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Fuzzy L^p-Spaces

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Abstract

The purpose of this paper is to introduce the fuzzy L^p-Spaces. We give some basic definitions and main properties of fuzzy spaces. The fuzzy Holder's inequality will be proved. Also we show that the dual of fuzzy L^p-spaces is fuzzy L^q-spaces, where the scalars p and q are conjugate exponents.

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1. Introduction and Preliminaries

In many cases of the modeling of the real world phenomena, fuzzy initial value problems appear naturally, since information about the behavior of a dynamical system is uncertain. In order to obtain a more adequate model, we have to take into account these uncertainties [1]. So it is important to develop fuzzy mathematics and introduce the formulas for the fuzzy operations. More recently, the literature on fuzzy numbers has grown in terms of contributions to the fuzzy arithmetic operations. In general, the arithmetic operations on fuzzy numbers can be approached either by the direct use of the membership function (by the Zadeh extension principle) or by the equivalent use of the α-cuts representation (introduced by Goetschel and Voxman in [2]). By this approach, it is possible to define a parametric representation of fuzzy numbers that allows a large variety of possible shapes (types of membership functions) and is very simple to implement, with the advantage of obtaining easily a much wider set for standard model the lower and upper extremal values of the α-cuts.
As we know that the subject of classical L^p-spaces have been extensively studied. They are special examples of Banach lattices. Several techniques have been applied to prove existence of solution of integral equations on L^p-spaces. In [3], the authors presented existence of solutions of integral equations of convolution type on L^p-spaces. In [4], existence and finding an approximation of a continuous solution of nonlinear integral equations of Volterra types is proposed. Also in [5] and [6], authors studied fuzzy metric spaces.

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In this work, we shall introduce the fuzzy L^p-spaces by using the α-cuts of fuzzy numbers and by using fuzzy integration. The organization of the paper is as follows: Section 2 contains a brief explanation about the fuzzy numbers and the notations that we will use. In Section 3 we introduce the fuzzy L^p-spaces, some basic definitions and main properties of these spaces is given in details and fuzzy Holder's inequality, is proved. Also the dual of these spaces are considered.

2. Preliminaries

In this section, the most basic notations, which are used in this paper will be introduced. We start with the essential concepts of fuzzy set theory. The notion of a fuzzy set is an extension of the classical notion of a set. In classical set theory, an element either belongs or does not belong to a given set. By contrast, in fuzzy set theory, an element has a degree of membership, which is a real number from $[0,1]$, in a given fuzzy set.
In the following definition the concept of fuzzy set is given:

Definition 2.1. The fuzzy set $\tilde{A}$ in X is the set of ordered pairs $\tilde{A} = \{(x,\mu)|x \in X\}$, where $\mu : X \to [0,1]$ is called the membership function associated with the fuzzy set $\tilde{A}$. The values of μ represents the degree of membership of x in $\tilde{A}$.

We now recall the definition of a fuzzy number.

Definition 2.2. A fuzzy number is a function $u : \mathbb{R} \to [0,1]$ satisfying the following properties.

a. u is upper semicontinuous on $\mathbb{R}$,

b. $u(x) = 0$ outside of some interval $[c,d]$,

c. there are the real numbers $a,b : c \leq a \leq b \leq d$, such that u is increasing on $[c,a]$, decreasing on $[b,d]$ and $u(x) = 1$ for each $x \in [a,b]$,

d. u is fuzzy convex set (that is $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}, \quad \forall x,y \in \mathbb{R}, \lambda \in [0,1]$).

The set of all fuzzy numbers is denoted by $\mathbb{R}_F$.
Fuzzy number can also be represented via their α-cut as follows:

Definition 2.3. For any $u \in \mathbb{R}_F$ the α-cut set of u, is denoted by $[u]^\alpha$, and is defined by $[u]^\alpha = \{x \in \mathbb{R} \mid u(x) \geq \alpha\}$, where $0 \leq \alpha \leq 1$. The notation,

$$[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha], \quad \alpha \in [0,1],$$

denotes the lower and upper extremal value of the α-cuts, in the other words

$$\underline{u}^\alpha = \min[u]^\alpha, \quad \overline{u}^\alpha = \max[u]^\alpha.$$

An arbitrary fuzzy number u is represented, in parametric form, by an ordered pair of functions $u = (\underline{u}, \overline{u})$, which define the end points of the α-cut, satisfying the three conditions:

a. $\underline{u}$ is a bounded non-decreasing left continuous function on $[0,1]$,

b. $\overline{u}$ is a bounded non-increasing left continuous function on $[0,1]$,

c. $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

For arbitrary $u = (\underline{u}, \overline{u})$, $v = (\underline{v}, \overline{v})$, we define addition $(u+v)$ and multiplication by real k as

$$(\underline{u+v})(r) = \underline{u}(r) + \underline{v}(r), \quad (\overline{u+v})(r) = \overline{u}(r) + \overline{v}(r),$$

$$ku = (k\underline{u}, k\overline{u}), \quad k \geq 0, \quad ku = (k\overline{u}, k\underline{u}), \quad k \leq 0.$$

Definition 2.4. For arbitrary fuzzy number $u = (\underline{u}, \overline{u})$, $v = (\underline{v}, \overline{v})$ the Hausdorff distance between these fuzzy numbers given by $D : \mathbb{R}_F \times \mathbb{R}_F \to R_+ \cup \{0\}$

$$D(u, v) = \sup_{r\in[0,1]} \max\{| \underline{u}(r) - \underline{v}(r) |, | \overline{u}(r) - \overline{v}(r) |\},$$

where D is a metric on $\mathbb{R}_F$ and has the following properties(see [7]).

a. $D(u \oplus w, v \oplus w) = D(u, v), \quad \forall u, v, w \in \mathbb{R}_F,$

b. $D(k \odot u, k \odot v) =| k | D(u, v), \forall k \in \mathbb{R}, u, v \in \mathbb{R}_F,$

c. $D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e), \forall u, v, w, e \in \mathbb{R}_F,$

d. $(\mathbb{R}_F, D)$ is a complete metric space.

Definition 2.5. The function $f : T \subseteq \mathbb{R} \to \mathbb{R}_F$ is called a fuzzy function, and the α-cut set of f is represented by

$$f(t, \alpha) = [\underline{f}(t, \alpha), \overline{f}(t, \alpha)], \quad \alpha \in [0, 1], t \in T,$$

where $\underline{f}(t, \alpha) = \underline{f(t)^\alpha}$, $\overline{f}(t, \alpha) = \overline{f(t)}^\alpha$.

A fuzzy function may have fuzzy domain and fuzzy range. So the function $f : \mathbb{R}_F \to \mathbb{R}_F$ is also a fuzzy function. More details about the fuzzy function can be seen in [2].

Definition 2.6. Let $f : \mathbb{R} \to \mathbb{R}_F$ be a fuzzy function. We say that f is continuous at $t_0 \in \mathbb{R}$, where we consider the natural metric on $\mathbb{R}$ and the defined Hausdorff distance on $\mathbb{R}_F$ as above, i.e. if for an arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that

$$| t - t_0 |< \delta \Rightarrow D(f(t), f(t_0)) < \epsilon.$$

Proposition 2.7. *The fuzzy function $f : \mathbb{R} \to \mathbb{R}_F$ is continuous if and only if $\underline{f}, \overline{f}$ are uniformly continuous functions respect to α.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}_F$ be a fuzzy continuous function at a point $t_0 \in \mathbb{R}$, according to the definition, for an arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that

$$| t - t_0 |< \delta \Rightarrow D(f(t), f(t_0)) < \epsilon,$$

$$D(f(t), f(t_0)) = \sup_{\alpha\in[0,1]} \max\{| \underline{f}(t, \alpha) - \underline{f}(t_0, \alpha) |, | \overline{f}(t, \alpha) - \overline{f}(t_0, \alpha) |\},$$

so we have,

$$\sup_{\alpha\in[0,1]} | \underline{f}(t, \alpha) - \underline{f}(t_0, \alpha) | \leq D(f(t), f(t_0)),$$

and

$$\sup_{\alpha\in[0,1]} | \overline{f}(t, \alpha) - \overline{f}(t_0, \alpha) | \leq D(f(t), f(t_0)).$$

And so $\underline{f}$ and similarly $\overline{f}$ are uniformly continuous functions on $\{(t_0, \alpha)|\alpha \in [0, 1]\}$.
Now let $\underline{f}, \overline{f}$ are uniformly continuous functions on $\{(t_0, \alpha)|\alpha \in [0, 1]\}$. So for each $\varepsilon > 0$ there exists $\delta > 0$ such that,

$$| t - t_0 |< \delta \Rightarrow \sup_{\alpha\in[0,1]} | \underline{f}(t, \alpha) - \underline{f}(t_0, \alpha) |< \epsilon, \sup_{\alpha\in[0,1]} | \overline{f}(t, \alpha) - \overline{f}(t_0, \alpha) |< \epsilon.$$

So,

$$\max(\sup_{\alpha\in[0,1]} | \underline{f}(t, \alpha) - \underline{f}(t_0, \alpha) |, \sup_{\alpha\in[0,1]} | \overline{f}(t, \alpha) - \overline{f}(t_0, \alpha) |) < \epsilon.$$

So we have,

$$D(f(t), f(t_0)) < \epsilon,$$

and the proof is complete. □

Definition 2.8. The fuzzy function $f : \mathbb{R} \to \mathbb{R}_F$ is called to be fuzzy bounded, if there exists $M > 0$ such that $\|f\|_{F.u} := \sup_{u\in\mathbb{R}} D(f(u), \hat{0}) \leq M$.

Proposition 2.9. *Let $f : [a, b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}_F$ be a fuzzy continuous function. Then it is fuzzy bounded.*

Boundedness of a fuzzy continuous function is trivial by the primary concepts of the metric spaces. In the following we consider the concept of integral of a fuzzy function.

Definition 2.10. Let $f : [a, b] \to \mathbb{R}_F$ be a fuzzy function. For each partition $p = \{x_1, x_2, \cdots, x_m\}$ of $[a, b]$ and for arbitrary $x_{i-1} \leq \xi_i \leq x_i$, $2 \leq i \leq m$, let $\mathbb{R}_P = \sum_{i=2}^{m} f(\xi_i)(x_i - x_{i-1})$. The define integral of f over $[a, b]$ is,

$$\int_a^b f(x,\alpha) = lim \mathbb{R}_P, \quad \max \mid x_i - x_{i-1} \mid \to 0,$$

provided that this limit exists in metric D.

If the function f is continuous, its define integral exists [2]. Furthermore:

$$\underline{\int_a^b f(x,\alpha)} = \int_a^b \underline{f}(x,\alpha),$$

and

$$\overline{\int_a^b f(x,\alpha)} = \int_a^b \overline{f}(x,\alpha),$$

More details about the properties of the fuzzy integral are given in [2].

3. Fuzzy L^p-spaces

In this section, we will introduce the fuzzy L^p-spaces for $1 \leq p \leq \infty$. Some basic definitions and main properties of these spaces will be given. The suitable norm and the dual of such spaces will be also considered.
Let X be a compact space, we define the set,

$$C_F(X) = \Big\{ f : X \longrightarrow \mathbb{R}_F;\ f \textit{ is continuous} \Big\}.$$

Since, for $f, g \in C_F(X)$ and $\alpha \in \mathbb{R}$, $\alpha f + g$ is continuous, we could say that $C_F(X)$ is a linear space.
we define the fuzzy uniform norm as follows,

$$\|f\|_{F\cdot u} = \sup_{\xi\in X} D(f(\xi), \hat{0}).$$

Since, each $f \in C_F(X)$ can be represented as $f = (\underline{f}, \overline{f})$ and as we proved before, the fuzzy function $f : X \to \mathbb{R}_F$ is uniformly continuous, if and only if $\underline{f}, \overline{f}$ are uniformly continuous functions on $X \times [0, 1]$. So,

$$\begin{aligned}
\|f\|_{F\cdot u} &= \sup_{\xi\in X} D(f(\xi), \hat{0}) \\
&= \sup_{\xi\in X} \sup_{\alpha\in[0,1]} \max\{\mid \underline{f}(\xi,\alpha) - 0) \mid, \mid \overline{f}(\xi,\alpha) - 0) \mid\} \\
&= \sup_{\xi\in X} \sup_{\alpha\in[0,1]} \max\{\mid \underline{f}(\xi,\alpha) \mid, \mid \overline{f}(\xi,\alpha) \mid\} \\
&= \max\{(\|\underline{f}\|_u, \|\overline{f}\|_u)\}.
\end{aligned}$$

Thus, we have,

$$C_F(X) = C(X \times [0, 1]) \oplus^{c_0} C(X \times [0, 1]).$$

In the next theorem we show that $C_F(X)$ is a Banach space.

Theorem 3.1. *$(C_F(X), \|.\|_{F\cdot u})$ is a Banach space.*

Proof. The proof of this theorem has been given in [8]. □

Similarly, for a locally compact space X, we define the set,

$$C_{c,F}(X) = \{f = (\underline{f}, \overline{f}), f : X \to \mathbb{R}_F, \underline{f}, \overline{f} \in C_c(X \times [0, 1])\},$$

where $C_c(X \times [0, 1])$ denotes all continuous compact support functions on $X \times [0, 1]$.
In the following the fuzzy L^p-spaces for $1 \le p \le \infty$ are introduced. Throughout this section (X, μ) is a Borel σ-finite measure space and λ is a Lebesgue measure on [0,1].
For $1 \le p \le \infty$ consider the set,

$$L^p_F((X, \mu)) = \{f = (\underline{f}, \overline{f}), f : X \to \mathbb{R}_F, \underline{f}, \overline{f} \in L^p(\mu \times \lambda)\}.$$

It is easy to see that $L^p_F(X, \mu)$ for $1 \le p \le \infty$ is a vector space. Now we put norm as follows:
For $1 \le p < \infty$ the fuzzy norm as follows:

$$\|f\|_{F\cdot p} = (\|\underline{f}\|_p^p + \|\overline{f}\|_p^p)^{\frac{1}{p}},$$

where,

$$\|\underline{f}\|_p = \left(\int |\underline{f}|^p d(\mu \times \lambda)\right)^{\frac{1}{p}},$$

and

$$\|\overline{f}\|_p = \left(\int |\overline{f}|^p d(\mu \times \lambda)\right)^{\frac{1}{p}}.$$

For $p = \infty$, we define,

$$\|f\|_{F\cdot\infty} = max\{(\|\underline{f}\|_\infty, \|\overline{f}\|_\infty)\},$$

we can see that they are the normed vector spaces of fuzzy functionnand according to the definition it is obvious that,

$$L^p_F(X, \mu) = L^p(X \times [0, 1]) \oplus^{l^p} L^p(X \times [0, 1]).$$

$\| \cdot \|_{F\cdot p}$ for $1 \le p \le \infty$ has the properties of a usual norm on $\mathbb{R}_F$, i.e. $\|f\|_{F\cdot p} \ge 0$, $\|\lambda f\|_{F\cdot p} = |\lambda|\|f\|_{F\cdot p}$ and $\|f + g\|_{F\cdot p} \le \|f\|_{F\cdot p} + \|g\|_{F\cdot p}$ for any $f, g \in L^p_F(X, \mu)$.
So with the definition of norm as above we have,

$$\begin{aligned}(\|\underline{f}\|_p^p + \|\overline{f}\|_p^p)^{\frac{1}{p}} &\le (\max\{(\|\underline{f}\|_p, \|\overline{f}\|_p)\}^p + \max\{(\|\overline{f}\|_p, \|\underline{f}\|_p)\}^p)^{\frac{1}{p}} \\ &= (2 \max\{(\|\underline{f}\|_p, \|\overline{f}\|_p)\}^p)^{\frac{1}{p}} \\ &= 2^{\frac{1}{p}}(\max\{(\|\underline{f}\|_p, \|\overline{f}\|_p)\}).\end{aligned}$$

So,

$$2^{\frac{-1}{p}}(\|\underline{f}\|_p^p + \|\overline{f}\|_p^p)^{\frac{1}{p}} \le max\{(\|\underline{f}\|_p, \|\overline{f}\|_p)\} \le (\|\underline{f}\|_p^p + \|\overline{f}\|_p^p)^{\frac{1}{p}}.$$

Now in the next theorem we show that for $1 \le p < \infty$, $C_{c,F}(X)$ in $\| \cdot \|_{F\cdot p}$ is dense in $L^p_F(X, \mu)$.

Theorem 3.2. *$\overline{(C_{c,F}(X), \| \cdot \|_{F\cdot u})} = L^p_F(X, \mu)$ in $\| \cdot \|_{F\cdot p}$*

Proof. Let $f \in L^p_F(X, \mu)$, then it can be represented as $f = (\underline{f}, \overline{f})$ such that $\underline{f}, \overline{f} \in L^p(\mu \times \lambda)$, so there exists a sequence of compact support functions $f_n = (\underline{f_n}, \overline{f_n})$, such that $\underline{f_n} \to \underline{f}$ *and* $\overline{f_n} \to \overline{f}$ in $\| \cdot \|_p$, so for $\varepsilon > 0$, there exists $n \in N$ such that for $n \ge N$ we have

$$\|\underline{f_n} - \underline{f}\|_p < \frac{\varepsilon}{2^{\frac{1}{p}}}, \qquad \|\overline{f_n} - \overline{f}\|_p < \frac{\varepsilon}{2^{\frac{1}{p}}}.$$

So we have,

$$\begin{aligned}\|f_n - f\|_{F.p} &= (\|\underline{f_n} - \underline{f}\|_p^p + \|\overline{f_n} - \overline{f}\|_p^p)^{\frac{1}{p}} \\ &< (\frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2})^{\frac{1}{p}} \\ &< \varepsilon.\end{aligned}$$

Consequently, $(\underline{f_n}, \overline{f_n}) \to (\underline{f}, \overline{f})$ with $\|\cdot\|_p$. So we infer that $L_F^p(X,\mu)$ is a complete normed space, so $(L_F^p(X,\mu), \|\cdot\|_{F.p})$ is a Banach space. □

Now in the following theorem we show that $L_F^\infty(X,\mu)$ is a Banach space, as well.

Theorem 3.3. *$(L_F^\infty(X,\mu), \|\cdot\|_{F.\infty})$ is a Banach space.*

Proof. Let $\{f_n = \underline{f}_n, \overline{f}_n\}$ be a cauchy sequence in $L_F^\infty(X,\mu)$. We claim that for some $f = (\underline{f}, \overline{f})$, $\lim \|f - f_n\|_{F.\infty} = 0$. As we know that $L^\infty(\mu \times \lambda)$ is a Banach space[9] so there exists some $\underline{f}, \overline{f} \in L^\infty(\mu \times \lambda)$ such that $\lim \|\underline{f} - \underline{f_n}\|_\infty = 0$ and $\lim \|\overline{f} - \overline{f_n}\|_\infty = 0$, since $\lim \| f - f_n\|_{F.\infty} = \max(\|\underline{f} - \underline{f_n}\|_\infty, \|\overline{f} - \overline{f_n}\|_\infty)$, this shows that $\lim \| f - f_n\|_{F.\infty} = 0$, and the proof of the theorem is complete. □

It is so important that the fuzzy L^p-spaces for $1 \le p \le \infty$, can be defined directly by fuzzy integration. Let f be a fuzzy function. So the fuzzy $\int \mid f \mid^p d\mu$ is defined, that is a fuzzy number and $D(\int \mid f \mid^p d\mu, \hat{0})^{\frac{1}{p}}$ can also be calculated. In the next theorem we show that if f is a fuzzy function in the fuzzy L^p-spaces, then $D(\int \mid f \mid^p d\mu, \hat{0})^{\frac{1}{p}}$ is finite, and vice versa.

Theorem 3.4. *$f \in L_F^p(X,\mu)$, if and only if $D(\int \mid f \mid^p d\mu, \hat{0})^{\frac{1}{p}} < \infty$.*

Proof. Let $f \in L_F^p(X,\mu)$. So f can be represented by $f = (\underline{f}, \overline{f})$ such that $\underline{f}, \overline{f} \in L^p(\mu \times \lambda)$, therefore $\left(\int |\underline{f}|^p d\mu \times \lambda\right)^{\frac{1}{p}} < \infty$ and $\left(\int |\overline{f}|^p d\mu \times \lambda\right)^{\frac{1}{p}} < \infty$. According to the definition the metric between fuzzy numbers we have,

$$D(\int \mid f \mid^p d\mu, \hat{0}) = \sup_{\alpha\in[0,1]} \max\{\int \mid \underline{f} \mid^p (x,\alpha) d\mu(x), \int \mid \overline{f} \mid^p (x,\alpha) d\mu(x)\}.$$

Now it is enough to show that $\sup_{\alpha\in[0,1]} \int \mid \underline{f} \mid^p (x,\alpha) d\mu(x) < \infty$. It is known that, if we take the integral of a function of two variable respect to one variable, then this integral is continuous respect to another variable so $\int \mid \underline{f} \mid^p (x,\alpha) d\mu(x), \int \mid \underline{f} \mid^p (x,\alpha) d\mu(x)$ are continuous respect to $\alpha \in [0,1]$ and by the compactness of $[0,1]$,

$$\sup_{\alpha\in[0,1]} \int \mid \underline{f} \mid^p (x,\alpha) d\mu(x) < \infty, \sup_{\alpha\in[0,1]} \int \mid \overline{f} \mid^p (x,\alpha) d\mu(x) < \infty.$$

And the result is achieved. Now let

$$D(\int \mid f \mid^p d\mu, \hat{0})^{\frac{1}{p}} < \infty,$$

so

$$\sup_{\alpha\in[0,1]} \int \mid \underline{f} \mid^p (x,\alpha) d\mu(x) < \infty, \sup_{\alpha\in[0,1]} \int \mid \overline{f} \mid^p (x,\alpha) d\mu(x) < \infty.$$

Also,

$$\int\int \mid \underline{f} \mid^p (x,\alpha) d\mu(x) d\lambda(\alpha) \le \sup_{\alpha\in[0,1]} \int \mid \underline{f} \mid^p (x,\alpha) d\mu(x).\lambda(\alpha) < \infty.$$

So, $\left(\int |\underline{f}|^p(x,\alpha) d(\mu \times \lambda)\right) < \infty$. Similarly, $\left(\int |\overline{f}|^p(x,\alpha) d(\mu \times \lambda)\right) < \infty$.
Consequently, $\underline{f}, \overline{f} \in L^p(\mu \times \lambda)$ and $f \in L_F^p(X,\mu)$. □

As we know that,

$$\|\underline{f}\|_p^p \leq D(\int \mid f \mid^p d\mu, \hat{0}),$$
$$\|\overline{f}\|_p^p \leq D(\int \mid f \mid^p d\mu, \hat{0}),$$
$$(\|\underline{f}\|_p^p + \|\overline{f}\|_p^p)^{\frac{1}{p}} \leq 2^{\frac{1}{p}} D(\int \mid f \mid^p d\mu, \hat{0})^{\frac{1}{p}},$$
$$\|f\|_{F.p} \leq 2^{\frac{1}{p}} D(\int \mid f \mid^p d\mu, \hat{0})^{\frac{1}{p}}$$
$$2^{\frac{-1}{p}} \|f\|_{F.p} \leq D(\int \mid f \mid^p d\mu, \hat{0})^{\frac{1}{p}}.$$

On the other hand,

$$D(\int \mid f \mid^p d\mu, \hat{0})^{\frac{1}{p}} \leq \|f\|_{F \cdot p}.$$

So we have,

$$2^{\frac{-1}{p}} \|f\|_{F.p} \leq D(\int \mid f \mid^p d\mu, \hat{0})^{\frac{1}{p}} \leq \|f\|_{F \cdot p}.$$

It means that the topology on the space of fuzzy integrable functions, inherited from the defined norm and the natural metric above, are the same.

3.1. Duality in fuzzy L^p-spaces

Riesz Representation Theorem is known on L_p-spaces. It asserts that if T is a bounded linear functional on L_p-space, $1 \leq p < \infty$, then there is a function $t \in L_q$, where $\frac{1}{p} + \frac{1}{q} = 1$, such that $T(s) = \int st$. In addition, $\|T\| = \|t\|_q$.The above integration are in the appropriate measure space. In slightly different terminology, the Riesz Representation Theorem states that the dual space of L_p-spaces are L_q-spaces. In this section, for $1 < p < \infty$, we discuss on the dual of fuzzy L^p-spaces. our claim is that the dual of fuzzy L^p-spaces is fuzzy L^q-spaces, for dual scalars $1 < p, q < \infty$.
In the next theorem an important inequality between L^p-norms in fuzzy spaces known as fuzzy Holder's inequality, is proved.

Lemma 3.5. *Let $1 < p < \infty$ and $1 < q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L_F^p(X, \mu)$ and $g \in L_F^q(X, \mu)$, then*

$$\int |\underline{f} \cdot \underline{g}| d(\mu \times \lambda) + \int |\overline{f}.\overline{g}| d(\mu \times \lambda) \leq \|f\|_{F \cdot p}.\|g\|_{F \cdot q}.$$

Proof. If $f = 0$ a.e. or $g = 0$ a.e. then the inequality is trivial. So let $f \neq 0$ a.e. and $g \neq 0$ a.e. Then $\|f\|_{F \cdot p} > 0$ and $\|g\|_{F \cdot q} > 0$. Since,

$$|\int \underline{f} \cdot \underline{g} d(\mu \times \lambda)| + |\int \overline{f} \cdot \overline{g} d(\mu \times \lambda)| \leq \int |\underline{f} \cdot \underline{g}| d(\mu \times \lambda) + \int |\overline{f} \cdot \overline{g}| d(\mu \times \lambda)$$
$$\leq \|\underline{f}\|_p \|\underline{g}\|_q + \|\overline{f}\|_p \|\overline{g}\|_q.$$

So we will get the result if we prove:

$$(\|\underline{f}\|_p \|\underline{g}\|_q + \|\overline{f}\|_p \|\overline{g}\|_q) \leq (\|\underline{f}\|_p^p + \|\overline{f}\|_p^p)^{\frac{1}{p}} . (\|\underline{g}\|_q^q + \|\overline{g}\|_q^q)^{\frac{1}{q}}.$$

Let $a_1 = \|\underline{f}\|_p$, $b_1 = \|g\|_q$, $a_2 = \|\overline{f}\|_p$ and $b_2 = \|\overline{g}\|_q$ on the other hand it is enough to prove that

$$a_1 b_1 + a_2 b_2 \leq (a_1^p + a_2^p)^{\frac{1}{p}} \cdot (b_1^q + b_2^q)^{\frac{1}{q}}, \quad (\frac{1}{p} + \frac{1}{q} = 1).$$

Using the Young's inequality, $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, put

$$\tilde{a_1} = \frac{a_1}{(a_1^p + a_2^p)^{\frac{1}{p}}}, \quad \tilde{a_2} = \frac{a_2}{(a_1^p + a_2^p)^{\frac{1}{p}}},$$
$$\tilde{b_1} = \frac{b_1}{(b_1^q + b_2^q)^{\frac{1}{q}}}, \quad \tilde{b_2} = \frac{b_2}{(b_1^q + b_2^q)^{\frac{1}{q}}},$$

so,

$$(\tilde{a_1})^p + (\tilde{a_2})^p = \frac{a_1^p}{(a_1^p + a_2^p)} + \frac{a_2^p}{(a_1^p + a_2^p)} = 1.$$

Similarly,

$$(\tilde{b_1})^q + (\tilde{b_2})^q = 1,$$

withthe Young's inequality,

$$\tilde{a_1}\tilde{b_1} \leq \frac{\tilde{a_1}^p}{p} + \frac{\tilde{b_1}^q}{q}, \quad \tilde{a_2}\tilde{b_2} \leq \frac{\tilde{a_2}^p}{p} + \frac{\tilde{b_2}^q}{q},$$

$$\tilde{a_1}\tilde{b_1} + \tilde{a_2}\tilde{b_2} \leq \frac{\tilde{a_1}^p + \tilde{a_2}^p}{p} + \frac{\tilde{b_1}^q + \tilde{b_2}^q}{q} = \frac{1}{p} + \frac{1}{q} = 1. \tag{3.1}$$

With substituting $\tilde{a_1}, \tilde{b_1}, \tilde{a_2}$ and $\tilde{b_2}$ in relation 3.1 we have

$$\frac{a_1}{(a_1^p + a_2^p)^{\frac{1}{p}}} \cdot \frac{b_1}{(b_1^q + b_2^q)^{\frac{1}{q}}} + \frac{a_2}{(a_1^p + a_2^p)^{\frac{1}{p}}} \cdot \frac{b_2}{(b_1^q + b_2^q)^{\frac{1}{q}}} = \frac{a_1b_1 + a_2b_2}{(a_1^p + a_2^p)^{\frac{1}{p}} \cdot (b_1^q + b_2^q)^{\frac{1}{q}}} \leq 1.$$

We get $a_1b_1 + a_2b_2 \leq (a_1^p + a_2^p)^{\frac{1}{p}} \cdot (b_1^q + b_2^q)^{\frac{1}{q}}$ so, the holder inequality is proved. □

For each $g = (\underline{g}, \overline{g}) \in L_F^q(X, \mu)$ put,

$$\Phi_g : L_F^p(X, \mu) \to \mathbb{R},$$
$$f = (\underline{f}, \overline{f}) \to \int \underline{f} \cdot \underline{g} d(\mu \times \lambda) + \int \overline{f} \cdot \overline{g} d(\mu \times \lambda).$$

It is clear that Φ_g is a linear functional. Based on the definition of norm on $L_F^p(X, \mu)$ and the above inequality, it can be drawn that ϕ_g is bounded. It means,

$$\|\Phi_g\| = \sup_{\|f\|_{F \cdot p} \leq 1} \|\Phi_g(f)\| \leq \|g\|_{F \cdot q} < \infty.$$

So we can consider the map:

$$\Phi : L_F^q(X, \mu) \to (L_F^p(X, \mu))^*,$$
$$g = (\underline{g}, \overline{g}) \to \Phi_g.$$

Consequently it is proved that $(L_F^p(X, \mu))^* \supseteq L_F^q(X, \mu)$. Now in the next theorem we show that $(L_F^p(X, \mu))^* = L_F^q(X, \mu)$.

Theorem 3.6. *The map ϕ defined above is an isometric isomorphism.*

Proof. Linearly and the boundedness is proved. It is enough to show that $(L_F^p(X, \mu))^* \subseteq L_F^q(X, \mu)$. For this purpose the basic idea is that, each functional on fuzzy L^p-space can be decompose into two functionals on L^p-spaces. Consider the following functions,

$$\pi_1 : L^p(X, \mu) \to L^p(X, \mu) \oplus^{l^p} L^p(X, \mu),$$
$$f \to (f, 0),$$

$$\pi_2 : L^p(X,\mu) \to L^p(X,\mu) \oplus^{l^p} L^p(X,\mu),$$
$$f \to (0,f),$$

In this case π_1 and π_2 are bounded, linear and one to one functions. Based on the definition of Φ, function

$$\Phi(\underline{f},\overline{f}) = \Phi(\underline{f},0) + \Phi(0,\overline{f}) = \Phi\circ\pi_1(\underline{f}) + \Phi\circ\pi_2(\overline{f}),$$

where $\Phi\circ\pi_1$ and $\Phi\circ\pi_2$ are also bounded and linear functions. According to the Riesz Representation Theorem,

$$\exists g^*, g^{**} \in L^q(X,\mu) \quad \text{s.t} \quad \Phi\circ\pi_1(\underline{f}) = \int \underline{f} g^* d(\mu\times\lambda),$$
$$\Phi\circ\pi_2(\overline{f}) = \int \overline{f} g^{**} d(\mu\times\lambda),$$

Let $g = (g^*, g^{**})$, so for $f = (\underline{f},\overline{f})$

$$\Phi(f) = \Phi(\underline{f},\overline{f}) = \int \underline{f} g^* d(\mu\times\lambda) + \int \overline{f} g^{**} d(\mu\times\lambda) = \Phi_{(g^*,g^{**})}(f),$$

in this case $g \in L^q_F(X,\mu)$. But we must show that $\|\Phi\| = \|g\|_{F\cdot q}$. In other words $\|\Phi\| = (\|g^*\|^q + \|g^{**}\|^q)^{\frac{1}{q}}$. By the fuzzy Holders inequality $\|\Phi\| \le (\|g^*\|^q + \|g^{**}\|^q)^{\frac{1}{q}}$.

Now let α^* and α^{**} be real numbers such that $\alpha^* g^* = |g^*|$ and $\alpha^{**} g^{**} = |g^{**}|$. Now,

$$E_n^* = \{x \in X, |\ g^*(x)| < n\},$$
$$E_n^{**} = \{x \in X, |\ g^{**}(x)| < n\}.$$

Let $\underline{f} := \chi_{E_n^*}|g^*|^{q-1}\alpha^*$, $\overline{f} := \chi_{E_n^{**}}|g^{**}|^{q-1}\alpha^{**}$. In this way on E_n, $|\underline{f}|^p = |g^*|^q$, $|\overline{f}|^p = |g^{**}|^q$

$$\begin{aligned}\int_{E_n^*} |g^*|^q d\mu + \int_{E_n^{**}} |g^{**}|^q d\mu &= \int \underline{f} g^* d(\mu\times\lambda) + \int \overline{f} g^{**} d(\mu\times\lambda) = \Phi(f)\\ &\le \|\Phi\|\|f\|_{F\cdot p}\\ &= \|\Phi\|(\int_{E_n^*} |g^*|^q d\mu + \int_{E_n^{**}} |g^{**}|^q d\mu)^{\frac{1}{p}},\end{aligned}$$

we have $1 - \frac{1}{p} = \frac{1}{q}$, so,

$$\left(\int_{E_n^*} |g^*|^q d\mu + \int_{E_n^{**}} |g^{**}|^q d\mu\right)^{\frac{1}{q}} \le \|\Phi\|.$$

By taking limit on n and using the monotone convergence theorem, the left hand side increase to

$$\left(\int |g^*|^q d\mu + \int |g^{**}|^q d\mu\right)^{\frac{1}{q}}.$$

So we will have $\|g\|_{F\cdot q} \le \|\Phi\|$ and we can infer that $\|\Phi\| = \|g\|_{F\cdot q}$. Consequently for $1 < p < \infty$, $(L^p_F(X,\mu))^* = L^q_F(X,\mu)$ in fuzzy spaces. □

4. Conclusions

In this paper, the fuzzy L^p-Spaces are introduced. Properties of these spaces are given in details. The suitable norm and the dual of such spaces are also considered.

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