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Pseudospectrum and condition pseudospectrum of non-archimedean matrices

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Abstract

In this paper, we introduce and study the notions of pseudospectrum, condition pseudospectrum of non-archimedean matrices and pseudospectrum of non-archimedean matrix pencil. Many results are proved and we give some examples.

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1. Introduction and Results

Throughout this paper, K is a non-archimedean (n.a) non trivially complete valued field with valuation $|\cdot|$, $\mathcal{M}_n(K)$ denote the space of all $n \times n$ matrices over K, $\mathbb{Q}_p$ is the field of p-adic numbers. Ingleton [5] showed that:

Theorem 1.1. *Let X be a n.a Banach space. For all $x \in X\backslash\{0\}$ there is $\xi \in X'$ such that $\xi(x) = 1$ and $\|\xi\| = \|x\|^{-1}$.*

We introduce the following definitions.

Definition 1.2. Let $A \in \mathcal{M}_n(K)$. The spectrum of matrix A is given by

$$\sigma(A) = \{\lambda \in K : (A - \lambda I) \textit{ is not invertible}\}.$$

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Definition 1.3. Let $A, B \in \mathcal{M}_n(K)$ and $\varepsilon > 0$. The pseudospectrum $\sigma_\epsilon(A)$ of matrix A is defined by

$$\sigma_\epsilon(A) = \sigma(A) \cup \{\lambda \in K : \ \|(A - \lambda I)^{-1}\| > \varepsilon^{-1}\}.$$

The pseudoresolvent $\rho_\epsilon(A)$ of matrix A is defined by

$$\rho_\epsilon(A) = \rho(A) \cap \{\lambda \in K : \ \|(A - \lambda I)^{-1}\| \leq \varepsilon^{-1}\},$$

by convention $\|(A - \lambda I)^{-1}\| = \infty$ if, and only if, $\lambda \in \sigma(A)$.

In particular case of [[1], Proposition 3.2], we have:

Proposition 1.4. *Let $A \in \mathcal{M}_n(K)$ and $\varepsilon > 0$, we have*

(i) $\sigma(A) = \bigcap\limits_{\varepsilon > 0} \sigma_\varepsilon(A)$.

(ii) For all ε_1 and ε_2 such that $0 < \varepsilon_1 < \varepsilon_2$, $\sigma(A) \subset \sigma_{\varepsilon_1}(A) \subset \sigma_{\varepsilon_2}(A)$.

In particular case of [[1], Theorem 3.4], we have:

Theorem 1.5. *Let X be a n.a finite dimensional over $\mathbb{Q}_p$ such that $\|X\| \subset |\mathbb{Q}_p|$, let $A \in \mathcal{L}(X)$ and $\varepsilon > 0$. Then,*

$$\sigma_\varepsilon(A) = \bigcup_{\|C\| < \varepsilon} \sigma(A + C).$$

We have some examples of non-archimedean pseudospectrum of matrices.

Example 1.6. Let $K = \mathbb{Q}_p$ and $\varepsilon > 0$. If

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \ \lambda_1, \lambda_2 \in \mathbb{Q}_p \backslash \{0\}.$$

Then

$$\begin{aligned} \sigma_\varepsilon(A) &= \sigma(A) \cup \{\lambda \in \mathbb{Q}_p : \|(A - \lambda I)^{-1}\| > \frac{1}{\varepsilon}\} \\ &= \{\lambda_1, \lambda_2\} \cup \{\lambda \in \mathbb{Q}_p : \max\{\frac{1}{|\lambda_1 - \lambda|}, \frac{1}{|\lambda_2 - \lambda|}\} > \frac{1}{\varepsilon}\}. \end{aligned}$$

Example 1.7. Let $K = \mathbb{Q}_p$ and $\varepsilon > 0$. If

$$A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

Then

$$\begin{aligned} \sigma_\varepsilon(A) &= \sigma(A) \cup \{\lambda \in \mathbb{Q}_p : \|(A - \lambda I)^{-1}\| > \frac{1}{\varepsilon}\} \\ &= \{1\} \cup \{\lambda \in \mathbb{Q}_p : \max\{\frac{1}{|1 - \lambda|}, \frac{|3|}{|(1 - \lambda)^2|}\} > \frac{1}{\varepsilon}\}. \end{aligned}$$

Example 1.8. Let $K = \mathbb{Q}_p$ and $\varepsilon > 0$. If

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

Consequently

$$\begin{aligned} \sigma_\varepsilon(A) &= \sigma(A) \cup \{\lambda \in \mathbb{Q}_p : \|(A - \lambda I)^{-1}\| > \frac{1}{\varepsilon}\} \\ &= \{0, 2\} \cup \{\lambda \in \mathbb{Q}_p : \max\{\frac{1}{|\lambda(2 - \lambda)|}, \frac{|1 - \lambda|}{|\lambda(2 - \lambda)|}\} > \frac{1}{\varepsilon}\}. \end{aligned}$$

The following definition is introduced.

Definition 1.9. Let $A \in \mathcal{M}_n(K)$ and $\varepsilon > 0$. The condition pseudospectrum of matrix A is given by

$$\Lambda_\epsilon(A) = \sigma(A) \cup \{\lambda \in K : \ \|(A - \lambda I)\| \|(A - \lambda I)^{-1}\| > \varepsilon^{-1}\}$$

by convention $\|(A - \lambda I)\| \|(A - \lambda I)^{-1}\| = \infty$, if and only if $\lambda \in \sigma(A)$.
The pseudoresolvent of matrix A is $K \backslash \Lambda_\epsilon(A)$.

In particular case of [2], Proposition 2.2 (i). We have:

Proposition 1.10. *Let $A \in \mathcal{M}_n(K)$. For all $\varepsilon > 0$, we have:*

(i) $\sigma(A) = \bigcap\limits_{0<\varepsilon<1} \Lambda_\varepsilon(A)$.

(ii) If $0 < \varepsilon_1 < \varepsilon_2$, *then* $\sigma(A) \subset \Lambda_{\varepsilon_1}(A) \subset \Lambda_{\varepsilon_2}(A)$.

From Theorem 2.1 of [2], we have:

Lemma 1.11. *Let X be a n.a finite dimensional Banach space over K, let $A \in \mathcal{L}(X)$ and $\varepsilon > 0$. Then $\lambda \in \Lambda_\epsilon(A) \setminus \sigma(A)$ if, and only if, there is $x \in X$ such that*

$$\|(A - \lambda I)x\| < \varepsilon \|A - \lambda I\| \|x\|.$$

The following theorem is a particular case of [[2], Proposition 2.3].

Theorem 1.12. *Let $A \in \mathcal{M}_n(K)$ be invertible and $A^{-1} \in \mathcal{M}_n(K)$ and $\varepsilon > 0$ and $k = \|A^{-1}\| \|A\|$. Then,*

$$\lambda \in \Lambda_\varepsilon(A^{-1}) \backslash \{0\} \; if \; and \; only \; if \; \frac{1}{\lambda} \in \Lambda_{\varepsilon k}(A) \backslash \{0\}.$$

In the following theorem, we investigate the relation between the condition pseudospectrum and the usual spectrum in K.

Theorem 1.13. *Let $A \in \mathcal{M}_n(K)$ and $\lambda \in K$ and $\varepsilon > 0$. If there is $C \in \mathcal{M}_n(K)$ such that $\|C\| < \varepsilon \|A - \lambda I\|$ and $\lambda \in \sigma(A + C)$. Then, $\lambda \in \Lambda_\varepsilon(A)$.*

Proof. Suppose that there is C such that $\|C\| < \varepsilon \|A - \lambda I\|$. If $\lambda \notin \Lambda_\varepsilon(A)$, hence $\lambda \in \rho(A)$ and $\|A - \lambda I\| \|(A - \lambda I)^{-1}\| \leq \varepsilon^{-1}$.
Consider D defined on X by

$$D = \sum_{n=0}^{\infty} (A - \lambda I)^{-1} \Big(- C(A - \lambda I)^{-1} \Big)^n. \tag{1.1}$$

One can see that $D = (A - \lambda I)^{-1}(I + C(A - \lambda I)^{-1})^{-1}$. Hence there is $y \in X$ such that

$$D(I + C(A - \lambda I)^{-1})y = (A - \lambda I)^{-1} y. \tag{1.2}$$

We put $x = (A - \lambda I)^{-1})y$. We have that for each $x \in X$, $D(A - \lambda I + C)x = x$. Moreover, for all $x \in X$, $(A - \lambda I + C)Dx = x$. Thus, we conclude that $(A - \lambda I + C)$ is invertible and $D = (A - \lambda I + C)^{-1}$, contradiction. Then $\lambda \in \Lambda_\varepsilon(A)$. □

We set $\mathcal{C}_\varepsilon(X) = \{C \in \mathcal{L}(X) : \|C\| < \varepsilon \|A - \lambda I\|\}$. The following result is a particular case of [[2], Theorem 2.4].

Theorem 1.14. *Let X be a n.a finite dimensional Banach space over $\mathbb{Q}_p$ such that $\|X\| \subset |\mathbb{Q}_p|$, let $A \in \mathcal{L}(X)$ and $\varepsilon > 0$. We have,*

$$\Lambda_\varepsilon(A) = \bigcup_{\mathcal{C}_\varepsilon(X)} \sigma(A+C).$$

Proof. Let $\lambda \in \bigcup_{\mathcal{C}_\varepsilon(X)} \sigma(A+C)$. If $\lambda \in \rho(A)$ and $\|A-\lambda B\|\|(A-\lambda B)^{-1}\| \leq \varepsilon^{-1}$. Consider D defined on X given by

$$D = \sum_{n=0}^{\infty}(A-\lambda I)^{-1}\Big(-C(A-\lambda I)^{-1}\Big)^n. \tag{1.3}$$

With simple calculation we have $D = (A-\lambda B)^{-1}(I + C(A-\lambda I)^{-1})^{-1}$. Then there is $y \in X$ such that $D(I+C(A-\lambda B)^{-1})y = (A-\lambda I)^{-1}y$. We set $x = (A-\lambda I)^{-1})y$. Hence for all $x \in X$, $D(A-\lambda I+C)x = x$. Consequently, for each $x \in X$, $(A-\lambda I+C)Dx = x$. Thus, $(A-\lambda I+C)$ is invertible and $D = (A-\lambda I+C)^{-1}$. Conversely, suppose that $\lambda \in \Lambda_\varepsilon(A)$. If $\lambda \in \sigma(A)$, we put $C = 0$. If $\lambda \in \Lambda_\varepsilon(A)$ and $\lambda \notin \sigma(A)$. By Lemma 1.11 and $\|X\| \subset |\mathbb{Q}_p|$, there is $x \in X$ such that $\|x\| = 1$ and $\|(A-\lambda I)x\| < \varepsilon\|A-\lambda I\|$.
By Theorem 1.1, there is $\phi \in X'$ such that $\phi(x) = 1$ and $\|\phi\| = \|x\|^{-1} = 1$. Consider the operator C defined on X by for all $y \in X$, $Cy = -\phi(y)(A-\lambda I)x$. We have that $\|C\| < \varepsilon\|A-\lambda I\|$. Hence, $\|C\| < \varepsilon\|A-\lambda I\|$ and $D(C) = X$. Thus, for all $x \in X$, $(A-\lambda I+C)x = 0$. So, $(A-\lambda I+C)$ is not invertible. Consequently, $\lambda \in \bigcup_{\mathcal{C}_\varepsilon(X)} \sigma(A+C)$. □

Example 1.15. Let $a, b \in \mathbb{Q}_p$ with $a \neq b$. If

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

then $\sigma(A) = \{a, b\}$ and

$$\|A-\lambda I\| = \max\{|a-\lambda|,\ |b-\lambda|\}.$$

Hence

$$\|A-\lambda I\|\|(A-\lambda I)^{-1}\| = \max\Big\{\frac{|a-\lambda|}{|b-\lambda|},\ \frac{|b-\lambda|}{|a-\lambda|}\Big\}.$$

Thus, the condition pseudospectrum of A is

$$\Lambda_\varepsilon(A) = \{a, b\} \cup \Big\{\lambda \in \mathbb{Q}_p :\ \frac{|a-\lambda|}{|b-\lambda|} > \frac{1}{\varepsilon}\Big\} \cup \Big\{\lambda \in \mathbb{Q}_p :\ \frac{|b-\lambda|}{|a-\lambda|} > \frac{1}{\varepsilon}\Big\}.$$

We have the following propositions.

Proposition 1.16. *Let $A \in \mathcal{M}_n(K)$ and for every $0 < \varepsilon < 1$ such that $\varepsilon < \|A-\lambda I\|$. Then,*

(1) $\lambda \in \Lambda_\varepsilon(A)$ if, and only if, $\lambda \in \sigma_{\varepsilon\|A-\lambda I\|}(A)$.

(2) $\lambda \in \sigma_\varepsilon(A)$ if, and only if, $\lambda \in \Lambda_{\frac{\varepsilon}{\|A-\lambda I\|}}(A)$.

Proof. (1) Let $\lambda \in \Lambda_\varepsilon(A)$, then $\lambda \in \sigma(A)$ and $\|(A-\lambda I)\|\|(A-\lambda I)^{-1}\| > \varepsilon^{-1}$. Hence $\lambda \in \sigma(A)$ *and* $\|(A-\lambda I)^{-1}\| > \frac{1}{\varepsilon\|(A-\lambda I)\|}$. Consequently, $\lambda \in \sigma_{\varepsilon\|A-\lambda I\|}(A)$. The converse is similar.

(2) Let $\lambda \in \sigma_\varepsilon(A)$, then, $\lambda \in \sigma(A)$ and $\|(A-\lambda I)^{-1}\| > \varepsilon^{-1}$. Thus

$$\lambda \in \sigma(A) \textit{ and } \|(A-\lambda I)\|\|(A-\lambda I)^{-1}\| > \frac{\|(A-\lambda I)\|}{\varepsilon}.$$

Then, $\lambda \in \sigma_{\frac{\varepsilon}{\|A-\lambda B\|}}(A)$. The converse is similar.

□

The following proposition is a particular case of [[2], Proposition 2.2 (iii)].

Proposition 1.17. *Let $A \in \mathcal{M}_n(K)$ and $\varepsilon > 0$. If $\alpha,\ \beta \in \mathbb{K}$, with $\beta \neq 0$ then $\Lambda_\varepsilon(\beta A + \alpha I) = \alpha + \beta\Lambda_\varepsilon(A)$.*

We have the following theorem.

Theorem 1.18. *Let $A, C, V \in \mathcal{M}_n(K)$ and V be invertible. If $C = V^{-1}AV$. Then, for all $0 < \varepsilon < 1$, $k = \|V^{-1}\|\|V\|$ and $0 < k^2\varepsilon < 1$, we have*

$$\Lambda_{\frac{\varepsilon}{k^2}}(C) \subseteq \Lambda_\varepsilon(A) \subseteq \Lambda_{k^2\varepsilon}(C).$$

Proof. Let $\lambda \in \Lambda_{\frac{\varepsilon}{k^2}}(C)$, we have $\lambda \in \sigma(C)\Big(= \sigma(A)\Big)$ and

$$\begin{aligned}
\frac{k^2}{\varepsilon} < \|C - \lambda I\|\|(C - \lambda I)^{-1}\| &= \|V^{-1}\Big(A - \lambda I\Big)V\|\|V^{-1}\Big(A - \lambda I\Big)^{-1}V\| \\
&\leq \Big(\|V\|\|V^{-1}\|\Big)^2\|\Big(A - \lambda I\Big)\|\|\Big(A - \lambda I\Big)^{-1}\| \\
&\leq k^2\|\Big(A - \lambda I\Big)\|\|\Big(A - \lambda I\Big)^{-1}\|.
\end{aligned}$$

Or, $k^2 > 0$, then, $\lambda \in \Lambda_\varepsilon(A, B)$. Hence, $\Lambda_{\frac{\varepsilon}{k^2}}(C) \subseteq \Lambda_\varepsilon(A)$.
Let $\lambda \in \Lambda_\varepsilon(A)$. Then, $\lambda \in \sigma(C)\Big(= \sigma(A)\Big)$ and

$$\begin{aligned}
\frac{1}{\varepsilon} < \|A - \lambda I\|\|(A - \lambda I)^{-1}\| &= \|V\Big(C - \lambda I\Big)V^{-1}\|\|V\Big(C - \lambda I\Big)^{-1}V^{-1}\| \\
&\leq \Big(\|V\|\|V^{-1}\|\Big)^2\|\Big(C - \lambda I\Big)\|\|\Big(C - \lambda I\Big)^{-1}\| \\
&\leq k^2\|\Big(C - \lambda I\Big)\|\|\Big(C - \lambda I\Big)^{-1}\|.
\end{aligned}$$

Hence, $\lambda \in \Lambda_{k^2\varepsilon}(C,\ B)$. Consequently $\Lambda_\varepsilon(A, B) \subseteq \Lambda_{k^2\varepsilon}(C, B)$. □

For $B = I$, we have the following example.

Example 1.19. Let $A \in B(K^n)$ be diagonal operator such that for all $i \in \{1, \cdots, n\}$, $Ae_i = a_ie_i$, with $(a_i)_{i\in\{1,\cdots,n\}} \subset \mathbb{Q}_p$. Then

$$\sigma(A) = \{a_i :\ i \in \{1, \cdots, n\}\}$$

and for all $\lambda \in \rho(A)$, we have:

$$\begin{aligned}
\|(A - \lambda)^{-1}\| &= \sup_{i\in\{1,\cdots,n\}} \frac{\|(A - \lambda)^{-1}e_i\|}{\|e_i\|} \\
&= \sup_{i\in\{1,\cdots,n\}} \left|\frac{1}{a_i - \lambda}\right| = \frac{1}{\inf\limits_{i\in\{1,\cdots,n\}} |a_i - \lambda|}.
\end{aligned}$$

Hence, for all $i, j \in \{1, \cdots, n\}$,

$$\left\{\lambda \in \mathbb{Q}_p : \|(A - \lambda)\|\|(A - \lambda)^{-1}\| > \frac{1}{\varepsilon}\right\} = \bigcup_{i,\ j\ i\neq j} \left\{\lambda \in \mathbb{Q}_p : \frac{|a_i - \lambda|}{|a_j - \lambda|} > \frac{1}{\varepsilon}\right\}.$$

Consequently,

$$\sigma_\varepsilon(A) = \{a_i :\ i \in \{1, \cdots, n\}\} \cup \bigcup_{i,\ j\ i\neq j} \left\{\lambda \in \mathbb{Q}_p : \frac{|a_i - \lambda|}{|a_j - \lambda|} > \frac{1}{\varepsilon}\right\}.$$

We introduce the following definitions.

Definition 1.20. Let $A, B \in \mathcal{M}_n(K)$, the spectrum $\sigma(A, B)$ of matrix pencil (A, B) or of the pair (A, B) is defined by

$$\begin{aligned} \sigma(A, B) &= \{\lambda \in K : \ A - \lambda B \textit{ is not invertible in } \mathcal{L}(X)\}, \\ &= \{\lambda \in K : \ 0 \in \sigma(A - \lambda B)\}. \end{aligned}$$

The resolvent set $\rho(A, B)$ of matrix pencil (A, B) is the complement of $\sigma(A, B)$ in $\mathbb{K}$ given by

$$\rho(A, B) = \{\lambda \in K : \ R_\lambda(A, B) = (A - \lambda B)^{-1} \textit{ exists in } \mathcal{M}_n(K)\}.$$

$R_\lambda(A, B)$ is called the resolvent of matrix pencil (A, B).

Definition 1.21. Let $A, B \in \mathcal{L}(X)$, the couple (A, B) is said to be regular, if $\rho(A, B) \neq \emptyset$.

For a regular couple (A, B), we have

Definition 1.22. Let $A, B \in \mathcal{M}_n(K)$ and $\varepsilon > 0$. The pseudospectrum of matrix pencil (A, B) on X is defined by

$$\sigma_\epsilon(A, B) = \sigma(A, B) \cup \{\lambda \in K : \ \|(A - \lambda B)^{-1}\| > \varepsilon^{-1}\}.$$

The pseudoresolvent of matrix pencil (A, B) is denoted by

$$\rho_\epsilon(A, B) = \rho(A, B) \cap \{\lambda \in K : \ \|(A - \lambda B)^{-1}\| \leq \varepsilon^{-1}\},$$

by convention $\|(A - \lambda B)^{-1}\| = \infty$ if, and only if, $\lambda \in \sigma(A, B)$.

We introduce a new definition of pseudospectrum of matrix pencil in non-archimedean case.

Definition 1.23. Let $A, B \in \mathcal{M}_n(K)$ and $\varepsilon > 0$. The pseudospectrum of matrix pencil (A, B) on X is defined by

$$\Sigma_\epsilon(A, B) = \sigma(A, B) \cup \{\lambda \in K : \ \|(A - \lambda B)^{-1}B\| > \varepsilon^{-1}\}.$$

By convention $\|(A - \lambda B)^{-1}B\| = \infty$ if, and only if, $\lambda \in \sigma(A, B)$.

Remark 1.24.

(i) If $B = I$, then, $\Sigma_\varepsilon(A, I) = \sigma_\varepsilon(A)$ where $\sigma_\varepsilon(A)$ is pseudospectrum of A.

In the rest of this paper, we assume that (A, B) is regular. The next proposition gives a comparison between $\sigma_\varepsilon(A, B)$ and $\Sigma_\varepsilon(A, B)$.

Proposition 1.25. *Let $A, B \in \mathcal{M}_n(K)$. Then for all $\varepsilon > 0$,*

$$\Sigma_\varepsilon(A, B) \subset \sigma_{\varepsilon\|B\|}(A, B).$$

Proof. Let $\varepsilon > 0$ and $\lambda \in \Sigma_\varepsilon(A, B)$, then $\lambda \in \sigma(A, B)$ and

$$\begin{aligned} \frac{1}{\varepsilon} &< \|(A - \lambda B)^{-1}B\| && (1.4) \\ &\leq \|(A - \lambda B)^{-1}\|\|B\|. && (1.5) \end{aligned}$$

Hence

$$\frac{1}{\|B\|\varepsilon} < \|(A - \lambda B)^{-1}\|.$$

Thus $\lambda \in \sigma_{\varepsilon\|B\|}(A, B)$. Consequently

$$\Sigma_\varepsilon(A, B) \subset \sigma_{\varepsilon\|B\|}(A, B).$$

□

We have the following lemma.

Lemma 1.26. *Let $A, B \in \mathcal{M}_n(K)$ such that $\|B\| = 1$. Then for all $\varepsilon > 0$,*

$$\Sigma_\varepsilon(A, B) \subset \sigma_\varepsilon(A, B).$$

We have the following theorem.

Theorem 1.27. *Let $A, B, C \in \mathcal{M}_n(K)$ such that C is invertible. Then*

(1) For all $\varepsilon > 0$, $\Sigma_\varepsilon(A, B) = \Sigma_\varepsilon(CA, CB)$.

(2) For all $\varepsilon > 0$, $\Sigma_\varepsilon(A, C) = \sigma_\varepsilon(C^{-1}A)$. In particular $C = I$, $\Sigma_\varepsilon(A, I) = \sigma_\varepsilon(A)$.

Proof. (1) For all $\lambda \in \rho(A, B)$, we have $(A-\lambda B)^{-1}C^{-1} = (CA-\lambda CB)^{-1}$. Then, $\sigma(A,\ B) = \sigma(CA,\ CB)$. In addition, it is clear that

$$\|(CA - \lambda CB)^{-1}CB\| = \|(A - \lambda B)^{-1}B\|. \tag{1.6}$$

Hence $\lambda \in \Sigma_\varepsilon(A,\ B)$, if, and only if, $\lambda \in \Sigma_\varepsilon(CA,\ CB)$.

(2) Assume that C is invertible, then $(A - \lambda C)^{-1}C = (C^{-1}A - \lambda I)^{-1}$. Then $\lambda \in \Sigma_\varepsilon(A,\ C)$, if and only if $\lambda \in \sigma_\varepsilon(C^{-1}A)$.

□

Proposition 1.28. *Let $A, B \in \mathcal{M}_n(K)$. For all $\varepsilon > 0$, we have*

(i) $\sigma(A, B) = \bigcap_{\varepsilon>0} \Sigma_\varepsilon(A, B)$.

(ii) If $0 < \varepsilon_1 < \varepsilon_2$, then $\sigma(A, B) \subset \Sigma_{\varepsilon_1}(A, B) \subset \Sigma_{\varepsilon_2}(A, B)$.

Proof. (i) By Definition 1.23, we have for all $\varepsilon > 0$, $\sigma(A,\ B) \subset \Sigma_\varepsilon(A,\ B)$. Conversely, if $\lambda \in \bigcap_{\varepsilon>0} \Sigma_\varepsilon(A,\ B)$, then for all $\varepsilon > 0$, $\lambda \in \Sigma_\varepsilon(A,\ B)$. If $\lambda \notin \sigma(A,\ B)$, then $\lambda \in \{\lambda \in K :\ \|(A - \lambda B)^{-1}B\| > \varepsilon^{-1}\}$, taking limits as $\varepsilon \to 0^+$, we get $\|(A - \lambda B)^{-1}B\| = \infty$. Thus $\lambda \in \sigma(A,\ B)$.

(ii) For $0 < \varepsilon_1 < \varepsilon_2$. Let $\lambda \in \sigma_{\varepsilon_1}(A,\ B)$, then $\|(A - \lambda B)^{-1}B\| > \varepsilon_1^{-1} > \varepsilon_2^{-1}$. Hence $\lambda \in \sigma_{\varepsilon_2}(A,\ B)$.

□

We have the following examples.

Example 1.29. Let $K = \mathbb{Q}_p$.

(i) If

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

One can see that for all $\lambda \in \mathbb{Q}_p$, $\det(A - \lambda B) = -2\lambda$, then $\sigma(A, B) = \{0\}$. Also we have

$$(A - \lambda B)^{-1}B = \begin{pmatrix} \frac{-1}{\lambda} & 0 \\ 0 & 0 \end{pmatrix},$$

thus, for all $\varepsilon > 0$, $\Sigma_\varepsilon(A, B) = \{0\} \cup \{\lambda \in \mathbb{Q}_p : |\lambda|_p < \varepsilon\}$.

(ii) If

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

Then, for all $\lambda \in \mathbb{Q}_p$, $\det(A - \lambda B) = (1-\lambda)(1-2\lambda)$, hence $\sigma(A,B) = \{\frac{1}{2}, 1\}$ and

$$\|(A-\lambda B)^{-1}B\| = \max\left\{\frac{1}{|\lambda - 1|}, \frac{1}{|\lambda-1||1-2\lambda|}, \frac{1}{|1-2\lambda|}\right\}.$$

Hence, the pseudospectrum of (A, B) is

$$\Sigma_\varepsilon(A,B) = \{\frac{1}{2}, 1\} \cup \left\{\lambda \in \mathbb{Q}_p : \max\left\{\frac{1}{|\lambda - 1|}, \frac{1}{|\lambda-1||1-2\lambda|}, \frac{1}{|1-2\lambda|}\right\} > \frac{1}{\varepsilon}\right\}.$$

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