$ a sequence such that :

$$\begin{aligned}u_n&\longrightarrow 0,\text{ in }B\\ Lu_n&\longrightarrow\left(f;u_0'\right),\text{ in }F\end{aligned} \tag{3.22}$$

The procedure is to find

$$f\equiv 0,\;\; u_0'\equiv 0.$$

Proof is similar to [21].

Denote by $\overline{L}$ the closure of L and $D\left(\overline{L}\right)$ its domain. □

Definition 3.3. The solution of $\overline{L}u=\mathcal{F}$ is called the strong generalized solution of the problem (3.1)

since the point of the graph of L is boundary of sequence of points of the graph of L, the energy inequality obtained by passing to the limit in (3.6)

$$\|u\|_B\leq C\left\|\overline{L}u\right\|_F. \tag{3.23}$$

This last entails the following corollaries

Corollary 3.4. *The range $R(L)$ of $\bar{L}$ is closed in F and $R(\bar{L}) = \overline{R(L)}$*

Proof. See [21] □

Proposition 3.5. *The problem (3.1) admit a strong generalized solution $u = L^{-1}\mathcal{F}$*

Proof. First we prove $R(L)$ is dense in F i.e. $\left(R(L)^{\perp} = \{0\}\right)$ for the special case, where $D(L) \equiv B$ is reduced to $D_0(L)$ with $D_0(L) = \{u,\ u \in D(L), qu = 0\}$, then we complete the proof by the density of the trace operator in any Hilbert space. □

Lemma 3.6. *For all $u \in D(L)$ and $\omega \in L^2(Qt)$ we have*

$$\int_{Qt} \mathcal{L} \cdot \omega \, dxdt = 0, \tag{3.24}$$

then ω annulled almost everywhere.

Proof. The scalar product of F is defined by

$$(Lu, \omega)_F = \int_{Qt} \mathcal{L} \cdot \omega \, dxdt + \int_0^1 qu \cdot \omega_0' dx, \tag{3.25}$$

the inequality (3.25) written as follow

$$\int_{Qt} D_t^{\alpha} u \cdot \omega \, dxdt = \int_{Qt} k \frac{\partial^2 u}{\partial x^2} \cdot \omega \, dxdt + \int_{Qt} \eta \frac{\partial^3 u}{\partial t \partial x^2} \cdot \omega \, dxdt, \tag{3.26}$$

by taking

$$\omega(x,t) = \int_x^1 \int_0^n u(\zeta, t) dn d\zeta, \tag{3.27}$$

we integrating by parts each terms we get

$$\int_{Qt} D_t^{\alpha} u \cdot \omega \, dxdt$$

$$\geqslant \frac{1}{2} \int_{Qt} {}_0^C D_t^{\alpha} \Big(\int_0^x u(\zeta, t) \, d\zeta \Big)^2 dxdt = \frac{1}{2} {}_0^C D_t^{\alpha} \left\| \Big(\int_0^x (1-\zeta) \frac{\partial u}{\partial \zeta} d\zeta \right\|_{L^2(Qt)}^2 . \tag{3.28}$$

Again from (3.26), it follows that

$$\int_{Qt} k \frac{\partial^2 u}{\partial x^2} \cdot \omega \, dxdt = -k \left\| u \right\|_{L^2(Qt)}^2, \tag{3.29}$$

the same for

$$\int_{Qt} \eta \frac{\partial^3 u}{\partial t \partial x^2} \cdot \omega dxdt = -\frac{\eta}{2} \left\| u(x, \tau) \right\|^2, \tag{3.30}$$

finally, we get

$$\frac{1}{2} \left\| {}_0^C D_t^{\alpha} \Big(\int_0^x (1-\zeta) \frac{\partial u}{\partial \zeta} d\zeta \right\|_{L^2(Qt)}^2 + k \left\| u \right\|_{L^2(Qt)}^2 + \frac{\eta}{2} \left\| u(x, \tau) \right\|_{L^2(0,1)}^2 \leq 0. \tag{3.31}$$

So we have $u = 0$ then ω definitely equal to zero. □

4. The nonlinear problem

We consider the homogeneous equation with the same conditions in (3.1)

$$ {}_0^C D_t^\alpha w - k\frac{\partial^2 w}{\partial x^2} - \eta\frac{\partial^3 w}{\partial t\partial x^2} = 0, \tag{4.1} $$

$$ \begin{cases} \frac{\partial w(x,0)}{\partial x} = u_0'(x) \\ \frac{\partial w(0,t)}{\partial t} = 0 \\ \int\limits_0^1 w(x,t)dx = 0 \end{cases} . \tag{4.2} $$

We put $y = u - w$ where u is the solution of problem (3.1), therefore y satisfies

$$ {}_0^C D_t^\alpha y - k\frac{\partial^2 y}{\partial x^2} - \eta\frac{\partial^3 y}{\partial t\partial x^2} = \check{f}\left(x,t,y,\frac{\partial y}{\partial x}\right), \tag{4.3} $$

and

$$ \begin{cases} \frac{\partial y(x,0)}{\partial x} = 0 \\ \frac{\partial y(0,t)}{\partial t} = 0 \\ \int\limits_0^1 y(x,t)dx = 0 \end{cases} , \tag{4.4} $$

where $\check{f}\left(x,t,y,\frac{\partial y}{\partial x}\right) = f\left(x,t,y+w,\frac{\partial y}{\partial x}+\frac{\partial w}{\partial x}\right)$ is a lipschitz function, as we mentioned before realize

$$ \left|\check{f}(x,t,p_1,q_1) - \check{f}(x,t,p_2,q_2)\right| \leq L\left(|p_1-p_2| + |q_1-q_2|\right), \tag{A} $$

for all $(x,t) \in Q_\tau$

The inhomogeneous nonlinear equation (4.3) with condition (4.4) is obtained from the homogeneous linear equation (4.1) without source with the same condition (2.6) via the transformation. They are equivalent and free to the same physics as mentioned above.

According the results of the previous section we deduce that the homogenous problem admit a unique solution that depending the initial condition.

Now we introduce the space function as

$$ \widetilde{C}^1(Qt) = \left\{\nu \in C^1(Qt),\ \text{such that}\ \frac{\partial^2\nu}{\partial x\partial t} \in C(Qt)\ \right\}, \tag{4.5} $$

assume that $\nu, y \in \widetilde{C}^1(Qt)$, trying to write the variational formula for (4.3) we consider the integro-differential operator

$$ M\nu = \int\limits_0^x \frac{\partial\nu}{\partial x}(\zeta,t)d\zeta, \tag{4.6} $$

satisfied the condition

$$ \begin{cases} \int\limits_0^1 \frac{\partial\nu}{\partial x}(x,t)dx = 0 \\ \frac{\partial\nu(x,T)}{\partial x} = 0 \end{cases} , \tag{4.7} $$

with

$$\frac{\partial y(x,0)}{\partial x} = 0, \int_0^1 y(x,t)dx = 0. \tag{4.8}$$

Firstly we have

$$\int_{Qt} D_t^{\alpha} y \cdot M\nu dxdt - \int_{Qt} k\frac{\partial^2 y}{\partial x^2} \cdot M\nu dxdt - \int_{Qt} \eta\frac{\partial^3 y}{\partial t\partial x^2} \cdot M\nu dxdt = \int_{Qt} \check{f} \cdot M\nu dxdt, \tag{4.9}$$

by using conditions on ν and y, the integration by parts each term on the equation (4.9), with quick computation, gives

$$-\int_{Qt} \frac{\partial^2 y}{\partial x^2}\left(\int_0^x \frac{\partial \nu}{\partial x}(\zeta,t)d\zeta\right) dxd = \int_0^{\tau}\int_0^1 \frac{\partial y}{\partial x}\frac{\partial \nu}{\partial x}dxdt, \tag{4.10}$$

also

$$\int_{Qt} \frac{\partial^3 y}{\partial t\partial x^2}\left(\int_0^x \frac{\partial \nu}{\partial x}(\zeta,t)d\zeta\right) dxdt = \int_0^1\int_0^T \frac{\partial y}{\partial x}\frac{\partial^2 \nu}{\partial t\partial x}dtdx, \tag{4.11}$$

the same for

$$\int_{Qt} {}_0^C D_t^{\alpha} y \left(\int_0^x \frac{\partial \nu}{\partial x}(\zeta,t)d\zeta\right) dxdt = -\int_{Qt} D_t^{\alpha}\left(\int_0^x y(\zeta,t)d\zeta\right)\frac{\partial \nu}{\partial x}dxdt,$$

finally, the rest term

$$\int_{Qt} f\left(\int_0^x \frac{\partial \nu}{\partial x}(\zeta,t)d\zeta\right) dxdt = \int_{Qt}\left(\int_0^x f(\zeta,t)d\zeta\right)\frac{\partial \nu}{\partial x}dxdt. \tag{4.12}$$

Substituting (4.10) – (4.12) into (4.9), and let $A(y,\nu)$ been the left-hand side of (4.12) that's yields

$$A(y,\nu) = \int_{Qt}\left(\int_0^x f(\zeta,t)d\zeta\right)\frac{\partial \nu}{\partial x}dxdt, \tag{4.13}$$

where

$$A(y,\nu) = -\int_{Qt} D_t^{\alpha}\left(\int_0^x y(\zeta,t)d\zeta\right)\frac{\partial \nu}{\partial x}dxdt + k\int_{Qt}\frac{\partial y}{\partial x}\frac{\partial \nu}{\partial x}dxdt - \eta\int_{Qt}\frac{\partial y}{\partial x}\frac{\partial^2 \nu}{\partial t\partial x}dtdx \tag{4.14}$$

Definition 4.1. The function $y \in L^2\left(0,T;H_{\sigma}^1(0,1)\right)$ called a weak solution of the problem (4.3) – (4.4) if

$$\begin{cases} A(y,\nu) = \int_{Qt}\left(\int_0^x f(\zeta,t)d\zeta\right)\frac{\partial \nu}{\partial x}dxdt \\ y \text{ verifies } \frac{\partial y(0,t)}{\partial t} = 0, \end{cases}$$

hold.

We construct an iteration sequence, starting with

$$y^{(0)} = 0, \tag{4.15}$$

then the sequence $\left(y^{(n)}\right)_{n\in\mathbb{N}}$ is defined as follows:

$$ {}_0^C D_t^{\alpha} y^{(n)} - k\frac{\partial^2 y^{(n)}}{\partial x^2} - \eta\frac{\partial^3 y^{(n)}}{\partial t\partial x^2} = \check{f}\left(x,t,y^{(n-1)},\frac{\partial y^{(n-1)}}{\partial x}\right) \tag{4.16}$$

$$\begin{cases} \frac{\partial y^{(n)}(x,0)}{\partial x} = 0 \\ \frac{\partial y^{(n)}(0,t)}{\partial t} = 0 \\ \int_0^1 y^{(n)}(x,t)dx = 0 \end{cases}, \tag{4.17}$$

where $y^{(n-1)}$ given for $n = 1, 2, ...$

Asserts that for fixed n, each problem $(4.16) - (4.17)$ has a unique solution $y^{(n)}(x,t)$, if we set $z^{(n)} = y^{(n+1)} - y^{(n)}$, then we have the new problem

$$ {}_0^C D_t^{\alpha} z^{(n)} - k\frac{\partial^2 z^{(n)}}{\partial x^2} - \eta\frac{\partial^3 z^{(n)}}{\partial t\partial x^2} = \phi^{(n-1)}(x,t) \tag{4.18}$$

$$\begin{cases} \frac{\partial z^{(n)}(x,0)}{\partial x} = 0 \\ \frac{\partial z^{(n)}(0,t)}{\partial t} = 0 \\ \int_0^1 z^{(n)}(x,t)dx = 0 \end{cases}, \tag{4.19}$$

where

$$\phi^{(n-1)}(x,t) = \check{f}\left(x,t,y^{(n)},\frac{\partial y^{(n)}}{\partial x}\right) - \check{f}\left(x,t,y^{(n-1)},\frac{\partial y^{(n-1)}}{\partial x}\right). \tag{4.20}$$

Lemma 4.2. *Assume that condition (A) holds, then there exist a positive constant*

$$C_1 = 2TCL^2\exp(T), \tag{4.21}$$

such that

$$\left\|z^{(n)}\right\|^2_{L^2(0,T;H^1_\sigma(0,1))} \le C_1\left\|z^{(n-1)}\right\|^2_{L^2(0,T;H^1_\sigma(0,1))}. \tag{4.22}$$

Proof. Multiplying Eq (4.18)by $\int_0^x (1-\zeta)\frac{\partial z^{(n)}}{\partial \zeta}d\zeta$ we get

$$\int_{Qt}\left({}_0^C D_t^{\alpha}\right)z^{(n)}\left(\int_0^x(1-\zeta)\frac{\partial z^{(n)}}{\partial\zeta}\right) - k\int_{Qt}\frac{\partial^2 z^{(n)}}{\partial x^2}\left(\int_0^x(1-\zeta)\frac{\partial z^{(n)}}{\partial\zeta}\right) - \eta\int_{Qt}\frac{\partial^3 z^{(n)}}{\partial t\partial x^2}\left(\int_0^x(1-\zeta)\frac{\partial z^{(n)}}{\partial\zeta}\right)$$

$$= \int_{Qt}\phi^{(n-1)}(x,t)\left(\int_0^x(1-\zeta)\frac{\partial z^{(n)}}{\partial\zeta}\right), \tag{4.23}$$

integrating over $(0,\tau)\times(0,1)$, we obtain

$$\frac{1}{2}{}_0^C D_t^{\alpha}\left\|\left(\int_0^x(1-\zeta)\frac{\partial z^{(n)}(\zeta,t)}{\partial\zeta}d\zeta\right\|^2_{L^2(Qt)} + \frac{\eta}{2}\left\|\frac{\partial z^{(n)}(x,\tau)}{\partial x}\right\|^2_{L^2_\sigma(0,1)}$$

$$\leq \frac{\epsilon}{2}\int_0^\tau \left\|\phi^{(n-1)}\right\|^2_{L^2(0,1)} dt + \left(\frac{1}{2\epsilon}\frac{1}{\sqrt{2}} - k\right)\int_0^\tau \left\|\frac{\partial z^{(n)}(x,t)}{\partial x}\right\|^2_{L^2_\sigma(0,1)} dt. \tag{4.24}$$

Applying the Gromwell's Lemma we find

$${}_0^C D_t^\alpha \left\|(\int_0^x (1-\zeta)\frac{\partial z^{(n)}}{\partial \zeta} d\zeta\right\|^2_{L^2(Qt)} + \left\|\frac{\partial z^{(n)}(x,\tau)}{\partial x}\right\|^2_{L^2_\sigma(0,1)}$$

$$\leq C\int_0^\tau \left\|\phi^{(n-1)}\right\|^2_{L^2(0,1)} dt \tag{4.25}$$

where

$$C = \frac{\max\left(\frac{\epsilon}{2}, \left(\frac{1}{2\epsilon}\frac{1}{\sqrt{2}} - k\right)\right)}{\min\left(\frac{1}{2}, \frac{\eta}{2}\right)} e^{\left(\frac{1}{2\epsilon}\frac{1}{\sqrt{2}} - k\right)T}, \tag{4.26}$$

adding this elementary inequality into(4.25) and eliminate its first element on the left-side

$$\left\|z^{(n)}(x,\tau)\right\|^2_{L^2_\sigma(0,1)} \leq \int_0^\tau \left\|z^{(n)}(x,t)\right\|^2_{L^2_\sigma(0,1)} dt + \int_0^\tau \left\|\frac{\partial z^{(n)}(x,t)}{\partial x}\right\|^2_{L^2_\sigma(0,1)} dt, \tag{4.27}$$

we get

$$\left\|z^{(n)}(x,\tau)\right\|^2_{H^1_\sigma(0,1)} \leq C\int_0^T \left\|\phi^{(n-1)}\right\|^2_{L^2(0,1)} dt + \int_0^\tau \left\|z^{(n)}(x,t)\right\|^2_{H^1_\sigma(0,1)} dt. \tag{4.28}$$

Now, since

$$\int_0^T \left\|\phi^{(n-1)}\right\|^2_{L^2(0,1)} = \int_0^T \left\|\check{f}\left(x,t,y^{(n)},\frac{\partial y^{(n)}}{\partial x}\right) - \check{f}\left(x,t,y^{(n-1)},\frac{\partial y^{(n-1)}}{\partial x}\right)\right\|^2_{L^2(0,1)}, \tag{4.29}$$

by hypothesis (A) we have

$$\leq L^2\int_0^T\int_0^1 \left(\left|z^{(n-1)}(x,t)\right| + \left|\frac{\partial z^{(n-1)}(x,t)}{\partial x}\right|\right)^2 dxdt \tag{4.30}$$

$$\leq 2L^2\int_0^T \left(\left\|z^{(n-1)}(x,t)\right\|^2_{L^2(0,1)} + \left\|\frac{\partial z^{(n-1)}(x,t)}{\partial x}\right\|^2_{L^2(0,1)}\right) dt$$

$$= 2L^2\int_0^T \left\|z^{(n-1)}(x,t)\right\|^2_{H^1_\sigma(0,1)} dt. \tag{4.31}$$

Hence, the inequality (4.28) becomes

$$\left\|z^{(n)}(x,\tau)\right\|^2_{H^1_\sigma(0,1)} \leq 2CL^2\int_0^T \left\|z^{(n-1)}(x,t)\right\|^2_{H^1_\sigma(0,1)} dt + \int_0^\tau \left\|z^{(n)}(x,t)\right\|^2_{H^1_\sigma(0,1)} dt. \tag{4.32}$$

Again, the Gronwall's lemma amounts to finding

$$\left\|z^{(n)}(x,\tau)\right\|^2_{H^1_\sigma(0,1)} \leq 2CL^2 \exp(T) \int_0^T \left\|z^{(n-1)}(x,t)\right\|^2_{H^1_\sigma(0,1)} dt, \tag{4.33}$$

integrating over $(0,\tau)$

$$\left\|z^{(n)}\right\|^2_{L^2(0,T;H^1_\sigma(0,1))} \leq C_1 \left\|z^{(n-1)}\right\|^2_{L^2(0,T;H^1_\sigma(0,1))} dt, \tag{4.34}$$

where

$$C_1 = 2TCL^2 \exp(T).$$

From the criteria of convergence of series, we see that the series $\sum_{n\geqslant 1} z^{(n)}$ is convergent if $\sqrt[2]{C_1} < 1$, Since $z^{(n)} = y^{(n+1)} - y^{(n)}$, then the sequence $\left(y^{(n)}\right)_{n\in\mathbb{N}}$ is defined by

$$\begin{aligned} y^{(n)}(x,t) &= \sum_{k=1}^{n-1} z^{(k)} + y^{(0)} \\ &= \sum_{k=1}^{n-1} \left(y^{(k+1)} - y^{(k)}\right) + y^{(0)}, \end{aligned} \tag{4.35}$$

which converges to an element y in $L^2\left(0,T;H^1_\sigma(0,1)\right)$.

Now to prove that this limit function y is a solution of the problem under consideration $(4.16)-(4.17)$, we should show that y verify the Definition 4.1.

Consider the weak formulation of the problem $(4.16)-(4.17)$

$$A(y,\nu) = \int_{Qt} \left(\int_0^x f(\zeta,t)d\zeta\right) \frac{\partial \nu}{\partial x} dxdt,$$

where

$$A(y,\nu) = -\int_{Qt} D_t^\alpha \left(\int_0^x y(\zeta,t)d\zeta\right) \frac{\partial \nu}{\partial x} dxdt + k\int_{Qt} \frac{\partial y}{\partial x}\frac{\partial \nu}{\partial x} dxdt - \eta \int_{Qt} \frac{\partial y}{\partial x}\frac{\partial^2 \nu}{\partial t \partial x} dtdx. \tag{4.36}$$

Trying to estimate the first term of (4.36) by integrating by parts

$$\int_{Qt} D_t^\alpha \left(\int_0^x y(\zeta,t)d\zeta\right) \frac{\partial \nu}{\partial x} dxdt = \int_{Qt} \left(\int_0^x y d\zeta\right) \left({}_tD^\alpha\right) \left(\frac{\partial \nu}{\partial x}\right) dxdt \tag{4.37}$$

then, becomes

$$\begin{aligned} &\leq \int_{Qt} \left(\int_0^x y d\zeta\right)^2 dxdt \int_{Qt} \left[\left({}_tD^\alpha\right)\left(\frac{\partial \nu}{\partial x}\right)\right]^2 dxdt \\ &\leq \frac{1}{2} \left\|\left({}_tD^\alpha\right)\left(\frac{\partial \nu}{\partial x}\right)\right\|_{L^2(Qt)} \int_{Qt} y^2 dxdt, \end{aligned}$$

Then, we have

$$A\left(y^{(n)} - y, \nu\right) \tag{4.38}$$

$$= -\int\limits_{Qt} D_t^{\alpha}\left(\int\limits_0^x y^{(n)} - y d\zeta\right)\frac{\partial \nu}{\partial x}dxdt + k\int\limits_{Qt}\frac{\partial\left(y^{(n)}-y\right)}{\partial x}\frac{\partial \nu}{\partial x}dxdt - \eta\int\limits_{Qt}\frac{\partial\left(y^{(n)}-y\right)}{\partial x}\frac{\partial^2 \nu}{\partial t\partial x}dtdx$$

$$\leq \frac{1}{2}\left\|\left({}_tD^{\alpha}\right)\left(\frac{\partial \nu}{\partial x}\right)\right\|_{L^2(Qt)}\left\|y^{(n)}-y\right\|_{L^2(Qt)} + k\left\|\frac{\partial\left(y^{(n)}-y\right)}{\partial x}\right\|_{L^2(Qt)}\left\|\frac{\partial \nu}{\partial x}\right\|_{L^2(Qt)}$$

$$+\eta\left\|\frac{\partial\left(y^{(n)}-y\right)}{\partial x}\right\|_{L^2(Qt)}\left\|\frac{\partial^2 \nu}{\partial t\partial x}\right\|_{L^2(Qt)}.$$

Furthermore

$$A\left(y^{(n)}-y,\nu\right)$$

$$= \int\limits_{Qt}\left(\int\limits_0^x f\left(\zeta,t,y^{(n-1)},\frac{\partial y^{(n-1)}}{\partial \zeta}\right)d\zeta\right)\frac{\partial \nu}{\partial x}dxdt - \int\limits_{Qt}\left(\int\limits_0^x f\left(\zeta,t,y,\frac{\partial y}{\partial \zeta}\right)d\zeta\right)\frac{\partial \nu}{\partial x}dxdt \tag{4.39}$$

$$\leq \frac{1}{2}L\left(\left\|y^{(n)}-y\right\|_{L^2(0,T;H^1(0,1))}\left\|\frac{\partial \nu}{\partial x}\right\|_{L^2(Qt)}\right), \tag{4.40}$$

passing to the limit into (4.38) – (4.40) we obtain

$$\lim_{n\longrightarrow\infty} A\left(y^{(n)}-y,\nu\right) = 0. \tag{4.41}$$

Finally we conclude that problem(4.16) – (4.17) has a weak solution. □

Lemma 4.3. *The problem* (4.3) – (4.4) *has a unique solution.*

Proof. Suppose that y_1, y_2 are two solutions of (4.3) – (4.4) ,such that $(z = y_1 - y_2\ ) \in L^2\left(0,T;H_\sigma^1(0,1)\right)$ then

$${}_0^C D_t^{\alpha} z - k\frac{\partial^2 z}{\partial x^2} - \eta\frac{\partial^3 z}{\partial t\partial x^2} = \phi(x,t) \tag{4.42}$$

with

$$\begin{cases} \frac{\partial z(x,0)}{\partial x} = 0 \\ \frac{\partial z(0,t)}{\partial t} = 0 \\ \int\limits_0^1 z(x,t)dx = 0 \end{cases}, \tag{4.43}$$

where

$$\phi(x,t) = \check{f}\left(x,t,y_1,\frac{\partial y_1}{\partial x}\right) - \check{f}\left(x,t,y_2,\frac{\partial y_2}{\partial x}\right).$$

Using the same integro-differential operator and making similar computation used in Lemma 4.2, we obtain

$$\|z\|_{L^2(0,T;H_\sigma^1(0,1))} \leq \sqrt{C_1}\,\|z\|_{L^2(0,T;H_\sigma^1(0,1))} \tag{4.44}$$

where

$$\sqrt{C_1} = \sqrt{2TCL^2\exp(T)} < 1, \tag{4.45}$$

then

$$\|z\|^2_{L^2(0,T;H_\sigma^1(0,1))} = 0. \tag{4.46}$$

we conclude that $y_1 = y_2$. □

5. Conclusions

In our work we have shown that the solution of the pseudoparabolic fractional differential equation of the order $0 < \alpha < 1$, depending on Caputo approach, via the energy inequality method, exist and unique. The proof was divided into two steps, For the first part, we started by the linear problem, the uniqueness of solution is achieved in accordance with a priori estimate, the existence of solution is realized consistent with the density of the operator associated with the linear problem. The rest of the proof is depended on the results obtained on the first, we used them to handle the nonlinear case by applying an iterative process. In addition this manuscript furnish a perspective to discuss the existence and uniqueness for a similar fractional partial differential equations, unifies the classical fractional operators :Riemann-Liouville and Caputo, with a highest fractional order, including also a numerical study.

References

[1] Bouziani, A. Solvability of nonlinear pseudoparabolic equation with a nonlocal boundary condition. Nonlinear Analysis: Theory, Methods & Applications. (2003) 55(7-8), 883-904. 1

[2] Merad, A., & Hadid, S. Analytical Solution of Non-Integer Extra-Ordinary Differential Equation Via Adomian Decomposition Method. Malaya Journal of Matematik. (2016) 4(1, 2016), 126-135. 1

[3] Wu, G. C., & Lee, E. W. M. Fractional variational iteration method and its application. Physics Letters A, (2010) 374(25), 2506-2509. 1

[4] Zhang, Y. A finite difference method for fractional partial differential equation. Applied Mathematics and Computation, (2009) 215(2), 524-529. 1

[5] Merazga, N., & Bouziani, A. Rothe method for a mixed problem with an integral condition for the two-dimensional diffusion equation. In Abstract and applied analysis, (2003) 16, 899-922. 1

[6] Abu Arqub, O. Application of residual power series method for the solution of time-fractional Schrödinger equations in one-dimensional space. Fundamenta Informaticae. (2019) 166(2), 87-110. 1

[7] Merad, A., Bouziani, A., Cenap, O. Z. E. L., & Kilicman, A. On solvability of the integrodifferential hyperbolic equation with purely nonlocal conditions. Acta Mathematica Scientia. (2015) 35(3), 601-609. 1

[8] Adil Khan, M., Iqbal, A., Suleman, M., & Chu, Y. M. Hermite–Hadamard type inequalities for fractional integrals via Green's function. Journal of Inequalities and Applications. (2018) 1, 1-15. 1

[9] Mainardi, F., Pagnini, G., & Gorenflo, R. Mellin transform and subordination laws in fractional diffusion processes. arXiv preprint math/0702133. (2007) 1

[10] Khastan, A., Nieto, J. J., & Rodríguez-López, R. Schauder fixed-point theorem in semilinear spaces and its application to fractional differential equations with uncertainty. Fixed Point Theory and Applications. (2014)1, 1-14. 1

[11] Lin, L., Liu, X., & Fang, H. Method of upper and lower solutions for fractional differential equations. Electronic Journal of Differential Equations. (2012) 100, 1-13. 1

[12] Momani, S., & Odibat, Z. Homotopy perturbation method for nonlinear partial differential equations of fractional order. Physics Letters A. (2007) 365(5-6), 345-350. 1

[13] Martin-Vaquero, J., & Merad, A. Existence, uniqueness and numerical solution of a fractional PDE with integral conditions. Nonlinear Analysis: Modelling and Control. (2019) 24(3), 368-386. 3.1

[14] Alikhanov, A. A. A priori estimates for solutions of boundary value problems for fractional-order equations. Differential equations. (2010) 46(5), 660-666. 3.1

[15] Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. Theory and applications of fractional differential equations. elsevier.2006. 2

[16] P.J. Chen, M.E. Gurtin, On a theory of heat conduction involving two temperatures, Z. Angew. Math.Phys.(1968) 19 614–627. 2

[17] G. Barenblbatt, Iu. P. Zheltov, and I. N. Kochina, Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks [Strata], J. Appl. Math. Mech. (1960) 24 , 1286-1303. 2

[18] T. W. Ting, Certain non-steady flows of second order fluids, Arch. Rational Mech. Anal.(1963) 14, 1-26. 2

[19] E. A. Milne, The diffusion of imprisoned radiation through a gas, J. London Math. Soc. (1926) 1, 40-51. 2

[20] T. B. Benjamin, J. L. Bona, and J. J. Mahony, Model equations for long waves in non-linear dispersive systems, Phil. Trans. Roy. Soc. London Ser. A, (1972) 272 , 47-78. 2

[21] Bouziani, A. Mixed problem with boundary integral conditions for a certain parabolic equation. Journal of Applied Mathematics and Stochastic Analysis. (1996) 9(3), 323-330. 3.1, 3.2, 3.2

[22] MZ Djibibe, A Merad. On solvability of the third pseudo-parabolic fractional equation with purely nonlocal conditions, Advances in Differential Equations and Control Processes. (2020) 23(1), 87-104

[23] MZ Djibibe ,B Soampa and A Merad, Two sided a priori estimates of a mixed problem with boundary integral conditions for a certain parabolic fractional equation, FJMS, (2020) 127(2,2020), 79-90.

[24] A. Necib, and A. Merad. Laplace transform and Homotopy perturbation methods for solving the pseudo-hyperbolic integro-differential problems with purely integral conditions, Kragujevac Journal of Mathematics. (2020) 44(2), 251-272.