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Correction on General Convergence Analysis for Two-Step Projection Methods and Applications to Variational Problems

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Abstract

The aim of this study is to illustrate that the main result of the paper [1] is incorrect by giving an counter-example. I also present and study a new algorithm 4.1 to correct the main result of [1]. The possible impact of this study is rather important, it puts a question mark on results in all references that have been cited This publication (203 times just in Google Scholar alone).

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1. Introduction

Variational inequalities play an important role in studying many valuable problems arising in physical, medical images , industry, economics, and so on (see [2, 3, 4, 5] among others). A large class of problem in fluid mechanic, boundary value problem, transportation and equilibrium problems can be studied by variational inequalities is another beneficial of variational inequalities. Verma in [1] consider a system of two nonlinear variational inequality (abbreviated as SNVI) problems as follows: determine elements $x^*, y^* \in K$ such that :

$$\begin{cases} \langle \rho T(y^*) + x^* - y^*, v - x^* \rangle \geq 0, \forall v \in K, \rho > 0 \\ \langle \eta T(x^*) + y^* - x^*, v - y^* \rangle \geq 0, \forall v \in K, \eta > 0 \end{cases} \tag{1.1}$$

In this notes new approximation schemes (Algorithm 4.1) are discussed for solving the problem (SNVI). The results obtained in the paper correct the main results in [1].

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2. Preliminaries

Throughout this paper we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. We recall:

Definition 2.1. A mapping $T : H \to H$ is called λ-Lipschitz continous if there exists a constant $\lambda > 0$, such that :

$$\forall x, y \in H : \|T(x) - T(y)\| \leq \lambda \|x - y\|.$$

Definition 2.2. A mapping $T : H \to H$ is called r-strongly monotonic if there exists a constant $r > 0$, such that :

$$\forall x, y \in H : \|T(x) - T(y)\| \geq r \|x - y\|^2.$$

Proposition 2.3. *Let K be a closed convex set in H, for given an element $z \in H$, $x \in K$ satisfies the inequality*

$$\langle x - z, y - x \rangle \geq 0, \forall y \in K$$

if and only if

$$x = P_K(z)$$

where P_K is a projection of H into K.

It is known that P_K is a nonexpansive mapping, i.e

$$\|P_K(x) - P_K(y)\| \leq \|x - y\|, \forall x, y \in H.$$

Using Proposition 2.3, we can easily show that, finding the solution $(x^*, y^*) \in K \times K$ of (1.1) is equivalent to finding $(x^*, y^*) \in K \times K$ such that

$$\begin{cases} x^* = (1 - \alpha_n) x^* + \alpha_n P_K [y^* - \rho T(y^*)] \\ y^* = P_K [x^* - \eta T(x^*)] \end{cases}$$

where $\alpha_n \in [0, 1]$ for all $n \geq 0$.
In the following section, we show that the results of Verma in [1] are incorrect.

3. About Verma's Paper

Verma used the following iterative algorithm for solving the problem (SNVI) [Algorithm 3.1 in [1]].
Algorithm 3.1. For arbitrary chosen initial points $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ using

$$\begin{cases} x_{n+1} = (1 - \alpha_n) x_n + \alpha_n P_K [y_n - \rho T(y_n)] \\ y_n = (1 - \beta_n) x_n + \beta_n P_K [x_n - \eta T(x_n)] \end{cases}$$

where $\alpha_n, \beta_n \in [0, 1]$ for all $n \geq 0$.

Theorem 3.1. *[Theorem 3.1 in [1]] Let H be a real Hilbert space and K a nonempty closed convex subset of H. Let $T : K \to H$ be strongly r-monotonic and μ-Lipschitz continous. Suppose that $x^*, y^* \in K$ form a solution to the SNVI problem. If*

$$\begin{cases} 0 < \rho < \dfrac{2r}{\mu^2} \\ 0 < \eta < \dfrac{2r}{\mu^2} \end{cases}$$

and $\alpha_n, \beta_n \in [0, 1]$, $\sum\limits_{n=0}^{\infty} \alpha_n \beta_n = \infty$, then for arbitrarily chosen initial points $x_0, y_0 \in K$, x_n and y_n obtained from Algorithm 3.1 converge strongly to x^* and y^* respectively.

Commentaries

The sequence y_n does not converge to y^* because:
If we take:

$$\begin{cases} 0 < \rho < \dfrac{2r}{\mu^2} \\ 0 < \eta < \dfrac{2r}{\mu^2} \\ \alpha_n = \beta_n = \dfrac{1}{2} \end{cases}$$

It is clear that $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$, and

$$\begin{cases} x_{n+1} = \frac{1}{2} x_n + \frac{1}{2} P_K \left[y_n - \rho T \left(y_n \right) \right] \\ y_n = \frac{1}{2} x_n + \frac{1}{2} P_K \left[x_n - \eta T \left(x_n \right) \right] \end{cases}$$

By using Theorem (3.1), we obtian:

$$\begin{cases} x^* = \frac{1}{2} x^* + \frac{1}{2} P_K \left[y^* - \rho T \left(y^* \right) \right] \\ y^* = \frac{1}{2} x^* + \frac{1}{2} P_K \left[x^* - \eta T \left(x^* \right) \right] \end{cases}$$

Which is equivalent to,

$$\begin{cases} x^* = P_K \left[y^* - \rho T \left(y^* \right) \right] \\ 2y^* - x^* = P_K \left[x^* - \eta T \left(x^* \right) \right] \end{cases}$$

Using Proposition (2.3), we arrive at

$$\begin{cases} \left\langle \rho T \left(y^* \right) + x^* - y^*, v - x^* \right\rangle \geq 0, \forall v \in K, \\ \left\langle \eta T \left(x^* \right) + 2y^* - 2x^*, v - 2y^* + x^* \right\rangle \geq 0, \forall v \in K. \end{cases}$$

Which is not the same problem SNVI.

Remark 3.2. Let us consider the following text quoted from the proof of (Theorem 3.1 in [1]):
Similary, we have

$$\begin{aligned} \left\| y_k - y^* \right\| &= \left\| \left(1 - \beta_k \right) \left(x_k - x^* \right) + \beta_n P_K \left[x_k - \eta T \left(x_k \right) \right] - \beta_n P_K \left[x^* - \eta T \left(x^* \right) \right] \right\| \\ &\leq \left(1 - \beta_k \right) \left\| x_k - x^* \right\| + \beta_k \left\| \left[x_k - x^* \right] - \eta \left[T \left(x_k \right) - T \left(x^* \right) \right] \right\| \\ &\leq \left(1 - \beta_k \right) \left\| x_k - x^* \right\| + \beta_k \left[1 - 2\eta r + \left(\eta \mu \right)^2 \right]^{\frac{1}{2}} \left\| x_k - x^* \right\| \\ &= \left(1 - \beta_k \right) \left\| x_k - x^* \right\| + \beta_k \sigma \left\| x_k - x^* \right\|, \end{aligned}$$

where $\sigma = \left[1 - 2\eta r + \left(\eta \mu \right)^2 \right]^{\frac{1}{2}} < 1$.

This remark implies that the mistake is not a typo.

4. Main result

Now we suggest and analyze the following iterative method for solving (1.1).

Algorithm 4.1. For arbitrary chosen initial points $x_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ using

$$\begin{cases} x_{n+1} = \left(1 - \alpha_n \right) x_n + \alpha_n P_K \left[y_n - \rho T \left(y_n \right) \right] \\ y_n = P_K \left[x_n - \eta T \left(x_n \right) \right] \end{cases}$$

where $\alpha_n \in [0, 1]$ for all $n \geq 0$.

Theorem 4.1. *Let (x^*, y^*) be the solution of (1.1). Suppose that $T : H \to H$ be strongly $r-$monotonic and $\mu-$Lipschitz continuous. If*

$$\begin{cases} 0 < \rho < \dfrac{2r}{\mu^2} \\ 0 < \eta < \dfrac{2r}{\mu^2} \end{cases} \tag{4.1}$$

and $\alpha_n \in [0,1]$, $\sum\limits_{n=0}^{\infty} \alpha_k = \infty$, then for arbitrarily chosen initial points $x_0 \in K$, x_n and y_n obtained from Algorithm 4.1 converge strongly to x^* and y^* respectively.

Proof. To prove the result, we need first to evaluate $\|x_{n+1} - x^*\|$ for all $n \geq 0$.

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\alpha_n)\, x_n + \alpha_n P_K [y_n - \rho T(y_n)] - (1-\alpha_n)\, x^* + \alpha_n P_K [y^* - \rho T(y^*)]\| \\ &\leq (1-\alpha_n)\, \|x_n - x^*\| + \alpha_n \|P_K [y_n - \rho T(y_n)] - P_K [y^* - \rho T(y^*)]\| \\ &\leq (1-\alpha_n)\, \|x_n - x^*\| + \alpha_n \|[y_n - y^*] - \rho [T(y_n) - T(y^*)]\| \end{aligned}$$

Since T is $r-$strongly monotonic, we have :

$$\begin{aligned} \|y_n - y^* - \rho [T(y_n) - T(y^*)]\|^2 &= \|y_n - y^*\|^2 - 2\rho \langle T(y_n) - T(y^*), y_n - y^* \rangle \\ &\quad + \rho^2 \|T(y_n) - T(y^*)\|^2 \\ &\leq -2\rho \left[r \|y_n - y^*\|^2 \right] + \|y_n - y^*\|^2 + \rho^2 \|T(y_n) - T(y^*)\|^2 \end{aligned}$$

Using the fact that T is Lipschitzian, we have:

$$\|y_n - y^* - \rho [T(y_n) - T(y^*)]\|^2 \leq \left[1 - 2\rho r + \rho^2 \mu^2\right] \|y_n - y^*\|^2$$

As a result, we have:

$$\|x_{n+1} - x^*\| \leq (1-\alpha_n)\, \|x_n - x^*\| + \alpha_n \theta_1 \|y_n - y^*\| \tag{4.2}$$

where $\theta_1 = \left[1 - 2\rho r + \rho^2 \mu^2\right]^{\frac{1}{2}}$
Now we evaluate $\|y_n - y^*\|$ for all $n \geq 0$.

$$\begin{aligned} \|y_n - y^*\| &= \|P_K [x_n - \eta T(x_n)] - P_K [x^* - \eta T(x^*)]\| \\ &\leq \|[x_n - x^*] - \eta [T(x_n) - T(x^*)]\| \end{aligned}$$

Similary, Since T is $r-$strongly and $\mu-$Lipschitz continuous mapping, we obtain :

$$\|y_n - y^*\| \leq \theta_2 \|x_n - x^*\|. \tag{4.3}$$

where $\theta_2 = \left[1 - 2\eta r + \eta^2 \mu^2\right]^{\frac{1}{2}}$
From assumption (4.1), it is clear that $\theta_1 < 1$ and $\theta_2 < 1$. From (4.2) and (4.3) It follows that,

$$\|x_{n+1} - x^*\| \leq (1-\alpha_n)\, \|x_n - x^*\| + \alpha_n \theta_1 \theta_2 \|x_n - x^*\|$$

wich implies that:

$$\|x_{n+1} - x^*\| \leq \prod_{k=0}^{k=n} (1 - (1 - \theta_1 \theta_2)\, \alpha_k)\, \|x_0 - x^*\|$$

Since $0 < \theta_1 \theta_2 < 1$ and $\sum\limits_{k=0}^{\infty} \alpha_k = \infty$ it implies in light of [6] that

$\lim\limits_{n \to +\infty} \prod\limits_{k=0}^{k=n} ((1 - (1 - \theta_1 \theta_2)\, \alpha_k)) = 0$ therefore $x_n \to x^*$ and $y_n \to y^*$. □

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