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On Split Equilibrium and Fixed Point Problems for Finite Family of Bregman Quasi-Nonexpansive Mappings in Banach spaces

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Abstract

In this paper, we introduce a trifunction split equilibrium problem using a generalized relaxed α-monotonicity in the framework of p-uniformly convex and uniformly smooth Banach spaces. We develop an iterative algorithm for approximating a common solution of split equilibrium problem and fixed point problem for finite family of Bregman quasi-nonexpansive mappings. Using our iterative algorithm, we state and prove a strong convergence theorem for approximating a common solution of the aforementioned problems. Our iterative scheme is design in such a way that it does not require any knowledge of the operator norm. We display a numerical example to show the applicability of our result. Our result extends and complements some related results in literature.

Keywords: Split Equilibium Problem, Bregman Quasi-Nonexpansive, Iterative scheme, Fixed point problem.
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1. Introduction

Let E be a real Banach space with norm $||.||$ and E^* be the dual space of E. Let $K(E) := \{x \in E : ||x|| = 1\}$ denote the unit sphere of E. The modulus of convexity is the function $\delta_E : (0, 2] \to [0, 1]$ defined

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by

$$\delta_E(\epsilon) = \inf\big\{1 - \frac{||x+y||}{2} : x, y \in K(E),\ ||x-y|| \geq \epsilon\big\}.$$

The space E is said to be uniformly convex, if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$. Let $p > 1$, then E is said to be p-uniformly convex (or to have a modulus of convexity of power type p) if there exists $c_p > 0$ such that $\delta_E(\epsilon) \geq c_p\epsilon^p$ for all $\epsilon \in (0,2]$. Note that every p-uniformly convex space is uniformly convex. The modulus of smoothness of E is the function $\rho_E : \mathbb{R}^+ := [0,\infty) \to \mathbb{R}^+$ defined by

$$\rho_E(\tau) = \sup\big\{\frac{||x+\tau y + ||x-\tau y||}{2} - 1 : x, y \in K(E)\big\}.$$

The space E is said to be uniformly smooth, if $\frac{\rho_E(\tau)}{\tau} \to 0$ as $\tau \to 0$. Let $q > 1$, then a Banach space E is said to be q-uniformly smooth if there exists $\kappa_q > 0$ such that $\rho_E(\tau) \leq \kappa_q\tau^q$ for all $\tau > 0$. Moreover, a Banach space E is p-uniformly convex if and only if E^* is q-uniformly smooth, where p and q satisfy $\frac{1}{p} + \frac{1}{q} = 1$, (see [8]).
Let $p > 1$ be a real number, the generalized duality mapping $J_E^p : E \to 2^{E^*}$ is defined by

$$J_E^p(x) = \{\overline{x} \in E^* : \langle x, \overline{x}\rangle = ||x||^p, ||\overline{x}|| = ||x||^{p-1}\},$$

where $\langle ., .\rangle$ denotes the duality pairing between elements of E and E^*. In particular, $J_E^p = J_E^2$ is called the normalized duality mapping.
If E is p-uniformly convex and uniformly smooth, then E^* is q-uniformly smooth and uniformly convex. In this case, the generalized duality mapping J_E^p is one-to-one, single-valued and satisfies $J_E^p = (J_{E^*}^q)^{-1}$, where $J_{E^*}^q$ is the generalized duality mapping of E^*. Furthermore, if E is uniformly smooth then the duality mapping J_E^p is norm-to-norm uniformly continuous on bounded subsets of E, (see [9] for more details).
Let $f : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function, then the Frenchel conjugate of f denoted as $f^* : E^* \to (-\infty, +\infty]$ is define as

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\},\ x^* \in E^*.$$

Let the domain of f be denoted as $(dom f) = \{x \in E : f(x) < +\infty\}$, hence for any $x \in int(dom f)$ and $y \in E$, we define the right-hand derivative of f at x in the direction y by

$$f^0(x, y) = \lim_{t\to 0^+} \frac{f(x+ty) - f(x)}{t}.$$

Definition 1.1. [6] Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $\Delta_f : E \times E \to [0, +\infty)$ defined by

$$\Delta_f(x, y) := f(y) - f(x) - \langle \bigtriangledown f(x), y - x\rangle$$

is called the Bregman distance with respect of f.

It is well-known that Bregman distance Δ_f does not satisfy the properties of a metric because Δ_f fail to satisfy the symmetric and triangular inequality property. Moreover, it is well known that the duality mapping J_E^p is the sub-differential of the functional $f_p(.) = \frac{1}{p}||.||^p$ for $p > 1$, see [7]. Then, the Bregman distance Δ_p is defined with respect to f_p as follows:

$$\begin{aligned}\Delta_p(x, y) &= \frac{1}{p}||y||^p - \frac{1}{p}||x||^p - \langle J_E^p x, y - x\rangle \\ &= \frac{1}{q}||x||^p - \langle J_E^p x, y\rangle + \frac{1}{p}||y||^p \\ &= \frac{1}{q}||x||^p - \frac{1}{q}||y||^p - \langle J_E^p x - J_E^p y, y\rangle. \end{aligned} \tag{1.1}$$

Let $Fix(T)$ denotes the set of fixed points of a mapping T from C into itself. That is $Fix(T) = \{x \in C : Tx = x\}$. A point $p \in C$ is said to be an asymptotic fixed point of T, if C contains a sequence $\{x_n\}_{n=1}^{\infty}$ which converges weakly to p and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. We denote by $\hat{Fix}(T)$, the set of asymptotic fixed points of T. Moreso, a mapping $T : C \to int(domf)$ is said to be
(i) Bregman relatively nonexpansive, if

$$\hat{Fix}(T) = Fix(T) \text{ and } \Delta_p(p, Tx) \leq \Delta_p(p, x),\ \forall\ x \in C,\ p \in Fix(T).$$

(ii) Bregman quasi-nonexpansive, if

$$Fix(T) \neq \emptyset \text{ and } \Delta_p(p, Tx) \leq \Delta_p(p, x),\ \forall\ x \in C,\ p \in Fix(T).$$

Definition 1.2. A function $F : C \times C \times C \to \mathbb{R}$ is said to be generalized relaxed α-monotone if for any $x, y \in C$, we have

$$F(y, x, y) - F(y, x, x) \geq \alpha(x, y), \tag{1.2}$$

where $\lim_{t\to 0} \frac{\alpha(x,ty+(1-t)x)}{t} = 0$.

Remark 1.3. If $\alpha \equiv 0$ in (1.2), we say that F is a generalized monotone mapping. Also, if $\alpha(x, y) = \beta(y-x)$, where $\beta : C \to \mathbb{R}$ with $\beta(t) = t\beta(z)$, for $t > 0,\ p \geq 1$, then we say that F is a relaxed β-monotone mapping.

Recall that a metric projection P_C from E onto C satisfies the following property:

$$||x - P_C x|| \leq \inf_{y\in C} ||x - y||,\ \forall\ x \in E.$$

It is well known that $P_C x$ is the unique minimizer of the norm distance. Moreover, $P_C x$ is characterized by the following properties:

$$\langle J_E^p(x - P_C x), y - P_C x\rangle \leq 0,\ \forall\ y \in C. \tag{1.3}$$

The Bregman projection from E onto C denoted by Π_C also satisfies the property

$$\Delta_p(x, \Pi_C(x)) = \inf_{y\in C} \Delta_p(x, y),\ \forall\ x \in E. \tag{1.4}$$

Also, if C is a nonempty, closed and convex subset of a p-uniformly convex and uniformly smooth Banach space E and $x \in E$. Then the following assertions holds:
(i) $z = \Pi_C x$ if and only if

$$\langle J_E^p(x) - J_E^p(z),\ y - z\rangle \leq 0,\ \forall\ y \in C; \tag{1.5}$$

(ii)

$$\Delta_p(\Pi_C x, y) + \Delta_p(x, \Pi_C x) \leq \Delta_p(x, y),\ \forall\ y \in C. \tag{1.6}$$

When considering the p-uniformly convex space, the Bregman distance and the metric distance have the following relation, (see [16]).

$$\tau||x - y||^p \leq \Delta_p(x, y) \leq \langle x - y, J_E^p(x) - J_E^p(y)\rangle, \tag{1.7}$$

where $\tau > 0$ is some fixed number.
Let C be a nonempty, closed and convex subset of a Banach space E. The Equilibrium Problem (EP)

which was introduced by Blum and Oettli [5] is a generalization of optimization and variational inequality problems. Given a bifunction $F : C \times C \to \mathbb{R}$, the EP is to find $x \in C$ such that

$$F(x, y) \geq 0, \ \forall \ y \in C. \tag{1.8}$$

We denote by $EP(F)$, the solution set of EP (1.8).
The EP has a great impact in the study of problems which arise in economics, finance, network, optimization, image reconstruction and operation research in a general unified ways. Many authors have considered the EP together with the fixed point problem (see [1, 2, 3, 4, 5, 10, 14] and the references contained in).
In 2013 Kazmi and Rizvi [11] introduced the the following Split Equilibrium Problem (SEP) in real Hilbert spaces: Let H_1 and H_2 be real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively, let $F_1 : C \times C \to \mathbb{R}, \ F_2 : Q \times Q \to \mathbb{R}$ be two nonlinear bifunctions and $A : H_1 \to H_2$ be a bounded linear operator, then the SEP is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \ \forall \ x \in C; \tag{1.9}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) \geq 0, \ \forall \ y \in Q. \tag{1.10}$$

They [11] introduced an iterative algorithm to approximate a common solution of SEP together with a variational inequality problem and fixed point problem of nonexpansive mapping in real Hilbert spaces.
In 2018 Abass et. al. [1] introduced a viscosity-type algorithm to approximate a common solution of SEP and fixed point problem of an infinite family of quasi-nonexpansive mappings in real Hilbert spaces. They proved the following strong convergence theorem:

Theorem 1.4. *Let H_1 and H_2 be two real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and D be a strongly positive bounded linear operator on H_1 with coefficient $\overline{\tau} > 0$. Let $T_i : C \to K(C), i = 1, 2, 3, ...,$ be an infinite family of quasi-nonexpansive multi-valued mappings and $F_1 : C \times C \to \mathbb{R}, \ F_2 : Q \times Q \to \mathbb{R}$ be bifunctions, where F_2 is upper semi-continuous in the first argument. Suppose $\Gamma := \bigcap_{i=1}^{\infty} Fix(T_i) \bigcap SEP \neq \emptyset$ and f is a contraction mapping with coefficient $\mu \in (0, 1)$. Let the sequences $\{u_n\}, \ \{y_n\}$ and $\{x_n\}$ be generated by*

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n + \xi_n A^*(T_{r_n}^{F_2} - I)Ax_n); \\ y_n = \lambda_0 u_n + \sum_{i=1}^{\infty} \lambda_i z_n^i; \\ x_{n+1} = \gamma_n \tau f(x_n) + (I - \gamma_n D)y_n, \ \forall \ n \geq 1; \end{cases}$$

where $z_n^i \in T_i u_n, \ r_n \in (0, \infty)$ and the step size ξ_n is chosen in such a way that for some $\epsilon > 0$,

$$\xi_n \in \left(\epsilon, \frac{||(T_{r_n}^{F_2} - I)Ax_n||^2}{||A^*(T_{r_n}^{F_2} - I)Ax_n||^2} - \epsilon \right);$$

for all $T_{r_n}^{F_2} Ax_n \neq Ax_n$ and $\xi_n = \xi$, otherwise (ξ being any nonnegative real number) with the sequence γ_n and r_n satisfying the following conditions:
(i) $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\gamma_n \in (0, 1), \ 0 < \tau < \frac{\overline{\tau}}{\mu}$ and $0 < \gamma_n < 2\mu$;
(iii) $\lambda_0, \lambda_i \in (0, 1)$ such that $\sum_{i=0}^{\infty} \lambda_i = 1$. Then, the sequence $\{x_n\}$ converges strongly to $q \in \Gamma$ which solves the variational inequality

$$\langle (D - \tau f)q, q - p \rangle \leq 0, \ \forall \ p \in \Gamma.$$

Very recently, Mahato et. al. [13] proved the existence results for Trifunction Equilibrium Problem (TEP) which was introduced by Prada et. al. [15] in Banach space. The TEP for the function $F : C \times C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(y, x, x) \geq 0, \ \forall \ y \in C; \tag{1.11}$$

where C is a nonempty, closed and convex subset of a Banach space E.

If $F(x,y,z) = \langle Az, x-y\rangle$, where $A : C \to E^*$ is a mapping. Then (1.11) reduces to the classical variational inequality problem, which is to find $x \in C$ such that

$$\langle Ax, y-x\rangle \geq 0,\ \forall\ y \in C. \tag{1.12}$$

They [13] introduced the following Hybrid iterative algorithm for approximating a common element of solutions of a system of TEP and the set of fixed points of an infinite family of quasi-ϕ-nonexpansive mappings in a uniformly smooth and uniformly convex Banach space as follows:

$$\begin{cases} x_0 = x \in C, C_0 = C,\ Q_0 = C; \\ z_n = J^{-1}(\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty}\alpha_{n,i}JT_ix_n); \\ y_n = J^{-1}(\delta_nJx_n + (1-\delta_n)Jz_n); \\ u_n = T^{F_m}_{r_{m,n}}\cdots T^{F_{m-1}}_{r_{m-1,n}}\cdots T^{F_2}_{r_{2,n}}T^{F_1}_{r_{1,n}}y_n; \\ \text{where } T^{F_m}_{r_{j,n}}y_n = \{z \in C : F_j(y,z,z) + \frac{1}{r_{j,n}}\langle y-z, Jz-Jy_n\rangle \geq 0, \forall\ y \in C\}, \\ C_n = \{w \in C_{n-1} : G(w,Ju_n) \leq G(w,Jy_n) \leq G(w,Jx_n)\},\ n \geq 1; \\ Q_n = \{w \in Q_{n-1} : \langle x_n - w, Jx - Jx_n\rangle + \rho f(w) - \rho f(x_n) \geq 0\},\ n \geq 1; \\ x_{n+1} = \Pi^f_{C_n\cap Q_n}x; \end{cases}$$

where $J : E \to E^*$ is the normalized duality mapping, C is a nonempty, bounded, closed and convex subset of a uniformly convex and uniformly smooth Banach space E, $\{\delta_n\}$ and $\{\alpha_n\}$ are sequences in $[0,1]$ such that

(i) $\sum\limits_{i=0}^{\infty}\alpha_{n,i} = 1,\ \forall\ n \geq 0$;
(ii) $\limsup\limits_{n\to\infty}\delta_n < 1$;
(iii) $\liminf\limits_{n\to\infty}\alpha_{n,0}\alpha_{n,i} > 0,\ \forall\ i$;
(iv) $\{r_{j,n}\} \subset [\epsilon,\infty)$ for some $\epsilon > 0$.

Motivated by the works of Abass et. al. [1], Mahato et. al. [13], Kazmi and Rizvi [11], we introduce a Split Trifunction Equilibrium Problem ($STEP$) as follows: Let E_1 and E_2 be two Banach spaces, C and Q be nonempty, closed and convex subsets of E_1 and E_2 respectively. Let $F_1 : C\times C\times C \to \mathbb{R}$, $F_2 : Q\times Q\times Q \to \mathbb{R}$ be two nonlinear trifunctions and $A : E_1 \to E_2$ be a bounded linear operator, then the $STEP$ is to find $x^* \in C$ such that

$$F_1(y, x^*, x^*) \geq 0,\ \forall\ y \in C; \tag{1.13}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } F_2(z, y^*, y^*) \geq 0,\ \forall\ z \in Q. \tag{1.14}$$

The inequalities (1.13) and (1.14) constitute a pair of TEP whose image ($y^* = Ax^*$) of solution of (1.13) in E_1 under a given bounded linear operator A, is the solution of (1.14) in E_2. We denote by $\Omega := \{p \in TEP(F_1) : Ap \in TEP(F_2)\}$ the set of solution of $STEP$ (1.13)-(1.14).
We introduce an Halpern-type algorithm to approximate a common solution of $STEP$ (1.13)-(1.14) together with a fixed point problem of a finite family of Bregman quasi-nonexpansive mappings in the framework of p-uniformly convex and uniformly smooth Banach spaces. A strong convergence result of the aforementioned problems was obtained and the iterative algorithm employed is design in such a way that it does not require any knowledge of the operator norm. We apply our result to solve optimization problem and also display a numerical example to show the applicability of our result. The result present in this paper extends and complements the results of [1], [11] and other related results in literature.

2. Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively.

Lemma 2.1. *[7] Let E be a Banach space and $x, y \in E$. If E is q-uniformly smooth, then there exists $C_q > 0$ such that*

$$||x-y||^q \leq ||x||^q - q\langle J_q^E(x), y\rangle + C_q||y||^q.$$

Lemma 2.2. *[12] Let E be a real p-uniformly convex and uniformly smooth Banach space. Let $z, x_k \in E$ $(k = 1, 2, ..., N)$ and $\alpha_k \in (0,1)$ with $\sum_{k=1}^{N} \alpha_k = 1$. Then, we have*

$$\Delta_p(J_q^{E^*}(\sum_{k=1}^{N} \alpha_k J_p^E(x_k)), z) \leq \sum_{k=1}^{N} \alpha_k \Delta_p(x_k, z) - \alpha_i\alpha_j g_r^*(||J_p^E(x_i) - J_p^E(x_j)||),$$

for all $i, j \in 1, 2, ..., N$ and $g_r^* : \mathbb{R}^+ \to \mathbb{R}^+$ being a strictly increasing function such that $g_r^*(0) = 0$.

Lemma 2.3. *[17] Let E be a real p-uniformly convex and uniformly smooth Banach space. Let $V_p : E^* \times E \to [0, +\infty)$ be defined by*

$$V_p(x^*, x) = \frac{1}{q}||x^*||^q - \langle x^*, x\rangle + \frac{1}{p}||x||^p, \ \forall \ x \in E, x^* \in E.$$

Then the following assertions hold:
(i) V_p is nonnegative and convex in the first variable.
(ii) $\Delta_p(J_q^{E^*}(x^*), x) = V_p(x^*, x), \ \forall \ x \in E, \ x^* \in E.$
(iii) $V_p(x^*, x) + \langle y^*, J_q^{E^*}(x^*) - x\rangle \leq V_p(x^* + y^*, x), \forall \ x \in E, \ x^*, y^* \in E.$

Lemma 2.4. *[8] Let E be a real p-uniformly convex and uniformly smooth Banach space. Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded sequences in E. Then the following assertions are equivalent:*
(i) $\lim_{n\to\infty} \Delta_p(x_n, y_n) = 0$;
(ii) $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

Let D be a nonempty bounded, closed, convex and bounded subset of a smooth strictly convex and reflexive Banach space E. For $r > 0$ and $z \in D$, consider the following problems: find $x \in D$ such that

$$F(y, x, x) + \frac{1}{r}\langle y - x, Jx - Jy\rangle \geq 0, \ \forall \ y \in D; \tag{2.1}$$

and find $x \in D$ such that

$$F(y, x, y) + \frac{1}{r}\langle y - x, Jx - Jy\rangle \geq \alpha(x, y), \ \forall \ y \in D. \tag{2.2}$$

Lemma 2.5. *[13] Let D be a nonempty bounded, closed, convex and bounded subset of a smooth strictly convex and reflexive Banach space E. Assume $F : D \times D \times D \to \mathbb{R}, \ z \in D$ be such that:*
(i) $F(y, x, .)$ is hemicontinuous;
(ii) $F(., x.z)$ is convex;
(iii) $F(x, x, z) = 0$;
(iv) F is generalized relaxed α-monotone;
(v) $\alpha(., y)$ is lower semicontinuous. Then the problems (2.1) *and* (2.2) *are equivalent and have solutions.*

Lemma 2.6. *[13] Let D be a nonempty closed and convex subset of a smooth strictly convex and reflexive Banach space E. Let $F : D \times D \times D \to \mathbb{R}$ with $z \in D$ and $r > 0$. Let all assumptions of Lemma 2.5 hold with $\alpha(x, y) + \alpha(y, x) \geq 0,\ \forall\ x, y \in D$. Define a mapping $T_r : E \to D$ as follows:*

$$T_r x = \{z \in D : F(y, z, z) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall\ y \in D\},\ \forall\ x \in E. \tag{2.3}$$

Then, the following holds:
(i) $T_r x$ is nonempty and single-valued;
(ii) $\langle T_r x - T_r y, JT_r x - Jx\rangle \leq \langle T_r x - T_r y, JT_r y - Jy\rangle$;
(iii) $Fix(T_r) = EP(F)$;
(iv) $\Delta_p(q, T_r x) + \Delta_p(T_r x, x) \leq \Delta_p(q, x),\ \forall\ q \in Fix(T_r),\ x \in E$;
(v) $EP(F)$ is closed and convex.

Lemma 2.7. *[18] Assume $\{a_n\}$ is a sequence of nonnegative real sequence such that*

$$a_{n+1} \leq (1 - \sigma_n)a_n + \sigma_n \delta_n,\ n > 0,$$

where $\{\sigma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a real sequence such that
(i) $\sum_{n=1}^{\infty} \sigma_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=1}^{\infty} |\sigma_n \delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

Theorem 3.1. *Let D and G be nonempty bounded closed convex subsets of uniformly convex and uniformly smooth Banach spaces E_1 and E_2 respectively, and $A : E_1 \to E_2$ be a bounded linear operator with $A^* : E_2^* \to E_1^*$ being the adjoint of A. Let $F_1 : D \times D \times D \to \mathbb{R},\ F_2 : G \times G \times G \to \mathbb{R}$ be trifunctions satisfying the assumptions of Lemma 2.5 and Lemma 2.6 with F_1 and F_2 being continuous. Let $\{T_i\}_{i=1}^N$ be a finite family of Bregman quasi-nonexpansive mapping such that $\Gamma := \bigcap_{i=1}^N Fix(T_i) \bigcap \Omega \neq \emptyset$, then the sequences $\{u_n\}$ and $\{x_n\}$ are generated iteratively by*

$$\begin{cases} u_n = T_{r_n}^{F_1}(J_{E_1^*}^q[J_{E_1}^p(x_n) - t_n A^* J_{E_2}^p(T_{r_n}^{F_2} - I)Ax_n]); \\ x_{n+1} = J_{E_1^*}^q[\beta_n J_{E_1}^p(u) + (1 - \beta_n)(\alpha_{n,0} J_{E_1}^p(u_n) + \sum_{i=1}^N \alpha_{n,i} J_{E_1}^p T_i u_n)],\ n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_{n,0}\},\ \{\alpha_{n,i}\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ such that $\sum_{i=0}^N \alpha_{n,i} = 1,\ r_n \subset (0, \infty)$ and the step size t_n is chosen in such a way that if $n \in \Theta := \{n : (T_{r_n}^{F_2} - I)Ax_n \neq 0\}$, then

$$t_n^{q-1} \in \left(0,\ \frac{q||(T_{r_n}^{F_2} - I)Ax_n||^p}{C_q||A^* J_{E_2}^p(T_{r_n}^{F_2} - I)Ax_n||^q}\right), \tag{3.2}$$

where C_q is the constant of smoothness of E_1. Otherwise, $t_n = t$ (t being any nonnegative real number) with the sequences $\{\alpha_{n,0}\},\ \{\alpha_{n,i}\},\ \{\beta_n\}$ and $\{r_n\}$ satisfying the following conditions:
(i) $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$;
(ii) $\liminf_{n\to\infty} r_n > 0$;
(iii) $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$, for all i. Then $\{x_n\}$ converges strongly to $\overline{x} \in \Pi_\Gamma u$.

Proof. Let $\overline{x} \in \Gamma$, then from Lemma 2.1, (1.11) and (3.1), we have that

$$\begin{aligned}
\Delta_p(u_n,\overline{x}) &= \Delta_p[T_{r_n}^{F_1}(J_{E_1^*}^p(x_n) - t_nA^*J_{E_2}^p(T_{r_n}^{F_2} - I)Ax_n),\overline{x}] \\
&\leq \Delta_p(J_{E_1^*}^q[J_{E_1}^p(x_n) - t_nA^*J_{E_2}^p(T_{r_n}^{F_2})Ax_n],\overline{x}) \\
&= \frac{1}{q}||J_{E_1}^p(x_n) - t_nA^*J_{E_2}^p(T_{r_n}^{F_2} - I)Ax_n||^q - \langle J_{E_1}^p(x_n),\overline{x}\rangle \\
&+ t_n\langle A^*J_{E_2}^p(T_{r_n}^{F_2} - I)Ax_n,\overline{x}\rangle + \frac{1}{p}||x||^p \\
&\leq \frac{1}{q}||J_{E_1}^p(x_n)||^q - t_n\langle x_n, A^*J_{E_2}^p(T_{r_n}^{F_2} - I)Ax_n\rangle + \frac{C_q}{q}t_n^q||A^*J_{E_2}^p(T_{r_n}^{F_2} - I)Ax_n||^q \\
&+ \frac{||x||^p}{p} - \langle J_{E_1}^p(x_n),\overline{x}\rangle + t_n\langle A^*J_{E_2}^p(T_{r_n}^{F_2} - I)Ax_n,\overline{x}\rangle \\
&= \frac{||x_n||^p}{q} - \langle J_{E_1}^p(x_n),\overline{x}\rangle + \frac{||\overline{x}||^p}{p} - t_n\langle x_n - \overline{x}, A^*J_{E_2}^p(T_{r_n}^{F_2} - I)Ax_n\rangle \\
&+ \frac{C_q}{q}t_n^q||A^*J_{E_2}^p(T_{r_n}^{F_2} - I)Ax_n||^q \\
&= \Delta_p(x_n,\overline{x}) - t_n\langle Ax_n - A\overline{x}, J_{E_2}^p(T_{r_n}^{F_2} - I)Ax_n\rangle + \frac{C_q}{q}t_n^q||A^*J_{E_2}^p(T_{r_n}^{F_2} - I)Ax_n||^q. \qquad (3.3)
\end{aligned}$$

But

$$\begin{aligned}
\langle Ax_n - A\overline{x}, J_{E_2}^p(T_{r_n}^{F_2} - I)Ax_n\rangle &= ||T_{r_n}^{F_2}Ax_n - Ax_n||^p + \langle J_{E_2}^p((T_{r_n}^{F_2} - I)Ax_n), A\overline{x} - T_{r_n}^{F_2}(Ax_n)\rangle \\
&\geq ||T_{r_n}^{F_2}Ax_n - Ax_n||^p. \qquad (3.4)
\end{aligned}$$

On substituting (3.4) into (3.3), we have that

$$\begin{aligned}
\Delta_p(u_n,\overline{x}) &\leq \Delta_p(x_n,\overline{x}) - t_n||(T_{r_n}^{F_2} - I)Ax_n||^p + \frac{C_q}{q}t_n^q||A^*J_{E_2}^p(T_{r_n}^{F_2} - I)Ax_n||^q \\
&= \Delta_p(x_n,\overline{x}) - t_n\left[||(T_{r_n}^{F_2} - I)Ax_n||^p - \frac{C_q}{q}t_n^{q-1}||A^*J_{E_2}^p(T_{r_n}^{F_2} - I)Ax_n||^q\right]. \qquad (3.5)
\end{aligned}$$

Using the choice of t_n in (3.5), we have that

$$\Delta_p(u_n,\overline{x}) \leq \Delta_p(x_n,\overline{x}). \qquad (3.6)$$

Let $y_n = \alpha_{n,0}J_{E_1}^p(u_n) + \sum_{i=1}^N \alpha_{n,i}J_{E_1}^pT_iu_n$, then we have from Lemma 2.2 and (3.6) that

$$\begin{aligned}
\Delta_p(y_n,\overline{x}) &= \Delta_p(J_{E_1^*}^p[\alpha_{n,0}J_{E_1}^p(u_n) + \sum_{i=1}^N \alpha_{n,i}J_{E_1}^pT_iu_n],\overline{x}) \\
&\leq \alpha_{n,0}\Delta_p(u_n,\overline{x}) + \sum_{i=1}^N \alpha_{n,i}\Delta_p(T_iu_n,\overline{x}) \\
&- \alpha_{n,0}\alpha_{n,i}g_r^*(||J_{E_1}^p(u_n) - J_{E_1}^p(T_iu_n)||) \\
&\leq \alpha_{n,0}\Delta_p(u_n,\overline{x}) + \sum_{i=1}^N \alpha_{n,i}\Delta_p(u_n,\overline{x}) \\
&- \alpha_{n,0}\alpha_{n,i}g_r^*(||J_{E_1}^p(u_n) - J_{E_1}^p(u_n)||) \\
&\leq \Delta_p(u_n,\overline{x}) \\
&\leq \Delta_p(x_n,\overline{x}). \qquad (3.7)
\end{aligned}$$

From (3.1) and (3.7), we have that

$$\begin{aligned}\Delta_p(x_{n+1},\overline{x}) &= \Delta_p(J^q_{E_1^*}[\beta_n J^p_{E_1}(u)+(1-\beta_n)J^p_{E_1}y_n],\overline{x})\\ &\leq \beta_n\Delta_p(u,\overline{x})+(1-\beta_n)\Delta_p(y_n,\overline{x})\\ &\leq \beta_n\Delta_p(u,\overline{x})+(1-\beta_n)\Delta_p(x_n,\overline{x})\\ &\leq \max\{\Delta_p(u,\overline{x}),\Delta_p(x_n,\overline{x})\}\\ &\vdots\\ &\leq \max\{\Delta_p(u,\overline{x}),\ \Delta_p(x_1,\overline{x})\}.\end{aligned}\tag{3.8}$$

Therefore, we conclude that $\Delta_p(x_n,\overline{x})$ is bounded. Consequently, $\Delta_p(u_n,\overline{x})$ and $\Delta_p(y_n,\overline{x})$ are bounded. From (3.1), Lemma 2.3 and (3.7), we have that

$$\begin{aligned}\Delta_p(x_{n+1},\overline{x}) &= \Delta_p(J^q_{E_1^*}[\beta_n J^p_{E_1}(u)+(1-\beta_n)(y_n)],\overline{x})\\ &= V_p(\beta_n J^p_{E_1}(u)+(1-\beta_n)J^p_{E_1}(y_n),\overline{x})\\ &\leq V_p(\beta_n J^p_{E_1}(u)+(1-\beta_n)J^p_{E_1}(y_n)-\beta_n(J^p_{E_1}(u)-J^p_{E_1}(\overline{x}),\overline{x}))\\ &-\langle-\beta_n(J^p_{E_1}(u)-J^p_{E_1}(\overline{x})),J^q_{E_1^*}[\beta_n J^p_{E_1}(u)+(1-\beta_n)J^p_{E_1}(y_n)]-\overline{x}\rangle\\ &= V_p(\beta_n J^p_{E_1}(\overline{x})+(1-\beta_n)J^p_{E_1}(y_n),\overline{x})+\beta_n\langle J^p_{E_1}(u)-J^p_{E_1}(\overline{x}),x_{n+1}-\overline{x}\rangle\\ &\leq \beta_n V_p(J^p_{E_1}(\overline{x}),\overline{x})+(1-\beta_n)V_p(J^p_{E_1}(y_n),\overline{x})+\beta_n\langle J^p_{E_1}(u)-J^p_{E_1}(\overline{x}),x_{n+1}-\overline{x}\rangle\\ &= \beta_n\Delta_p(\overline{x},\overline{x})+(1-\beta_n)\Delta_p(y_n,\overline{x})+\beta_n\langle J^p_{E_1}(u)-J^p_{E_1}(\overline{x}),x_{n+1}-\overline{x}\rangle\\ &\leq (1-\beta_n)\Delta_p(u_n,\overline{x})-\alpha_{n,0}\alpha_{n,i}g^*_r(||J^p_{E_1}(u_n)-J^p_{E_1}(T_iu_n)||)+\beta_n\langle J^p_{E_1}(u)-J^p_{E_1}(\overline{x}),x_{n+1}-\overline{x}\rangle\\ &\leq (1-\beta_n)\Delta_p(x_n,\overline{x})-\alpha_{n,0}\alpha_{n,i}(1-\beta_n)g^*_r(||J^p_{E_1}(u_n)-J^p_{E_1}(u_n)||)\\ &-t_n(1-\beta_n)\left[||(T^{F_2}_{r_n}-I)Ax_n||^p-\frac{C_q}{q}t_n^{q-1}||A^*J^p_{E_2}(T^{F_2}_{r_n}-I)Ax_n||^q\right]+\beta_n\langle J^p_{E_1}(u)-J^p_{E_1}(\overline{x}),x_{n+1}-\overline{x}\rangle.\end{aligned}\tag{3.9}$$

We now divide our proof into two cases:

CASE 1: Suppose $\Delta_p(x_n,\overline{x})$ is monotone non-increasing, then $\Delta_p(x_n,\overline{x})$ is convergent. Hence,

$$\lim_{n\to\infty}(\Delta_p(x_n,\overline{x})-\Delta_p(x_{n+1},\overline{x}))=0.$$

From (3.9), it follows that

$$\begin{aligned}t_n(1-\beta_n)\left[||(T^{F_2}_{r_n}-I)Ax_n||^p-\frac{C_q}{q}t_n^{q-1}||A^*J^p_{E_2}(T^{F_2}_{r_n}-I)Ax_n||^q\right] &\leq (1-\beta_n)\Delta_p(x_n,\overline{x})-\Delta_p(x_{n+1},\overline{x})\\ &+\beta_n\langle J^p_{E_1}(u)-J^p_{E_1}(\overline{x}),x_{n+1}-\overline{x}\rangle\to 0,\ \text{as } n\to\ \infty.\end{aligned}\tag{3.10}$$

By the choice of the stepsize t_n, there exists a very small number $\epsilon>0$ such that

$$0<t_n^{q-1}\leq\frac{q||(T^{F_2}_{r_n}-I)Ax_n||^p}{C_q||A^*J^p_{E_2}(T^{F_2}_{r_n}-I)Ax_n||^q}-\epsilon,$$

this implies that

$$t_n^{q-1}\leq[C_q||A^*J^p_{E_2}(T^{F_2}_{r_n}-I)Ax_n||^q]\leq q||(T^{F_2}_{r_n}-I)Ax_n||^p-\epsilon[C_q||A^*J^p_{E_2}(T^{F_2}_{r_n}-I)Ax_n||^q].\tag{3.11}$$

Therefore, we have from (3.11) that

$$\lim_{n\to\infty} C_q||A^*J^p_{E_2}(T^{F_2}_{r_n} - I)Ax_n||^q = 0. \tag{3.12}$$

This implies that

$$\lim_{n\to\infty} ||A^*J^p_{E_2}(T^{F_2}_{r_n} - I)Ax_n||^q = 0. \tag{3.13}$$

Also, from (3.10), we have that

$$\lim_{n\to\infty} ||(T^{F_2}_{r_n} - I)Ax_n||^q = 0. \tag{3.14}$$

Also, from (3.9), (3.12), (3.13) and condition (i) and (iii) of (3.1), we have that

$$\begin{aligned}\alpha_{n,0}\alpha_{n,i}(1-\beta_n)g^*_r(||J^p_{E_1}(u_n) - J^p_{E_1}(u_n)||) &\leq (1-\beta_n)\Delta_p(x_n,\overline{x}) - \Delta_p(x_{n+1},\overline{x}) + \beta_n\langle J^p_{E_1}(u) - J^p_{E_1}(\overline{x}), x_{n+1}-\overline{x}\rangle \\ &\quad - t_n(1-\beta_n)\left[||(T^{F_2}_{r_n} - I)Ax_n||^p - \frac{C_q}{q}t_n^{q-1}||A^*J^p_{E_2}(T^{F_2}_{r_n} - I)Ax_n||^q\right] \to 0 \\ &\quad \text{as } n \to \infty.\end{aligned} \tag{3.15}$$

Hence, we obtain that

$$\lim_{n\to\infty} (||J^p_{E_1}(u_n) - J^p_{E_1}(u_n)||) = 0. \tag{3.16}$$

Since $J^q_{E_1^*}$ is norm-to-norm uniformly continuous on bounded subset of E_1^*, we have

$$\lim_{n\to\infty} ||u_n - T_iu_n|| = 0. \tag{3.17}$$

From (3.17) and condition (i) of (3.1), we have that

$$\Delta_p(y_n,u_n) = \sum_{i=1}^{N}\alpha_{n,i}\Delta_p(u_n,T_iu_n) \to 0 \text{ as } n\to\infty. \tag{3.18}$$

From (3.8) and condition (i) of (3.1), we have that

$$\Delta_p(x_{n+1},y_n) \leq \beta_n\Delta_p(u,y_n) \to 0 \text{ as } n \to \infty. \tag{3.19}$$

Now, let $a_n = J^q_{E_1^*}[J^p_{E_1}(x_n) - t_nA^*J^p_{E_2}(T^{F_2}_{r_n} - I)Ax_n]$, following the same approach as in (3.5), we obtain

$$\Delta_p(a_n,\overline{x}) \leq \Delta_p(x_n,\overline{x}). \tag{3.20}$$

From the definition of a_n, we obtain that

$$\begin{aligned}0 &\leq ||J^p_{E_1}(a_n) - J^p_{E_1}(x_n)|| \\ &\leq t_n||A^*||||J^p_{E_2}((T^{F_2}_{r_n} - I)Ax_n)|| \\ &= t_n||A^*||||((T^{F_2}_{r_n} - I)Ax_n)||^{p-1} \to 0 \text{ as } n\to\infty.\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\Delta_p(a_n,x_n) \to 0 = \lim_{n\to\infty}||a_n - x_n||. \tag{3.21}$$

Also, from the definition of a_n and (3.1), (3.6), (3.8), (3.20) and condition (i) of (3.1), we have that

$$\begin{aligned}\Delta_p(u_n, a_n) &= \Delta_p(a_n, T_{r_n}^{F_1} a_n)\\ &\leq \Delta_p(a_n, \overline{x}) - \Delta_p(T_{r_n}^{F_1} a_n, \overline{x})\\ &= \Delta_p(a_n, \overline{x}) - \Delta_p(u_n, \overline{x})\\ &= \Delta_p(a_n, \overline{x}) - \Delta_p(x_{n+1}, \overline{x}) + \Delta_p(x_{n+1}, \overline{x}) - \Delta_p(u_n, \overline{x})\\ &\leq \Delta_p(x_n, \overline{x}) - \Delta_p(x_{n+1}, \overline{x}) + \beta_n \Delta_p(u, \overline{x}) + (1-\beta_n)\Delta_p(u_n, \overline{x}) - \Delta_p(u_n, \overline{x}).\end{aligned}$$

Hence, from condition (i) of (3.1), we obtain that

$$\lim_{n\to\infty} \Delta_p(u_n, a_n) = 0 = \lim_{n\to\infty} ||u_n - a_n||. \tag{3.22}$$

From (3.21) and (3.22), we obtain that

$$\liminf_{n\to\infty} \Delta_p(u_n, x_n) = 0 = \lim_{n\to\infty} ||u_n - x_n||. \tag{3.23}$$

From (3.18) and (3.23), we have that

$$\lim_{n\to\infty} \Delta_p(y_n, x_n) = 0 = \lim_{n\to\infty} ||y_n - x_n||. \tag{3.24}$$

From (3.19) and (3.24), we obtain that

$$\lim_{n\to\infty} \Delta_p(x_{n+1}, x_n) = 0 = \lim_{n\to\infty} ||x_{n+1} - x_n||. \tag{3.25}$$

Since $\{x_n\}$ is bounded in E_1, there exists a subsequence $\{x_{n_j}\}$ which converges weakly to x^*. Since $\cap_{i=1}^N Fix(T_i) = \cap_{i=1}^N \hat{Fix}(T_i)$, we have from (3.17) that $x^* \in \cap_{i=1}^N Fix(T_i)$. Next, we show that $x^* \in \Omega$. Since $u_n = T_{r_n}^{F_1}(J_{E_1^*}^q(J_{E_1}^p(x_n) - t_n A^* J_{E_2}^p(T_{r_n}^{F_2} - I)Ax_n))$, and $\{r_n\} \subset (0, \infty)$, we have from Lemma 2.6 that

$$F_1(y, u_n, u_n) + \frac{1}{r_n}\langle y - u_n, J_{E_1}^p u_n - x_n - t_n A^* J_{E_2}^p (T_{r_n}^{F_2} - I)Ax_n\rangle \geq 0,$$

for all $y \in C$, which implies that

$$F_1(y, u_n, u_n) + \frac{1}{r_n}\langle y - u_n, J_{E_1}^p u_n - J_{E_1}^p x_n\rangle - \frac{1}{r_n}\langle y - u_n, J_{E_1}^p t_n A^* J_{E_2}^p (T_{r_n}^{F_2} - I)Ax_n\rangle \geq 0,$$

for all $y \in C$. Using generalized relaxed α-monotonicity of F_1, we have

$$\begin{aligned}\frac{1}{r_n}\langle y - u_n, J_{E_1}^p u_n - J_{E_1}^p x_n\rangle - \frac{1}{r_n}\langle y - u_n, J_{E_1}^p t_n A^* J_{E_2}^p (T_{r_n}^{F_2} - I)Ax_n\rangle &\geq -F_1(y, u_n, u_n)\\ &\geq \alpha(u_n, y) - F_1(y, u_n, y)\end{aligned}$$

for all $y \in C$.
Using (3.13), (3.23) and condition (ii) of (3.1) in the above inequality, we obtain that

$$\alpha(u_n, y) - F_1(y, u_n, y) \leq 0, \ \forall\ y \in C. \tag{3.26}$$

Since $\{u_n\}$ is bounded, it converges weakly to $x^* \in C$, hence we have from (3.26)

$$\alpha(x^*, y) - F_1(y, x^*, y) \leq 0, \ \forall\ x^* \in C.$$

For any $t \in (0, 1]$ and $y \in C$, let $y_t = ty + (1-t)x^*$. Since $y_t \in C$, hence we have that

$$\alpha(x^*, y_t) - F_1(y_t, x^*, y_t) \leq 0, \tag{3.27}$$

this implies that

$$\begin{aligned}\alpha(x^*, y_t) &\leq F_1(y_t, x^*, y_t)\\ &\leq tF_1(y, x^*, y_t) + (1-t)F_1(x^*, x^*, y_t)\\ &= tF_1(y, x^*, y_t)\\ &\Longrightarrow F_1(y, x^*, y_t) \geq \frac{\alpha(x^*, y_t)}{t}. \end{aligned} \tag{3.28}$$

Since $F_1(y, x, ..)$ is hemicontinuous, taking $t \to 0$, we obtain

$$F_1(y, x^*, x^*) \geq 0. \tag{3.29}$$

This implies that $x^* \in SEP(F_1)$. Since A is a bounded linear operator, $Ax_{n_j} \rightharpoonup Ax^*$. From (3.14), we have that

$$T^{F_2}_{r_{n_j}} Ax_{n_j} \rightharpoonup Ax^*, \tag{3.30}$$

as $j \to \infty$. By the definition of $T^{F_2}_{r_{n_j}} Ax_{n_j}$, we have

$$F_2(y, T_{r_{n_j}}, y) + \frac{1}{r_{n_j}}\langle y - T^{F_2}_{r_{n_j}} Ax_{n_j}, J^p_{r_{n_j}} Ax_{n_j} - Ax_{n_j}\rangle \geq 0, \tag{3.31}$$

for all $y \in C$. Since F_2 is upper-hemicontinuous in the first argument and from (3.30), it follows that

$$F_2(y, Ax^*, Ax^*) \geq 0, \ \forall\, y \in C.$$

This implies that $Ax^* \in SEP(F_2)$, hence $x^* \in \Gamma$.
Next, we show that $\{x_n\}$ converges strongly to x^*. From (3.9), we have

$$\Delta_p(x_{n+1}, \overline{x}) \leq (1-\beta_n)\Delta_p(x_n, \overline{x}) + \beta_n\langle J^p_{E_1}(u) - J^p_{E_1}(\overline{x}), x_{n+1} - \overline{x}\rangle. \tag{3.32}$$

Now, we show that $\limsup_{n\to\infty}\langle J^p_{E_1}(u) - J^p_{E_1}(\overline{x}), x_{n+1} - \overline{x}\rangle \leq 0$. Since $\{x_n\}$ is bounded, we choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to x^*. Using (3.25), we also have that x_{n+1} converges weakly to x^*. Hence,

$$\begin{aligned}\limsup_{n\to\infty}\langle J^p_{E_1}(u) - J^p_{E_1}(\overline{x}), x_{n+1} - \overline{x}\rangle &= \lim_{n_j\to\infty}\langle J^p_{E_1}(u) - J^p_{E_1}(\overline{x}), x_{n_j+1} - \overline{x}\rangle\\ &= \langle J^p_{E_1}(u) - J^p_{E_1}(\overline{x}), x^* - \overline{x}\rangle \leq 0. \end{aligned} \tag{3.33}$$

Using Lemma (2.7) in (3.32), we have that $\Delta_p(x_n, x^*) \to 0,\ n \to\ \infty$. Therefore, $x_n \longrightarrow x^*$.

Case 2: Assume that $\{\Delta_p(x_n, \overline{x})\}_{n\in\mathbb{N}}$ is not monotonically decreasing sequence. Let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\{j \in \mathbb{N} : j \leq n, \Delta_p(x_{n_j}) \leq \Delta_p(x_{n_j+1})\}.$$

Obviously, τ is a non-decreasing sequence, such that $\tau(n) \to \infty asn \to \infty$, then

$$0 \leq \Delta_p(x_{\tau(n)}, x^*) \leq \Delta_p(x_{\tau(n)+1}, x^*),\ \forall\, n \geq n_0. \tag{3.34}$$

Following the same approach as in (3.9), it is easy to see that

$$\lim_{\tau(n)\to\infty} ||A^* J^p_{E_2}(T^{F_2}_{r_n} - I)Ax_{\tau(n)}|| = 0$$

Also, from (3.25) and (3.33)

$$\lim_{\tau(n)\to\infty} ||x_{\tau(n)+1} - x_{\tau(n)}|| = 0, \tag{3.35}$$

and

$$\lim_{\tau(n)\to\infty} \langle J^p_{E_1}(u) - J^p_{E_1}(\overline{x}), x_{\tau(n)+1} - \overline{x}\rangle \leq 0. \tag{3.36}$$

Using (3.9), we have that

$$\lim_{\tau(n)\to\infty} \Delta_p(u_{\tau(n)}, T_i u_{\tau(n)}) = 0. \tag{3.37}$$

From (3.8), we have that

$$\Delta_p(x_{\tau(n)+1}, x^*) \leq (1-\beta_{\tau(n)})\Delta_p(x_{\tau(n)}, x^*) + \beta_{\tau(n)}\langle J^p_{E_1}(u) - J^p_{E_1}(x^*), x_{\tau(n)+1} - x^*\rangle.$$

This implies that

$$\limsup_{\tau(n)\to\infty} \Delta_p(x_{\tau(n)}, x^*) \leq 0.$$

Hence,

$$\lim_{\tau(n)\to\infty} \Delta_p(x_{\tau(n)}, x^*) = 0.$$

Therefore, it follows from (1.7) that

$$\begin{aligned} 0 \leq \Delta_p(x_{\tau(n)+1}, x^*) &\leq \langle x_{\tau(n)+1} - x^*\rangle, J^p_{E_1}(x_{\tau(n)+1}) - J^p_{E_1}(x^*)\rangle \\ &\leq ||x_{\tau(n)+1} - x^*||\ ||J^p_{E_1}(x_{\tau(n)+1}) - J^p_{E_1}(x^*)|| \to 0,\ n \to\ \infty. \end{aligned}$$

Moreso, for $n \geq n_0$, it is easy to see that $\Delta_p(x_{\tau(n)}, x^*) \leq \Delta_p(x_{\tau(n)+1}, x^*)$ if $n \neq \tau(n)$ (that is $\tau(n) < n$), because $\Delta_p(x_{n_j}, x^*) \geq \Delta_p(x_{n_j+1}, x^*)$ for $\tau(n)+1 \leq j \leq n$. Hence, we obtain for all $n \geq n_0$,

$$\begin{aligned} 0 \leq \Delta_p(x_n, x^*) &\leq \max\{\Delta_p(x_{\tau(n)}, x^*),\ \Delta_p(x_{\tau(n)+1}, x^*)\}. \\ &= \Delta_p(x_{\tau(n)+1}, x^*). \end{aligned}$$

Hence, $\lim\limits_{n\to\infty} \Delta_p(x_n, x^*) = 0$, which implies that $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to x^*. This completes the proof. □

Remark 3.2.

1. The iterative scheme considered in this article has an advantage over the one considered in [13] in the sense that we do not use any projection of a point on the intersection of closed and convex sets which creates some difficulties in a practical calculation of the iterative sequence. The Halpern iteration considered in this article provides more flexibility in defining the algorithm parameters which is important for the numerical implementation perspective.

2. The result discussed in this article extends and generalizes the results of [1, 2] from Hilbert spaces to p-uniformly convex Banach spaces which are also uniformly smooth.

We give the following consequence of our main result as follows:.

Corollary 3.3. *Let D and G be nonempty bounded closed convex subsets of uniformly convex and uniformly smooth Banach spaces E_1 and E_2 respectively, and $A : E_1 \to E_2$ be a bounded linear operator with $A^* : E_2^* \to E_1^*$ being the adjoint of A. Let $F_1 : D \times D \times D \to \mathbb{R},\ F_2 : G \times G \times G \to \mathbb{R}$ be trifunctions satisfying the assumptions of Lemma 2.5 and Lemma 2.6 with F_1 and F_2 being continuous. Let T be a Bregman relatively nonexpansive mapping such that $\Gamma := Fix(T) \bigcap \Omega \neq \emptyset$, then the sequences $\{u_n\}$ and $\{x_n\}$ are generated iteratively by*

$$\begin{cases} u_n = T_{r_n}^{F_1}(J_{E_1^*}^q[J_{E_1}^p(x_n) - t_n A^* J_{E_2}^p(T_{r_n}^{F_2} - I)Ax_n]); \\ x_{n+1} = J_{E_1^*}^q[\beta_n J_{E_1}^p(u) + (1-\beta_n)(\alpha_n J_{E_1}^p(u_n) + (1-\alpha_n)J_{E_1}^p Tu_n)],\ n \geq 1, \end{cases} \tag{3.38}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ such that $r_n \subset (0,\infty)$ and the step size t_n is chosen in such a way that if $n \in \Theta := \{n : (T_{r_n}^{F_2} - I)Ax_n \neq 0\}$, then

$$t_n^{q-1} \in \left(0,\ \frac{q||(T_{r_n}^{F_2} - I)Ax_n||^p}{C_q||A^* J_{E_2}^p(T_{r_n}^{F_2} - I)Ax_n||^q}\right), \tag{3.39}$$

where C_q is the constant of smoothness of E_1. Otherwise, $t_n = t$ (t being any nonnegative real number) with the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\liminf_{n\to\infty} r_n > 0$;

(iii) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. *Then $\{x_n\}$ converges strongly to $\overline{x} \in \Pi_\Gamma u$.*

Corollary 3.4. *Let D be a nonempty bounded closed convex subset of a uniformly convex and uniformly smooth Banach space E and let $F : D \times D \times D \to \mathbb{R}$ be a trifunction satisfying the assumptions of Lemma 2.5 and Lemma 2.6 with F_1 being continuous. Let T be a Bregman relatively nonexpansive mapping such that $\Gamma := Fix(T) \bigcap EP(F) \neq \emptyset$, then the sequences $\{u_n\}$ and $\{x_n\}$ are generated iteratively by*

$$\begin{cases} u_n = J_{E_1^*}^q T_{r_n}^F; \\ x_{n+1} = J_{E_1^*}^q[\beta_n J_{E_1}^p(u) + (1-\beta_n)(\alpha_n J_{E_1}^p(u_n) + (1-\alpha_n)J_{E_1}^p Tu_n)],\ n \geq 1, \end{cases} \tag{3.40}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ such that $r_n \subset (0,\infty)$, where the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\liminf_{n\to\infty} r_n > 0$;

(iii) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. *Then $\{x_n\}$ converges strongly to $\overline{x} \in \Pi_\Gamma u$.*

We also consider a bifunction equilibrium problem.

Corollary 3.5. *Let D and G be nonempty bounded closed convex subsets of uniformly convex and uniformly smooth Banach spaces E_1 and E_2 respectively, and $A : E_1 \to E_2$ be a bounded linear operator with $A^* : E_2^* \to E_1^*$ being the adjoint of A. Let $F_1 : D \times D \to \mathbb{R},\ F_2 : G \times G \to \mathbb{R}$ be bifunctions satisfying the assumptions of Lemma 2.5 and Lemma 2.6 with F_1 and F_2 being continuous. Let $\{T_i\}_{i=1}^N$ be a finite family Bregman quasi nonexpansive mapping such that $\Gamma := Fix(T) \bigcap \Omega \neq \emptyset$, then the sequences $\{u_n\}$ and $\{x_n\}$ are generated iteratively by*

$$\begin{cases} u_n = T_{r_n}^{F_1}(J_{E_1^*}^q[J_{E_1}^p(x_n) - t_n A^* J_{E_2}^p(T_{r_n}^{F_2} - I)Ax_n]); \\ x_{n+1} = J_{E_1^*}^q[\beta_n J_{E_1}^p(u) + (1-\beta_n)(\alpha_n J_{E_1}^p(u_n) + (1-\alpha_n)J_{E_1}^p Tu_n)],\ n \geq 1, \end{cases} \tag{3.41}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0,1)$ *such that* $r_n \subset (0,\infty)$ *and the step size* t_n *is chosen in such a way that if* $n \in \Theta := \{n : (T_{r_n}^{F_2} - I)Ax_n \neq 0\}$*, then*

$$t_n^{q-1} \in \left(0,\ \frac{q||(T_{r_n}^{F_2} - I)Ax_n||^p}{C_q||A^* J_{E_2}^p (T_{r_n}^{F_2} - I)Ax_n||^q}\right), \tag{3.42}$$

where C_q *is the constant of smoothness of* E_1*. Otherwise,* $t_n = t$ *(t being any nonnegative real number) with the sequences* $\{\alpha_n\}$*,* $\{\beta_n\}$ *and* $\{r_n\}$ *satisfying the following conditions:*
(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
(ii) $\liminf_{n\to\infty} r_n > 0$;
(iii) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$*. Then* $\{x_n\}$ *converges strongly to* $\overline{x} \in \Pi_\Gamma u$*.*

All codes were written on a personal laptop HP ENVY core i5-5200U with MATLAB 2019b.

4. Numerical Example

Example 4.1. Let $E_1 = E_2 = \mathbb{R}$ and $D = G = \mathbb{R}$. Let $F_1(x,y) = -15x^2 + xy + 14y^2$, then we derive our resolvent function $T_r^{F_1}$, using Lemma 2.6 as follows:

$$\begin{aligned} F_1(x,y) + \frac{1}{r}(y-x)(x-t) \geq 0 & \\ &\Leftrightarrow -15rx^2 + rxy + 14ry^2 + xy - yt - x^2 + xt \geq 0 \\ &\Leftrightarrow 14ry^2 + rxy + xy - yt - 15rx^2 - x^2 + xt \geq 0 \\ &\Leftrightarrow 14ry^2 + (rx + x - t)y - (15rx^2 + x^2 - xt) \geq 0. \end{aligned}$$

Let $Q(y) = 14ry^2 + (rx + x - t)y - (15rx^2 + x^2 - rt)$. Then, Q is a quadratic function of y with coefficient $a = 14r$, $b = rx + x - t$, $c = -15rx^2 - x^2 + rt$. We compute the discriminant of $Q(y)$ as follows:

$$\begin{aligned} \triangle = b^2 - 4ac &= (rx + x - t)^2 - 4(14r)(-15rx^2 - x^2 + rt) \\ &= r^2x^2 + rx^2 - rxt + rx^2 + x^2 - xt - rxt - xt \\ &+ t^2 + 840r^2x^2 + 56rx^2 - 56rxt \\ &= 841r^2x^2 + 58rx^2 - 58rxt - 2xt + x^2 + t^2 \\ &= t^2 - 58rxt - 2xt + 841r^2x^2 + 58rx^2 + x^2 \\ &= t^2 - 2((29r+1)x)t + x^2 + 841r^2x^2 + 58rx^2 \\ &= t^2 - 2((29r+1)x)t + ((29r+1)x)^2 \geq 0. \end{aligned}$$

Thus, $\triangle \geq 0\ \forall\ t \in \mathbb{R}$ and it has at most one solution in $\mathbb{R}$, then $\triangle \leq 0$, $T_{r_n}^{F_1}(t) = \frac{t}{29r_n+1}$. Let $F_2(x,y) = -19x^2 + xy + 18y^2, Ax = x$ and $A^*x = x$. Following the same process used in obtaining $T_{r_n}^{F_1}$, we have that

$$T_{r_n}^{F_2}(t) = \frac{t}{37r_n + 1}.$$

Furthermore, define $T : \mathbb{R} \to \mathbb{R}$ by $T = \frac{x}{3}, \forall\ x \in \mathbb{R}$. Let $t_n = 1$, $r_n = \frac{1}{2}$, $\alpha_n = \frac{n}{3n+5}$ and $\beta_n = \frac{1}{2n+1}$. Then (3.41) becomes

$$\begin{cases} u_n = T_{r_n}^{F_1}[J_{E_1}^p(x_n) - t_n A^* J_{E_2}^p (T_{r_n}^{F_2} - I)Ax_n] \\ x_{n+1} = J_{E_1^*}^q[\frac{1}{2n+1}J_{E_1}^p(u) + \frac{2n}{2n+1}(\frac{n}{3n+5}J_{E_1}^p(u_n) + \frac{2n+5}{3n+5}J_{E_1}^p \frac{x}{3}u_n)]. \end{cases}$$

Case 1: $x_1 = (0.9)$ and $u = 0.7$.
Case 2: $x_1 = (0.5)$ and $u = 0.4$.
Case 3: $x_1 = (0.8)$ and $u = 0.5$.

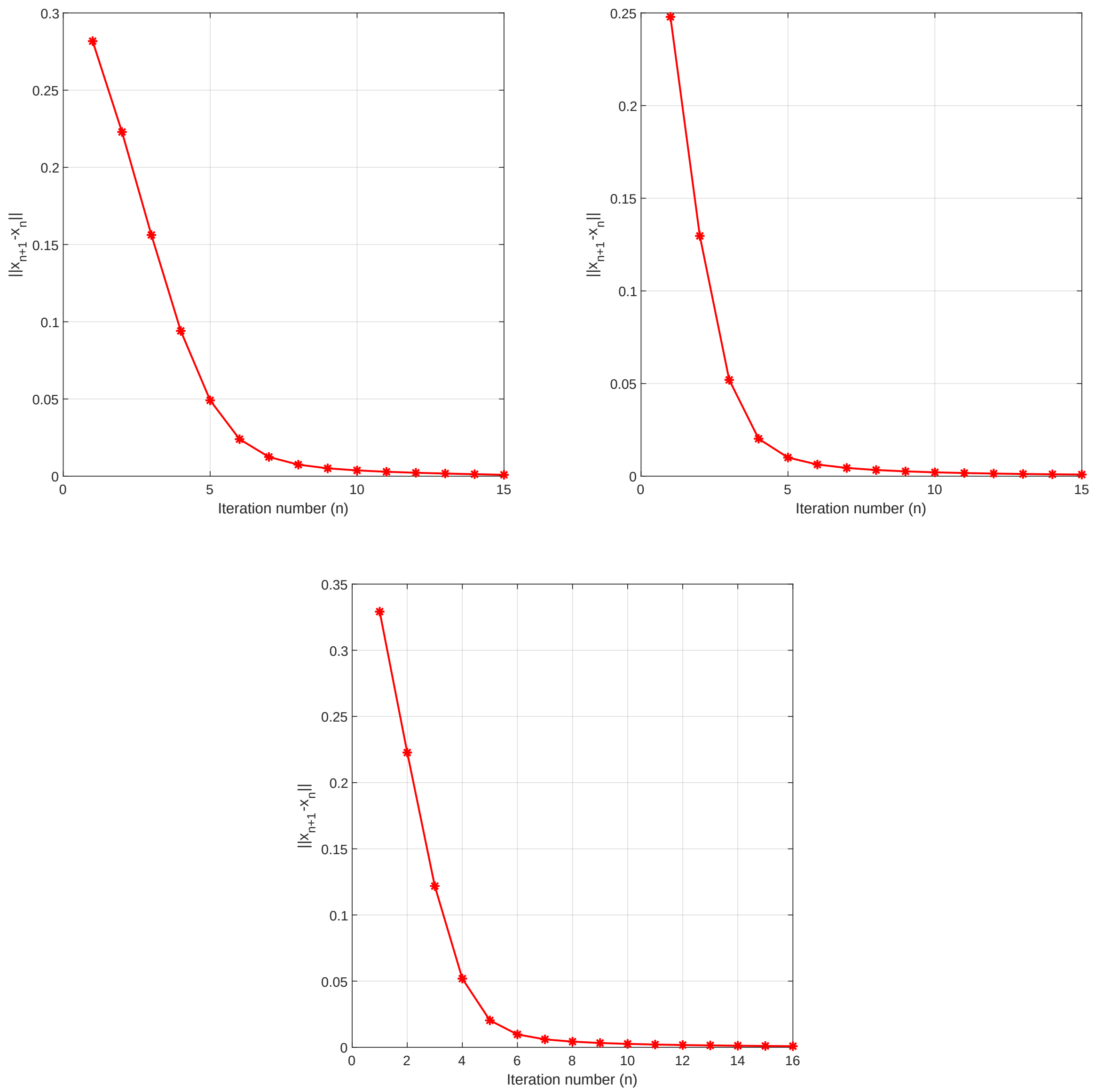

Figure 1: Example 4.1. Top left: Case 1, Top right: Case 2, Bottom: Case 3.

Example 4.2. Let $E_1 = E_2 = D = G = \mathbb{R}^3$. Define the trifunctions $F_1 : D \times D \times D \to \mathbb{R}$ and $F_2 : G \times G \times G \to \mathbb{R}^3$ respectively by

$$F_1(x, y, z) = \langle A^T Az, x - y\rangle, \quad \forall\, x, y, z \in \mathbb{R}^3,$$

where

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

and A^T is the transpose of matrix of A and

$$F_2(x, y, z) = \langle B^T Bz, x - y\rangle, \quad \forall\, x, y, z \in \mathbb{R}^3,$$

where

$$B = \begin{pmatrix} 2 & -1 & 3 \\ -1 & 2 & 3 \\ 1 & -1 & 3 \end{pmatrix}$$

and B^T is the transpose of matrix of B. From the definitions of F_1 and F_2 we see that

$$u = T_r^{F_1}(x) = \frac{x}{I + rA^tA}$$

and

$$v = T_r^{F_2}(y) = \frac{y}{I + rB^tB},$$

respectively for u and v in D and G. Let $N = 1$ and define the mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x) = (\frac{1}{2}x_1, x_2, \sin x_3), \ \forall\, x = (x_1, x_2, x_3)^T$. Let $i = 1$ and choose 0.5, $t_n = 1$, $r_n = \frac{1}{2}$, $\alpha_n = \frac{n}{3n+5}$ and $\beta_n = \frac{1}{2n+1}$. The results of this experiment for initial values of x_1 are displayed below as cases

(I) $x_1 = [2, 3, 5]^T$;

(II) $x_1 = [10, 20, 10]^T$;

(III) $x_1 = [0.5, 0.2, 0.125]^T$;

(IV) $x_1 = [-10, 15, -20]^T$.

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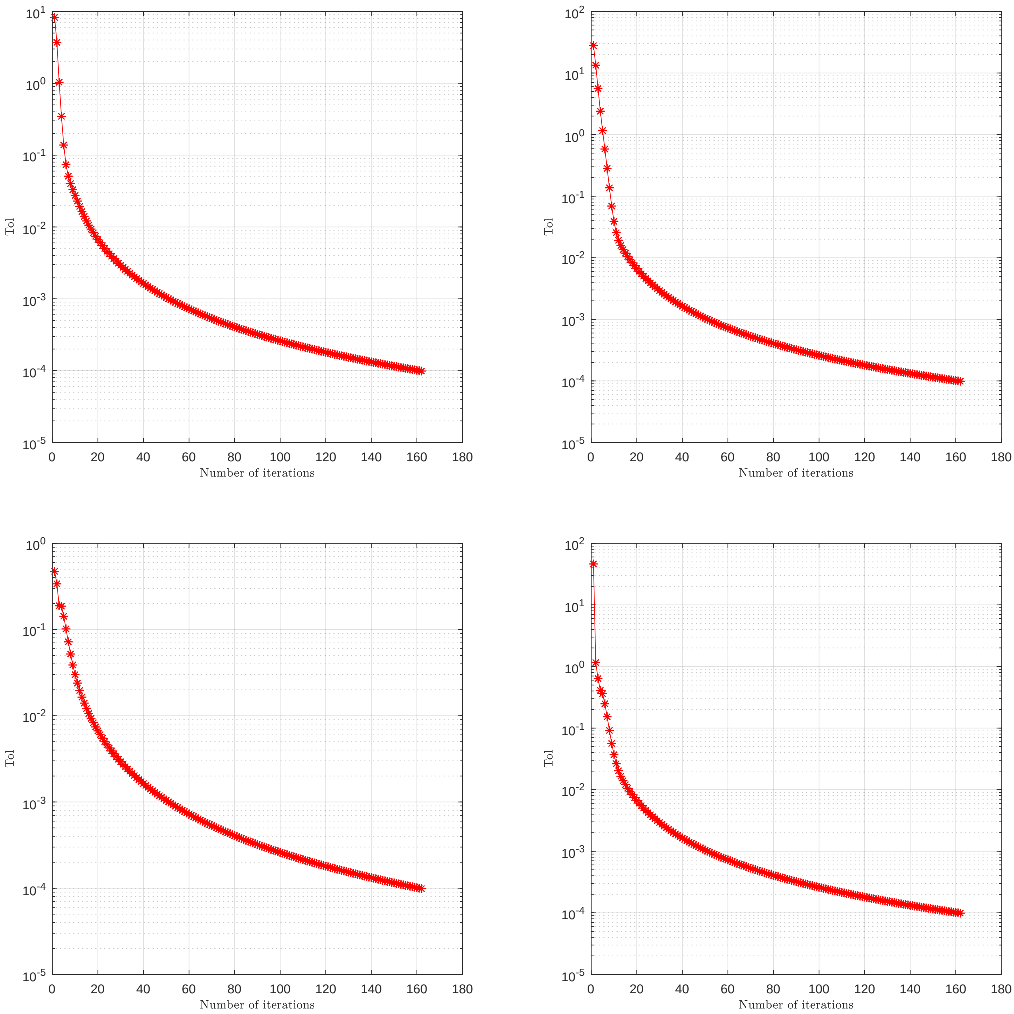

Figure 2: Example 4.2. Top left: I, Top right: II, Bottom left: III, Bottom right IV.

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